D.C. Versus Copositive Bounds for Standard QP

Kurt M. Anstreicher and Samuel Burer
Dept. of Management Sciences
University of Iowa

May 29, 2003

Abstract

The *standard quadratic program* (QPS) is $\min_{x \in \Delta} x^T Q x$, where $\Delta \subset \Re^n$ is the simplex $\Delta = \{x \geq 0 \mid \sum_{i=1}^n x_i = 1\}$. QPS can be used to formulate combinatorial problems such as the maximum stable set problem, and also arises in global optimization algorithms for general quadratic programming when the search space is partitioned using simplices. One class of "d.c." (for "difference between convex") bounds for QPS is based on writing $Q = S - T$, where S and T are both positive semidefinite, and bounding $x^T S x$ (convex on Δ) and $-x^T T x$ (concave on Δ) separately. We show that the maximum possible such bound can be obtained by solving a semidefinite programming (SDP) problem. The dual of this SDP problem corresponds to adding a simple constraint to the well-known Shor relaxation of QPS. We show that the max d.c. bound is dominated by another known bound based on a copositive relaxation of QPS, also obtainable via SDP at comparable computational expense. For the application of QPS to bounding the stability number of a graph, we use a novel formulation of the Lovasz ϑ number to compare ϑ, Schrijver's ϑ', and the max d.c. bound.

1 Introduction

Consider the *quadratic program on the simplex*

$$\begin{aligned} \text{QPS}: \qquad & \min \quad x^T Q x \\ & \text{s.t.} \quad x \in \Delta, \end{aligned}$$

where Q is symmetric, $\Delta = \{x \in \Re^n_+ \,|\, e^T x = 1\}$, $\Re^n_+$ denotes the non-negative orthant, and e is the vector of ones. The problem QPS is often refered to as the *standard quadratic program*. It is easy to show that in the general case QPS is NP-Hard; for example the problem of computing the maximum stable set in a graph can be posed as an instance of QPS. QPS also arises naturally in global optimization algorithms for general quadratic programming when simplices, rather than upper and lower bounds on variables, are used to partition the search space.

A number of recent papers have considered the construction of approximate solutions and/or lower bounds for QPS. Let v_{QPS} denote the solution value in QPS, and let $\bar{v}_{\text{QPS}} = \max_{x \in \Delta} x^T Q x$. Nesterov [7] constructs an approximate solution x satisfying

$$x^T Q x - v_{\text{QPS}} \leq \epsilon(\bar{v}_{\text{QPS}} - v_{\text{QPS}}) \tag{1}$$

for some $0 \leq \epsilon \leq 1$. In [7] two approaches yield approximations with $\epsilon = \frac{2}{3}$ and $\epsilon = \frac{1}{2}$. Bomze and de Klerk [2] consider families of linear programming (LP) and semidefinite programming (SDP) relaxations of QPS based on approximations of the cone of copositive matrices. For $r = 0, 1, \ldots$ either approach obtains a lower bound $v^r \leq v_{\text{QPS}}$ such that

$$v_{\text{QPS}} - v^r \leq \frac{1}{r+1}(\bar{v}_{\text{QPS}} - v_{\text{QPS}}). \tag{2}$$

In [2] it is also shown that a discretization of Δ, closely related to the LP bound, obtains a feasible solution $x \in \Delta$ satisfying (1) with $\epsilon = \frac{1}{r+2}$.

Bomze [1] suggests the use of "d.c." (for "difference between convex") bounds for QPS, based on writing $Q = S - T$, where $S \succeq 0$, $T \succeq 0$. It is then obvious that a lower bound for QPS is given by

$$v(S, T) = \min_{x \in \Delta} x^T S x + \min_{x \in \Delta} -x^T T x, \tag{3}$$

where the first minimization is convex and the second is concave. For any (S,T) the value of $v(S,T)$ is efficiently computable; in fact obtainable in polynomial time in the case where the entries of Q are rational [8]. Bomze considers several approaches for choosing a good (S,T), including maximizing a certain SDP approximation of $v(S,T)$.

In this paper we describe further results for the d.c. bounds considered by Bomze [1]. In the next section we show that for a given Q the bound $v(S,T)$ can be directly maximized by solving an SDP, resulting in an optimal bound of this type. The required SDP is no more complex than that used as an approximation of $v(S,T)$ in [1]. We also show that the dual of the SDP that gives the max d.c. bound has a surprising interpretation as a strengthening of the well-known Shor relaxation of QPS. In Section 3 we show that the optimal d.c. bound is itself dominated by the $r=0$ SDP bound from [2], which can be obtained at comparable computational effort. In Section 4 we consider in more detail the application of QPS to bound the size of the maximum stable set in a graph. We give a novel formulation of the Lovasz ϑ number that illustrates the relationship between ϑ, Schrijver's ϑ', and the max d.c. bound. In the last section we give some computational results, using a set of test problems considered in [2].

Notation. All matrices are symmetric. For matrices A and B we use $A \succeq B$ to denote that $A-B$ is positive semidefinite, and $A \geq 0$ to denote that A is componentwise nonnegative. The matrix inner product is written $A \bullet B = \mathrm{tr}(AB)$, where $\mathrm{tr}(\cdot)$ denotes the trace. For a matrix A, $\underline{\lambda}(A)$ and $\overline{\lambda}(A)$ denote the minimal and maximal eigenvalues, respectively. We use e to denote a vector of arbitrary dimension with each component equal to one, and $E = ee^T$. If A is a matrix and a is a vector, then $\mathrm{diag}(A)$ is the vector of diagonal components of A, and $\mathrm{Diag}(a)$ is the diagonal matrix with $\mathrm{diag}(\mathrm{Diag}(a)) = a$.

2 An optimal d.c. bound

The class of "d.c." bounds for QPS considered in [1] is based on writing $Q = S - T$, where $S \succeq 0$ and $T \succeq 0$. A lower bound on v_{QPS} is then given by $v(S,T)$, from (3). In [1] it is suggested that to obtain a good choice of (S,T) one could use the fact that

$$v(S,T) \geq v'(S,T) = \frac{1}{n}\underline{\lambda}(S) - \overline{\lambda}(T).$$

The problem of maximizing $v'(S,T)$ can be posed as an SDP. This SDP can be approximately solved to yield matrices $S \succeq 0$ and $T \succeq 0$ which can then be used to compute $v(S,T)$.

In this section we will show that the use of $v'(\cdot,\cdot)$ as a surrogate for $v(\cdot,\cdot)$ is unnecessary, and instead $v(\cdot,\cdot)$ can be directly maximized by solving an SDP. To obtain the required SDP we will use the fact that if $Q \succeq 0$, then QPS is equivalent to the Shor relaxation

$$\begin{array}{llll} \text{SQPS}: & \min & Q \bullet X \\ & \text{s.t.} & \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \succeq 0 \\ & & x \in \Delta. \end{array}$$

(Note that if $Q \not\succeq 0$ then the solution value in SQPS is $-\infty$.) It is straightforward to show that the dual of SQPS is

$$\begin{array}{llll} \text{DQPS}: & \max & \mu - \sigma \\ & \text{s.t.} & \begin{pmatrix} \sigma & s^T \\ s & Q \end{pmatrix} \succeq 0 \\ & & 2s + \mu e \leq 0. \end{array}$$

For any $Q \succeq 0$ the solution values of SQPS and DQPS are equal, and are both attained, since SQPS is equivalent to QPS and satisfies a Slater condition [8].

Now suppose that $Q = S - T$, where $S \succeq 0$, $T \succeq 0$. Then the first term in (3) can be expressed using DQPS, with S in place of Q, and the second is simply equal to $\min\{-t_{ii} \,|\, i = 1,\dots,n\} = \max\{-\theta \,|\, \theta \geq t_{ii},\ i = 1,\dots,n\}$. Combining the two terms written as maximizations, and considering S to be a variable, we arrive at the *optimal d.c. bound*

$$\begin{array}{lll} \text{DQPS}_{\text{DC}}: & \sup & \mu - \sigma - \theta \\ & \text{s.t.} & \begin{pmatrix} \sigma & s^T \\ s & S \end{pmatrix} \succeq 0 \\ & & 2s + \mu e \leq 0 \\ & & S \succeq Q, \quad \theta e \geq \operatorname{diag}(S - Q). \end{array}$$

The dual of DQPS_{DC} has a surprisingly simple interpretation as a strengthening of the Shor relaxation SQPS. It is straightforward to show that the dual of DQPS_{DC} is

$$\begin{array}{ll} \min & Q \bullet X \\ \text{s.t.} & \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \succeq 0 \\ & X \preceq \text{Diag}(y) \\ & x \in \Delta, \; y \in \Delta. \end{array}$$

Note that if (X, x, y) are feasible in this problem, then

$$1 = e^T(xx^T)e \leq e^T Xe \leq e^T \,\text{Diag}(y)e = 1, \tag{4}$$

and therefore $e^T(X - xx^T)e = e^T(\text{Diag}(y) - X)e = 0$. It follows that $Xe = x$ and $Xe = y$, so $x = y$ and the dual of DQPS_{DC} can be written in the simplified form

$$\text{SQPS}_{\text{DC}}: \qquad \begin{array}{ll} \min & Q \bullet X \\ \text{s.t.} & \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \succeq 0 \\ & X \preceq \text{Diag}(x) \\ & x \in \Delta. \end{array}$$

Thus the optimal d.c. bound corresponds exactly to adding the constraint $X \preceq \text{Diag}(x)$ to SQPS. Note that if $x \in \Delta$ then $xx^T \preceq \text{Diag}(x)$ holds (for example through an application of the Gerschgorin circle theorem), so this added constraint is certainly valid. The objective value in DQPS_{DC} is bounded and the feasible region satisfies a Slater condition, so the optimal values of SQPS_{DC} and DQPS_{DC} are equal and the value is attained in SQPS_{DC}. However SQPS_{DC} does *not* satisfy a Slater condition, because $Xe = x$ implies that

$$\begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \begin{pmatrix} -1 \\ e \end{pmatrix} = 0, \tag{5}$$

for any feasible solution. Note that it is obvious that the level sets in DQPS_{DC} are unbounded, since $v(S + \lambda E, T + \lambda E) = v(S, T)$ for any $\lambda \geq 0$.

3 Comparison with a copositive bound

In this section we will show that the optimal d.c. bound obtained in the previous section is dominated by another known bound for QPS. Consider a second semidefinite relaxation of QPS,

$$\begin{aligned} \mathrm{SQPS}^0_{\mathrm{CP}}: \qquad \min \quad & Q \bullet X \\ \text{s.t.} \quad & E \bullet X = 1 \\ & X \in \mathcal{K}_0^*, \end{aligned}$$

where $\mathcal{K}_0^* = \{X \mid X \succeq 0, X \geq 0\}$. A problem of the form $\mathrm{SQPS}^0_{\mathrm{CP}}$ is sometimes referred to as the "strengthened Shor relaxation" of QPS. The dual of $\mathrm{SQPS}^0_{\mathrm{CP}}$ is

$$\begin{aligned} \mathrm{DQPS}^0_{\mathrm{CP}}: \qquad \max \quad & \lambda \\ \text{s.t.} \quad & Q - \lambda E \in \mathcal{K}_0, \end{aligned}$$

where $\mathcal{K}_0 = \{X = S + P \mid S \succeq 0, P \geq 0\}$. The solution values in $\mathrm{SQPS}^0_{\mathrm{CP}}$ and $\mathrm{DQPS}^0_{\mathrm{CP}}$ are equal, and are both attained, since the feasible region in $\mathrm{SQPS}^0_{\mathrm{CP}}$ is compact and satisfies a Slater condition [8].

It is known that if $\mathcal{K}_0$ and $\mathcal{K}_0^*$ are replaced by the cone of symmetric copositive matrices and its dual, respectively, then $\mathrm{DQPS}^0_{\mathrm{CP}}$ and $\mathrm{SQPS}^0_{\mathrm{CP}}$ are equivalent to QPS [3]. Unfortunately these cones are not computationally tractable. However, it has been shown that there is a family of cones with SDP representations $\mathcal{K}_r$, $r \geq 0$, so that for any given Q, $\mathrm{SQPS}^0_{\mathrm{CP}}$ and $\mathrm{DQPS}^0_{\mathrm{CP}}$ are equivalent to QPS if $\mathcal{K}_0$ is replaced with $\mathcal{K}_r$, for r sufficiently large [4, 9]. In [2] it is shown that the use of the cone $\mathcal{K}_r$ in place of $\mathcal{K}_0$ produces a lower bound $v^r = v^r_{\mathrm{CP}}$ satisfying (2).

Theorem 1 *Let v_{DC} and v^0_{CP} denote the solution values in* $\mathrm{SQPS}_{\mathrm{DC}}$ *and* $\mathrm{SQPS}^0_{\mathrm{CP}}$*, respectively. Then $v^0_{\mathrm{CP}} \geq v_{\mathrm{DC}}$.*

Proof: Let X be a feasible solution in $\mathrm{SQPS}^0_{\mathrm{CP}}$, and let $x = Xe$. Then $x \in \Delta$. We claim that $xx^T \preceq X \preceq \mathrm{Diag}(x)$, so that (X, x) is feasible in $\mathrm{SQPS}_{\mathrm{DC}}$, which suffices to prove the theorem. The first claim, that $X \preceq \mathrm{Diag}(x)$, follows immediately from the nonnegativity of X and the Gerschgorin circle theorem. To see the second, let $X = WW^T$, where such W exists because $X \succeq 0$. Then

$$E \bullet X = \mathrm{tr}(EWW^T) = \mathrm{tr}(W^T EW) = 1,$$

and $W^T EW \succeq 0$. Hence $\overline{\lambda}(W^T EW) \leq 1$, implying that $I - W^T EW \succeq 0$. It follows that

$$W(I - W^T EW)W^T = X - Xee^T X = X - xx^T \succeq 0,$$

as claimed. $\square$

Note that if $Q \succeq 0$, then it is obvious that $v_{\mathrm{DC}} = v_{\mathrm{QPS}}$, since $v_{\mathrm{QPS}} = v(S,T)$ for $S = Q, T = 0$. From Theorem 1 it follows that $v^0_{\mathrm{CP}} = v_{\mathrm{QPS}}$ as well. This was not obvious *a-priori* since the constraint $X \succeq xx^T$ does not appear in $\mathrm{SQPS}^0_{\mathrm{CP}}$.

It is easy to construct cases where $v_{\mathrm{DC}} < v^0_{\mathrm{CP}}$. A very simple example uses the matrix

$$Q = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

For this Q it is clear that $v_{\mathrm{QPS}} = 0$, and $v^0_{\mathrm{CP}} = 0$ as well, since $\lambda = 0$ is feasible in $\mathrm{DQPS}^0_{\mathrm{CP}}$. However $v_{\mathrm{DC}} = \frac{-1}{8}$, where the optimal solution in $\mathrm{SQPS}_{\mathrm{DC}}$ is

$$X = \frac{1}{16}\begin{pmatrix} 3 & -1 & 2 \\ -1 & 3 & 2 \\ 2 & 2 & 4 \end{pmatrix}.$$

4 Bounding the stability number of a graph

In this section we consider in more detail the application of QPS to determining the size of the maximum stable set in a graph. It is well known that if $Q = I + A$, where A is the incidence matrix of a graph G, then $v_{\mathrm{QPS}} = 1/\alpha(G)$, where $\alpha(G)$ is the size of the maximum stable set [6]. Thus a lower bound on v_{QPS} provides an upper bound on the stability number. It is also known that for such problems $1/v^0_{\mathrm{CP}} = \vartheta'$, where ϑ' is Schrijver's strengthening of the Lovasz ϑ number [4]. The relationship between ϑ and ϑ' is well known, and that between

v_{DC} and v^0_{CP} is described in the previous section. The relationship between ϑ and v_{DC} is less clear, despite the fact that there are a variety of equivalent formulations of ϑ [5]. In this section we give a formulation of ϑ via a lower bound on QPS that facilitates a comparison with v_{DC}.

Let G be a graph with vertex set $\{1, 2, \ldots, n\}$, edge set $\mathcal{E}$ and incidence matrix A. We use $X_{\mathcal{E}} = 0$ (respectively $X_{\mathcal{E}} \geq 0$) to denote that $X_{ij} = 0$ (respectively $X_{ij} \geq 0$) for all $(i, j) \in \mathcal{E}$. The edges not in G are denoted $\bar{\mathcal{E}} = \{(i, j) \,|\, (i, j) \notin \mathcal{E}\}$. The Lovasz ϑ number can be defined via the SDP

$$\begin{aligned} \vartheta = \quad \max \quad & E \bullet X \\ \text{s.t.} \quad & \mathrm{tr}(X) = 1 \\ & X_{\mathcal{E}} = 0, \; X \succeq 0, \end{aligned}$$

while Schrijver's ϑ' is given by the problem

$$\begin{aligned} \vartheta' = \quad \max \quad & E \bullet X \\ \text{s.t.} \quad & \mathrm{tr}(X) = 1 \\ & A \bullet X = 0 \\ & X \geq 0, \; X \succeq 0. \end{aligned}$$

It is clear from the above formulations that ϑ' is a strengthening of ϑ obtained by adding the constraint $X \geq 0$. It is well known that

$$\alpha(G) \leq \vartheta' \leq \vartheta \leq \chi(\bar{G}),$$

where $\chi(\bar{G})$ is the coloring number of the complement of G.

In [4] the relationship $1/v^0_{\mathrm{CP}} = \vartheta'$ is established by first showing that

$$\begin{aligned} \vartheta' = \quad \max \quad & E \bullet X \\ \text{s.t.} \quad & (I + A) \bullet X = 1 \\ & X \in \mathcal{K}^*_0, \end{aligned}$$

which corresponds to simply replacing the two equality constraints in the original formulation for ϑ' with their sum. Swapping objective and constraint then yields an instance of $\mathrm{SQPS}^0_{\mathrm{CP}}$, with $Q = I + A$, whose optimal value is $1/\vartheta'$. Let $\mathcal{K}^*_{\mathcal{E}} = \{X \mid X_{\mathcal{E}} \geq 0,\ X \succeq 0\}$ and define

$$\begin{aligned} v_{\mathcal{E}} = \quad \min \quad & Q \bullet X \\ \text{s.t.} \quad & E \bullet X = 1 \\ & X \in \mathcal{K}^*_{\mathcal{E}}. \end{aligned}$$

Noting that ϑ may be reformulated as

$$\begin{aligned} \vartheta = \quad \max \quad & E \bullet X \\ \text{s.t.} \quad & \mathrm{tr}(X) = 1 \\ & A \bullet X = 0 \\ & X_{\mathcal{E}} \geq 0,\ X \succeq 0, \end{aligned}$$

the exact same argument as in [4] implies the following theorem.

Theorem 2 *Let G be a graph with vertex set $\{1, 2, \ldots, n\}$, edge set $\mathcal{E}$ and incidence matrix A. Then $\vartheta = 1/v_{\mathcal{E}}$, where $Q = I + A$.*

Using Theorem 2 we can give a clear comparison between the bounds $\vartheta = 1/v_{\mathcal{E}}$, $\vartheta' = 1/v^0_{\mathrm{CP}}$, and $1/v_{\mathrm{DC}}$ on the stability number. All three arise from optimization problems of the form

$$\begin{aligned} \min \quad & Q \bullet X \\ \text{s.t.} \quad & E \bullet X = 1 \\ & X \succeq 0, \end{aligned}$$

with additional constraints that vary for the three bounds. The added constraints corresponding to each of the three bounds are as follows.

$$\begin{aligned} \vartheta \;=\; & 1/v_{\mathcal{E}}\colon && X_{\mathcal{E}} \geq 0, \\ \vartheta' \;=\; & 1/v^0_{\mathrm{CP}}\colon && X \geq 0, \\ & 1/v_{\mathrm{DC}}\colon && Xe \geq 0,\ X \preceq \mathrm{Diag}(Xe). \end{aligned}$$

In the case of v_{DC}, note that $E \bullet X = 1$ follows from (4), and $X \succeq Xee^TX$ then holds as argued in the proof of Theorem 1. We have been unable to find an instance G where $v_{\mathrm{DC}} > v_{\mathcal{E}}$, but we are also unable to prove that $v_{\mathrm{DC}} \le v_{\mathcal{E}}$.

5 Computational results

In order to computationally compare v_{DC} with other bounds for QPS we considered a set of test problems used in [2]. The first two, problems 5.1 and 5.2, arise in estimating the maximum stable set on the graphs corresponding to the pentagon and the complement of the icosohedron, respectively. The next two problems, 5.3 and 5.4, arise from applications in population genetics and portfolio optimization, respectively. See [2] for more details and references. In Table 1 we give the values of several bounds for these four problems. The bounds v^0_{CP} and v^1_{CP} are the SDP bounds for $r = 0$ and $r = 1$, and v^1_{LP} is the LP bound for $r = 1$, all from [2]. Opt denotes the optimal solution value. All figures are rounded to 4 digits after the decimal point. Some caution is required in computing v_{DC} due to the fact that all feasible solutions of $\mathrm{SQPS}_{\mathrm{DC}}$ are singular (5). We obtained values of v_{DC} using the self-dual SDP code SeDuMi [11], which remains stable on problems of this type. (Alternatively one could use the reformulation of v_{DC} described at the end of the previous section.)

We know that in all cases $v_{\mathrm{DC}} \le v^0_{\mathrm{CP}} \le v^1_{\mathrm{CP}}$ and $v^1_{\mathrm{LP}} \le v^1_{\mathrm{CP}}$ must hold. For the problems considered in Table 1 it is interesting to note that $v^1_{\mathrm{LP}} < v_{\mathrm{DC}}$ throughout, and $v_{\mathrm{DC}} < v^0_{\mathrm{CP}}$ except on problem 5.4. It is also worth noting that the Lovasz ϑ number gives the same bound on problems 5.1 and 5.2 as $\vartheta' = 1/v^0_{\mathrm{CP}}$.

In addition to problems 5.1-5.4, [2] considers 20 problems based on estimating the maximum stable set for random graphs on 12 vertices constructed so as to have a maximum stable set of size 6. In [2] it is reported that for all 20 instances the QPS problem resulted in a bound v^1_{LP} of zero, and a value of v^1_{CP} equal to the true optimum ($v_{\mathrm{QPS}} = \frac{1}{6}$). We constructed 20 similar instances and obtained positive values for v_{DC} in all cases. The resulting bounds on the max stable set were however quite poor; greater than 12 in all but one case. (We have found other instances of random graphs where $v_{\mathrm{DC}} < 0$.) We also found that the Lovasz ϑ

Table 1: Comparison of bounds for instances of QPS

Problem [2]	n	v^1_{LP}	v_{DC}	v^0_{CP}	v^1_{CP}	Opt
5.1	5	0.3333	0.3528	0.4472	0.5000	0.5000
5.2	12	0.0000	0.0243	0.3090	0.3090	0.3333
5.3	5	-21.0000	-17.0096	-16.3333	-16.3333	-16.3333
5.4	5	0.3015	0.4839	0.4839	0.4839	0.4839

number gave the exact value $\vartheta = \alpha(G) = 6$ for all 20 instances, so that no improvement was possible from the more complex bounds v^0_{CP} and v^1_{CP}.

In addition to the quality of the various bounds it is interesting to consider the relative computational effort of obtaining them. SeDuMi requires the conversion of SQPS_{DC} and $\text{SQPS}^0_{\text{CP}}$ to a standard self-dual conic form, in which each variable is constrained to be in at most one cone. For any variable in the original form of the problem that is in two or more cones simultaneously, the conversion to standard form is achieved by introducing auxiliary variables, which are then themselves constrained to be in a single cone, as well as equality constraints linking the original and auxiliary variables. For example, in the case of $\text{SQPS}^0_{\text{CP}}$, which has $X \succeq 0$ and $X \geq 0$, we introduce $Y \geq 0$ and set $X = Y$. After the conversion, it is not difficult to see that SQPS_{DC} corresponds to optimization over the cone $\mathcal{S}^{n+1}_+ \times \mathcal{S}^n_+ \times \Re^n_+$ with $n(n+1)/2 + n + 2$ equality constraints, while $\text{SQPS}^0_{\text{CP}}$ is over the cone $\mathcal{S}^n_+ \times \Re^{n(n+1)/2}_+$ with $n(n+1)/2 + 1$ equality constraints. Here $\mathcal{S}^p_+$ is the cone of $p \times p$ symmetric positive semidefinite matrices. From a theoretical standpoint, SQPS_{DC} is simpler to solve using an interior-point algorithm since it has a barrier parameter that is $O(n)$, as opposed to $O(n^2)$ for $\text{SQPS}^0_{\text{CP}}$ [8]. In practice, however, it is well known that the number of iterations required by interior-point algorithms is almost independent of the barrier parameter. From a computational perspective, the work in each iteration for either problem is dominated by the time needed to form and factor the Schur complement matrix for calculating the Newton direction, which can be seen to be proportional to the cube of the number of equality constraints. This work is of the same order for SQPS_{DC} and $\text{SQPS}^0_{\text{CP}}$. In particular, neither problem appears to have an inherent structure – for example, a sparse Schur complement matrix – that would allow faster calculation of the Newton direction.

Hence, one would expect similar computational effort for solving both problems in practice.

In the specific case of calculating a bound on the stability number of a graph, it is also worth mentioning that the optimization problem that defines $v_{\mathcal{E}}$ is over the cone $\mathcal{S}^n_+ \times \Re^{|\mathcal{E}|}_+$ with $|\mathcal{E}|+1$ constraints. This problem is computationally cheaper than both $\mathrm{SQPS}_{\mathrm{DC}}$ and $\mathrm{SQPS}^0_{\mathrm{CP}}$, especially when the underlying graph is sparse.

References

[1] I.M. Bomze, Branch-and-bound approaches to standard quadratic optimization problems. *J. Global Optim.* 22 (2002), 27-37.

[2] I.M. Bomze and E. de Klerk, Solving standard quadratic optimization problems via linear, semidefinite, and copositive programming. *J. Global Optim.* 24 (2002), 163-185.

[3] I.M. Bomze, M. Dür, E. de Klerk, C. Roos, A. Quist and T. Terlaky, On copositive programming and standard quadratic optimization problems. *J. Global Optim.* 18 (2000), 301-320.

[4] E. de Klerk and D.V. Pasechnik, Approximation of the stability number of a graph via copositve programming. *SIAM J. Optim.* 12 (2002), 875-892.

[5] D.E. Knuth, The sandwich theorem. *Electronic J. Combin.* 1 (1994), 1-48.

[6] T.S. Motzkin and E.G. Straus, Maxima for graphs and a new proof of a theorem of Túran. *Canadian J. Math.* 17 (1965), 533-540.

[7] Y.E. Nesterov, Global quadratic optimization on the sets with simplex structure. Discussion Paper 9915, Center for Operations Research and Econometrics (CORE), Catholic University of Louvain, Louvain-la-Neuve, Belgium, 1999.

[8] Y. Nesterov and A. Nemirovskii, *Interior-Point Polynomial Algorithms in Convex Programming.* SIAM, Philadelphia, 1994.

[9] P.A. Parilo, Structured semidefinite programs and semi-algebraic geometry methods in robustness and optimization. Ph.D. thesis, California Institute of Technology, Pasadena, CA, 2000.

[10] A. Schrijver, A comparison of the Delsarte and Lovasz bounds. *IEEE Trans. Inform. Theory* 25 (1979), 425-429.

[11] J.F. Sturm, Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones. *Optim. Methods and Software* 11-12 (1999), 625-653.