Test Problems for Lipschitz Univariate Global Optimization with Multiextremal Constraints

Domenico Famularo
DEIS, Università degli Studi della Calabria, Via Pietro Bucci 41C-42C, 87036 Rende (CS), ITALY
e-mail - `famularo@deis.unical.it`

Yaroslav D. Sergeyev
ISI-CNR, c/o DEIS, Università degli Studi della Calabria, Via Pietro Bucci 41C-42C, 87036 Rende (CS), ITALY and Software Department, University of Nizhni Novgorod, Gagarin Av. 23, Nizhni Novgorod, RUSSIAN FEDERATION
e-mail - `yaro@si.deis.unical.it`

Paolo Pugliese
DEIS, Università degli Studi della Calabria, Via Pietro Bucci 41C-42C, 87036 Rende (CS), ITALY
e-mail - `pugliese@deis.unical.it`

Abstract. In this paper, Lipschitz univariate constrained global optimization problems where both the objective function and constraints can be multiextremal are considered. Two sets of test problems are introduced, in the first one both the objective function and constraints are differentiable functions and in the second one they are non-differentiable. Each series of tests contains 3 problems with one constraint, 4 problems with 2 constraints, 3 problems with 3 constraints, and one infeasible problem with 2 constraints. All the problems are shown in Figures. Lipschitz constants and global solutions are given. For each problem it is indicated whether the optimum is located on the boundary or inside a feasible subregion and the number of disjoint feasible subregions is given. Results of numerical experiments executed with the introduced test problems using Pijavskii's method combined with a non-differentiable penalty function are presented.

Keywords: Global optimization, multiextremal constraints, test problems, numerical experiments.

1. Introduction

In this paper we consider the global optimization problem with nonlinear constraints

$$\min\{f(x) : x \in [a,b], \quad g_j(x) \le 0, \quad 1 \le j \le m\}, \tag{1}$$

where $f(x)$ and $g_j(x), 1 \le j \le m$, are multiextremal Lipschitz functions (to unify the description process we shall use the designation $g_{m+1}(x) \triangleq f(x)$). More precisely, the functions $g_j(x), 1 \le j \le m+1$, satisfy the Lipschitz condition in the form

$$| g_j(x') - g_j(x'') | \le L_j \, | x' - x'' |, \quad x', x'' \in Q_j, \quad 1 \le j \le m+1. \tag{2}$$

where the constants

$$0 < L_j < \infty, \quad 1 \leq j \leq m+1, \tag{3}$$

are known. Since the functions $g_j(x), 1 \leq j \leq m$, are supposed to be multiextremal, the subdomains $Q_j \subset [a,b], 2 \leq j \leq m+1$, can have a few disjoint subregions each. In the following we shall suppose that all the sets $Q_j, 2 \leq j \leq m+1$, either are empty or consist of a finite number of disjoint intervals of a finite positive length.

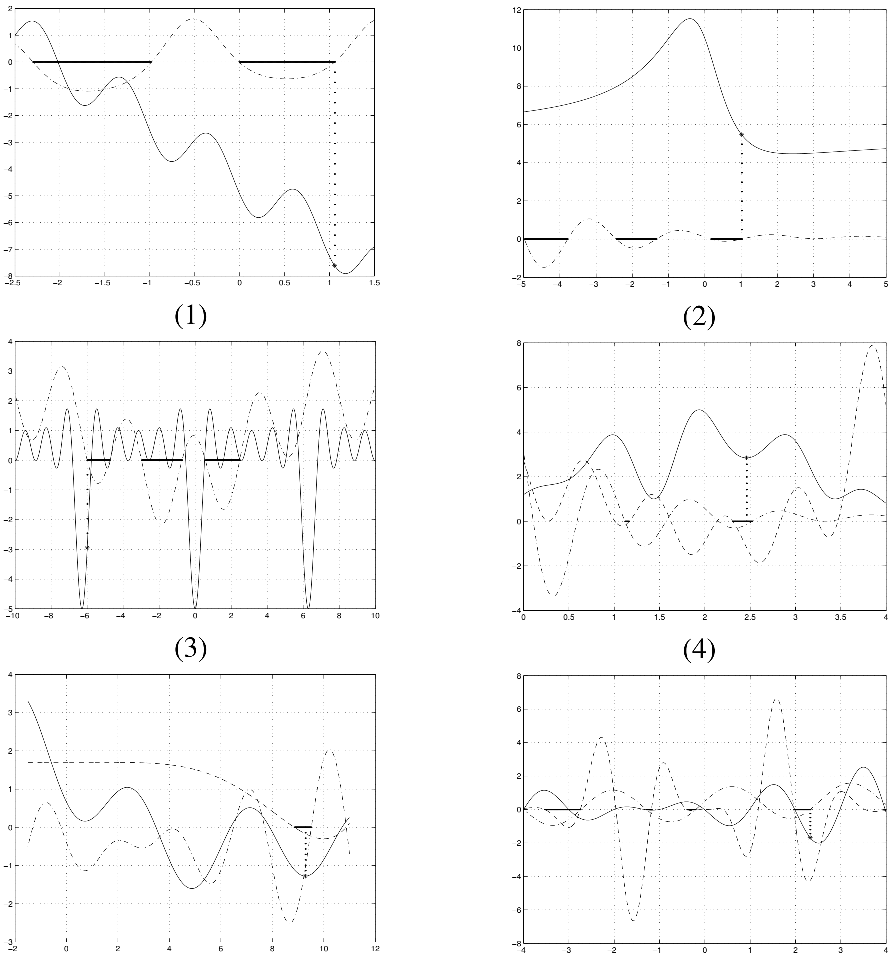
The recent literature in constrained optimization (Bomze et al., 1997; Floudas et al., 1999; Floudas and Pardalos, 1984; Hansen, Jaumard, and Lu, 1992a; Hansen, Jaumard, and Lu, 1992b; Horst and Pardalos, 1995; Horst and Tuy, 1993; Mockus, 1988; Mockus et al., 1996; Nocedal and Wright, 1999; Strongin and Sergeyev, 2000; Sun and Li, 1999) practically does not contain sets of tests with multiextremal constraints. This paper introduces problems for a systematic comparison of numerical algorithms developed for solving the global optimization problems with multiextremal constraints. Performance of the method of Pijavskii (Pijavskii, 1972; Hansen, Jaumard, and Lu, 1992a; Horst and Pardalos, 1995) combined with a non-differentiable penalty function on the introduced test problems is shown.

Two series of problems (ten feasible and one infeasible problem each) have been developed. The first series of tests is based on problems where both the objective function and the constraints are differentiable. The second series consists of problems where both the objective function and constraint are non-differentiable. Each series of tests contains:

- 3 problems with one constraint;
- 4 problems with 2 constraints;
- 3 problems with 3 constraints;
- one infeasible problem with 2 constraints.

For each problem the number of disjoint feasible subregions is presented. It is indicated whether the optimum is located on the boundary or inside a feasible subregion. All the problems are shown in Figures 1 and 2 (Differentiable problems) and Figures 3 and 4 (Non-differentiable problems). The constraints are drawn by dotted/mix-dotted lines and the objective function is drawn by a solid line. The feasible region is described by a collection of bold segments on the x axis and the global solution is represented by an asterisk located on the graph of the objective function.

In Table I (Differentiable problems) and Table II (Non-differentiable problems) the Lipschitz constants of the objective function and the constraints are reported together with an approximation of the global solution $(x^*, f(x^*))$.



(1) (2) (3) (4) (5) (6)

Figure 1. Differentiable Problems (1)-(6)

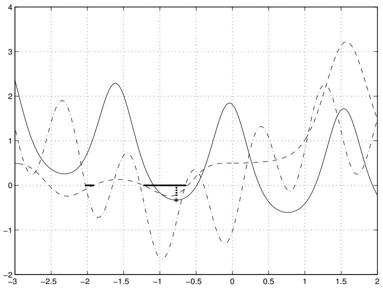
All these quantities have been computed by sweeping iteratively the search interval $[a, b]$ by the step $10^{-6}(b-a)$. Note that all the constrained problems have a unique global solution.

2. Differentiable problems

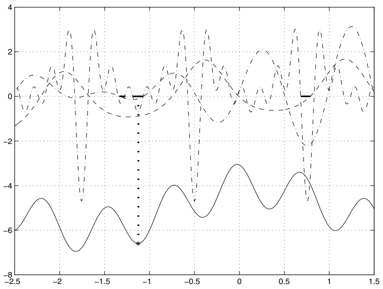
In this section the problems where both the objective function and the constraints are differentiable are described (see Table I).

Problem 1

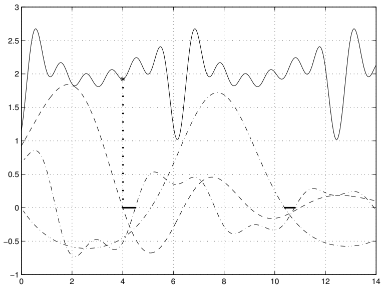
$$\min_{x\in[-2.5,1.5]} f(x) = -\frac{13}{6}x + \sin\left(\frac{13}{4}(2x+5)\right) - \frac{53}{12}$$



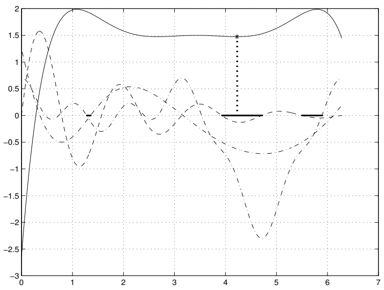
(7)



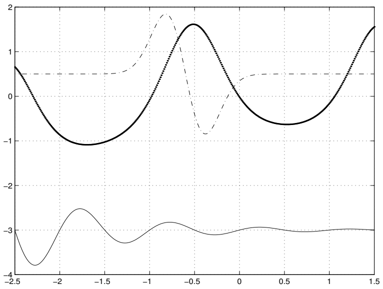
(8)



(9)



(10)



(11) (Infeasible)

Figure 2. Differentiable Problems (7)-(11)

subject to

$$g_1(x) = \exp(-\sin(3x)) - \frac{1}{10}\left(x - \frac{1}{2}\right)^2 - 1 \leq 0$$

The problem has 2 disjoint feasible subregions and the global optimum x^* is located on the boundary of one of the feasible subregions (see Figure 1-(1)).

Problem 2

$$\min_{x\in[-5,5]} f(x) = \frac{11x^2 - 10x + 21}{2\,(x^2+1)}$$

Table I. Differentiable problems. Lipschitz constants and global solutions.

	Lipschitz Constants				Global Solutions	
Pr.	$g_1(x)$	$g_2(x)$	$g_3(x)$	$f(x)$	x^*	$f(x^*)$
1	4.640837	–	–	8.666667	1.05738	−7.61284448
2	2.513269	–	–	6.372595	1.016	5.46063488
3	3.124504	–	–	13.201241	−5.9921	−2.94600839
4	29.731102	35.390605	–	12.893183	2.45956	2.8408089
5	5.654617	0.931981	–	2.021595	8.85725	−1.27299809
6	2.480000	25.108154	–	8.835339	2.32396	−1.6851399
7	8.332010	5.359309	–	6.387862	−0.774575	−0.33007413
8	20.184982	90.598898	6.372137	10.415012	−1.12724	−6.60059665
9	0.873861	1.682731	1.254588	3.843648	4.0000	1.92218867
10	3.170468	4.329008	7.999984	12.442132	4.2250023	1.474
11	4.640837	10.000000	–	6.283173	–	–

subject to

$$g_1(x) \ = \ \frac{1}{20} - \exp\left(-\frac{2}{5}(x+5)\right)\sin\left(\frac{4}{5}\pi(x+5)\right) \leq 0$$

The problem has 3 disjoint feasible subregions and the global optimum x^* is located on the boundary of one of the feasible subregions (see Figure 1-(2)).

Problem 3

$$\min_{x\in[-10,10]} f(x) = -\sum_{i=1}^{5}\cos(ix)$$

subject to

$$g_1(x) \ = \ \frac{3}{2}\left(\cos\left(\frac{7}{20}(x+10)\right) - \sin\left(\frac{7}{4}(x+10)\right) + \frac{1}{2}\right) \leq 0$$

The problem has 3 disjoint feasible subregions and the global optimum x^* is located on the boundary of one of the feasible subregions (see Figure 1-(3)).

Problem 4

$$\min_{x\in[0,4]} f(x) = 4\sin\left(\frac{\pi}{4}x + \frac{1}{20}\right)\cdot$$

$$\left(\sin^3\left(\frac{\pi}{2}x + \frac{1}{10}\right) + \cos^3\left(\frac{\pi}{2}x + \frac{1}{10}\right)\right)^2 + 1$$

subject to

$$\begin{aligned} g_1(x) &= \frac{6}{25} - 2\sum_{i=1}^{5} \cos\left(\frac{5}{4}(i+1)x + i\right) \le 0 \\ g_2(x) &= \frac{9}{50} - \frac{9}{2}\exp\left(-\left(x - \frac{1}{10}\right)\right)\sin\left(2\pi\left(x - \frac{1}{10}\right)\right) \le 0 \end{aligned}$$

The problem has 2 disjoint feasible subregions and the global optimum x^* is located inside one of the feasible subregions (see Figure 1-(4)).

Problem 5

$$\min_{x\in[-1.5,11]} f(x) = \sin(0.423531x + 3.13531) + \sin\left(\frac{10}{3}(0.423531x + 3.13531)\right)$$
$$+ \log(0.423531x + 3.13531) + 0.36634 - 0.355766x$$

subject to

$$\begin{aligned} g_1(x) &= \frac{17}{25} - \frac{2}{29763.233} \cdot \\ &\quad \left(-\frac{1}{6}x^6 + \frac{52}{25}x^5 - \frac{39}{80} - \frac{71}{10}x^3 + \frac{79}{20}x^2 + x - \frac{1}{10}\right) \le 0 \\ g_2(x) &= -\frac{14}{125}(3x - 8)\sin\left(\frac{252}{125}\left(x + \frac{3}{2}\right)\right) - \frac{1}{2} \le 0 \end{aligned}$$

The problem has one disjoint feasible region and the global optimum x^* is located inside the feasible region (see Figure 1-(5)).

Problem 6

$$\min_{x\in[-4,4]} f(x) = -\frac{7}{40}(3x + 4)\sin\left(\frac{63}{20}(x + 4)\right)$$

subject to

$$\begin{aligned} g_1(x) &= 40\left(\cos(4x)(x - \sin(x))\exp\left(-\frac{x^2}{2}\right)\right) \le 0 \\ g_2(x) &= \frac{2}{25}(x + 4) - \sin\left(\frac{12}{5}(x + 4)\right) \le 0 \end{aligned}$$

The problem has 4 disjoint feasible subregions and the global optimum x^* is located on the boundary of one of the feasible subregions (see Figure 1-(6)).

Problem 7

$$\min_{x\in[-3,2]} f(x) = \exp(-\cos(4x - 3)) + \frac{1}{250}(4x - 3)^2 - 1$$

subject to

$$
\begin{aligned}
g_1(x) &= \sin^3(x)\exp(-\sin(3x)) + \frac{1}{2} \leq 0 \\
g_2(x) &= \cos\left(\frac{7}{5}(x+3)\right) - \sin(7(x+3)) + \frac{3}{10} \leq 0
\end{aligned}
$$

The problem has 2 disjoint feasible subregions and the global optimum x^* is located inside one of the feasible subregions (see Figure 2-(7)).

Problem 8

$$
\min_{x\in[-2.5,1.5]} f(x) = \cos\left(\frac{7}{4}x + \frac{241}{40}\right) - \sin\left(\frac{35}{4}x + \frac{241}{8}\right) - 5
$$

subject to

$$
\begin{aligned}
g_1(x) &= \exp(-\sin(4x)) - \frac{1}{10}\left(x - \frac{1}{2}\right)^2 - 1 \leq 0 \\
g_2(x) &= \frac{3}{10} - \sum_{i=1}^{5} \cos\left(5(i+1)\left(x + \frac{1}{2}\right)\right) \leq 0 \\
g_3(x) &= \left(-\frac{21}{20}x - \frac{13}{8}\right)\sin\left(\frac{63}{10}x + \frac{63}{4}\right) + \frac{1}{5} \leq 0
\end{aligned}
$$

The problem has 3 disjoint feasible subregions and the global optimum x^* is located inside one of the feasible subregions (see Figure 2-(8)).

Problem 9

$$
\min_{x\in[0,14]} f(x) = \sum_{i=1}^{5} \frac{1}{5}\sin((i+1)x - 1) + 2
$$

subject to

$$
\begin{aligned}
g_1(x) &= \frac{1}{40}(x-4)\left(x - \frac{32}{5}\right)(x-9)(x-11)\cdot \\
&\quad \exp\left(-\frac{1}{10}\left(x - \frac{13}{2}\right)^2\right) \leq 0 \\
g_2(x) &= \left(\sin^3(x+1) + \cos^3(x+1)\right)\exp\left(-\frac{x+1}{10}\right) \leq 0 \\
g_3(x) &= \exp\left(-\cos\left(\frac{3}{5}\left(x - \frac{5}{2}\right)\right)\right) + \frac{1}{10}\left(\frac{3}{25}x - \frac{4}{5}\right)^2 - 1 \leq 0
\end{aligned}
$$

The problem has 2 disjoint feasible subregions and the global optimum x^* is located on the boundary of one of the feasible subregions (see Figure 2-(9)).

Problem 10

$$\begin{aligned}\min_{x\in[0,2\pi]} \quad f(x) &= -\frac{1}{500}\left(\frac{4}{\pi}\left(x-\frac{3}{10}\right)-4\right)^6+\frac{3}{100}\left(\frac{4}{\pi}\left(x-\frac{3}{10}\right)-4\right)^4\\ &\quad -\frac{27}{500}\left(\frac{4}{\pi}\left(x-\frac{3}{10}\right)-4\right)^2+\frac{3}{2}\end{aligned}$$

subject to

$$\begin{aligned} g_1(x) &= 2\exp\left(-\frac{2}{\pi}x\right)\sin(4x)\le 0\\ g_2(x) &= -\left(\frac{2}{\pi}x-\frac{1}{2}\right)^2\frac{\left(-\left(\frac{2}{\pi}x-\frac{1}{2}\right)^2+5\left(\frac{2}{\pi}x-\frac{1}{2}\right)-6\right)}{\left(\left(\frac{2}{\pi}x-\frac{1}{2}\right)^2+1\right)}-\frac{1}{2}\le 0\\ g_3(x) &= \sin^3(x)+\cos^3(2x)-\frac{3}{10}\le 0\end{aligned}$$

The problem has 2 disjoint feasible subregions and the global optimum x^* is located inside one of the feasible subregions (see Figure 2-(10)).

Problem 11

$$\min_{x\in[-2.5,1.5]} \quad f(x) = -\exp\left(-\left(x+\frac{5}{2}\right)\right)\sin\left(2\pi\left(x+\frac{5}{2}\right)\right)-3$$

subject to

$$\begin{aligned} g_1(x) &= \exp\left(-\sin(3*x)\right)-\frac{1}{10}\left(x-\frac{1}{2}\right)^2-1\le 0\\ g_2(x) &= \frac{1}{2}-\left(10\exp\left(-10\left(x+\frac{3}{5}\right)\right)\sin\left(x+\frac{3}{5}\right)\right)\le 0\end{aligned}$$

The problem is infeasible (see Figure 2-(11)).

3. Non-differentiable problems

In this section the problems where both the objective function and the constraints are non-differentiable are described in Table II and Figures 3 and 4.

Problem 1

$$\min_{x\in[-5,3]} f(x) = \left|\frac{x^2-10x+11}{2\,(x^2+1)}\right|+\left|\frac{3x^2+4x+1}{x^2+1}\right|$$

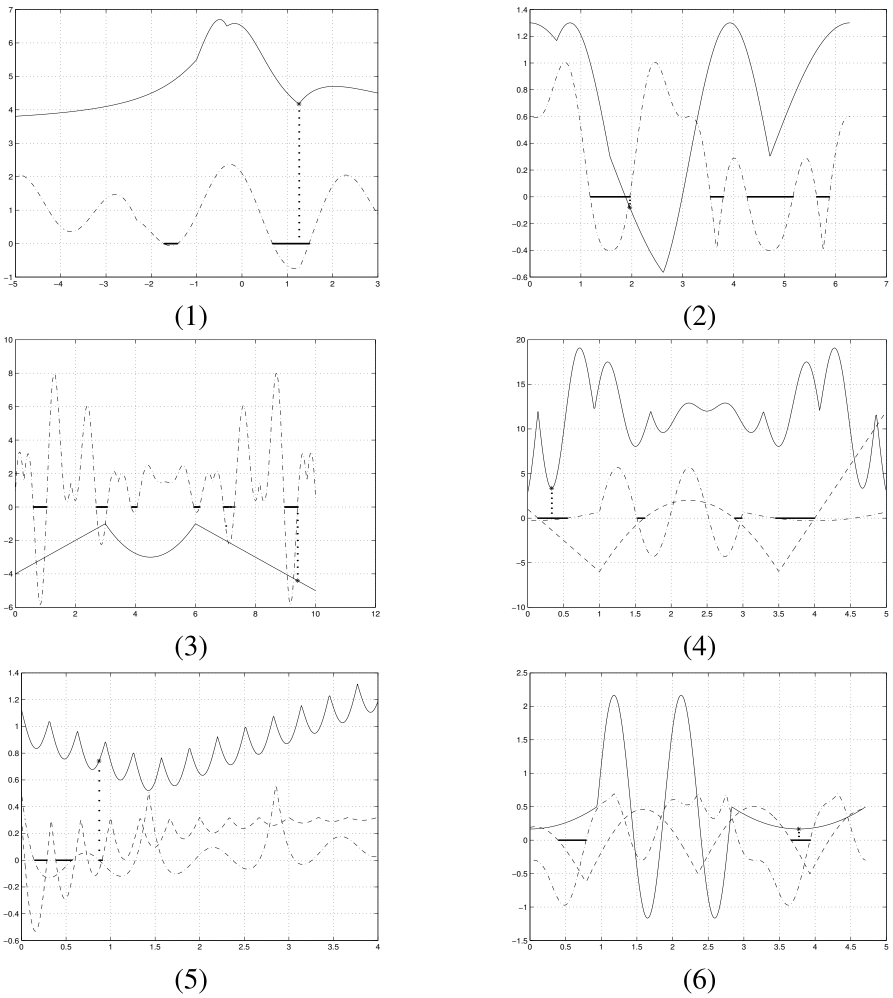


Figure 3. Non-Differentiable problems (1)-(6)

subject to

$$g_1(x) = \left|\sin\left(\frac{7}{554}(69x+347)\right)+\cos\left(\frac{7}{554}(69x+347)\right)\right| + \cos\left(\frac{21}{554}(69x+347)\right) \leq 0$$

The problem has 2 disjoint feasible subregions and the global optimum x^* is located inside one of the feasible subregions (see Figure 3-(1)). Note also that the minimum is located on a point at which the derivative is not continuous.

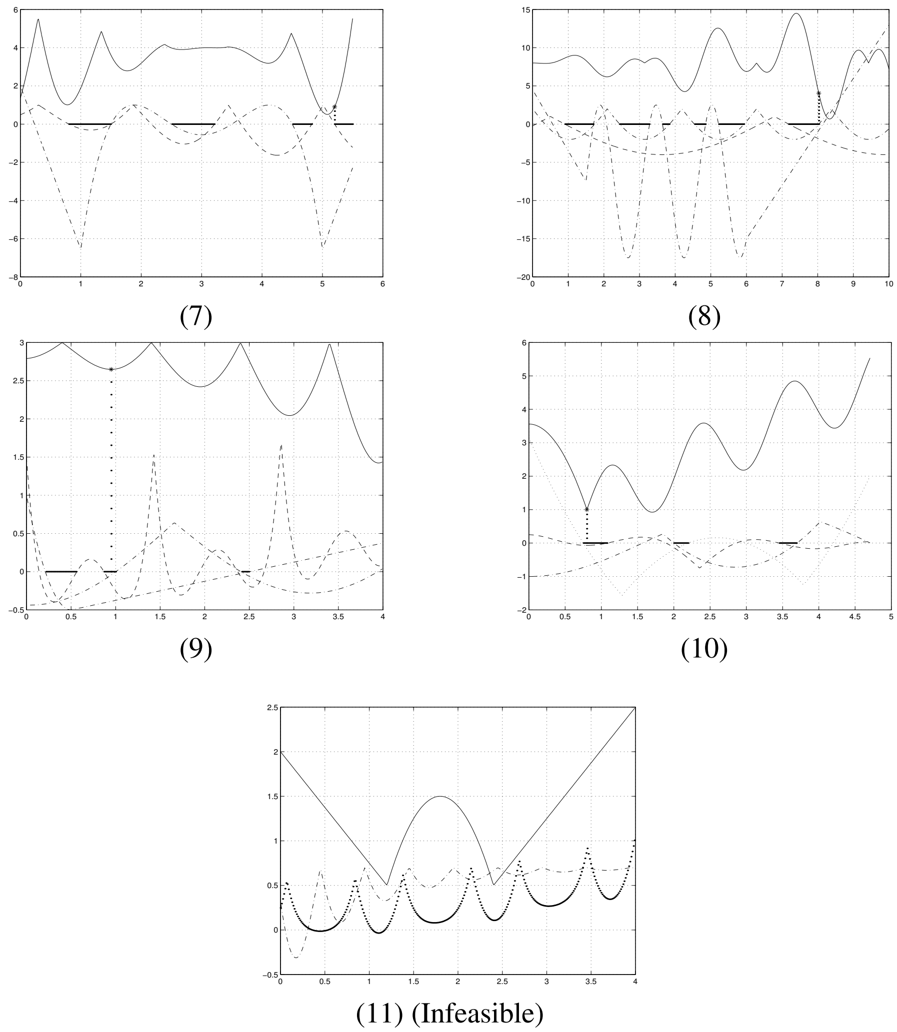


Figure 4. Non-Differentiable problems (7)-(11)

Problem 2

$$\min_{x\in[0,2\pi]} f(x) = \max\{\sin(2x), \cos(x)\} + \frac{3}{10}$$

subject to

$$g_1(x) \;=\; \left|\sin^3(2x) + \cos^3(x)\right| - \tfrac{2}{5} \le 0$$

The problem has 4 disjoint feasible subregions and the global optimum x^* is located on the boundary of one of the feasible subregions (see Figure 3-(2)).

Table II. Non-differentiable problems. Lipschitz constants and global solutions.

	Lipschitz Constants				Global Solutions	
Pr.	$g_1(x)$	$g_2(x)$	$g_3(x)$	$f(x)$	x^*	$f(x^*)$
1	3.808540	–	–	3.499998	1.25832	4.17418934
2	3.404631	–	–	2.000000	1.95966267	−0.07913964
3	47.250828	–	–	2.666662	9.40115	−4.40115
4	31.415927	12.799987	–	75.819889	0.33295	3.3461957
5	5.557050	9.424759	–	2.750000	0.86992	0.74162802
6	4.577345	2.166549	–	11.111111	3.76991118	0.16666667
7	21.999866	5.436564	–	23.400533	5.20115750	0.90278234
8	40.000000	6.000000	2.500000	23.625414	8.0285	4.0470244
9	1.050000	5.999988	16.671149	4.007294	0.95024	2.64804101
10	1.887454	2.334828	4.949998	6.399980	0.79999872	1.00000822
11	5.205608	6.921230	–	3.333328	–	–

Problem 3

$$\min_{x\in[0,10]} f(x) = \begin{cases} x-4 & x \le 3 \\ \frac{8}{9}x^2 - 8x + 15 & 3 < x \le 6 \\ -x+5 & x > 6 \end{cases}$$

subject to

$$g_1(x) = \frac{3}{2} - \cos(6(x-5))\,|2(x-5)\sin(2(x-5))| \le 0$$

The problem has 6 disjoint feasible subregions and the global optimum x^* is located on the boundary of one of the feasible subregions (see Figure 3-(3)).

Problem 4

$$\min_{x\in[0,5]} f(x) = 2\cos(4x-10)\,|(4x-10)\sin(4x-10)| + 12$$

subject to $g_1(x) \le 0,\ g_2(x) \le 0$ where

$$g_1(x) = \begin{cases} -7x+1 & x \le 1 \\ -\frac{128}{25}x^2 + \frac{576}{25}x - \frac{598}{25} & 1 < x \le \frac{7}{2} \\ 12(x-4) & x > \frac{7}{2} \end{cases}$$

$$g_2(x) = \begin{cases} x^2 - \frac{3}{10} & x \leq 1 \\ 5\sin(2\pi x) + \frac{7}{10} & 1 < x \leq 3 \\ x^2 - 8x + \frac{163}{10} & x > 3 \end{cases}$$

The problem has 4 disjoint feasible subregions and the global optimum x^* is located inside one of the feasible subregions (see Figure 3-(4)).

Problem 5

$$\min_{x\in[0,4]} \quad f(x) = \frac{1}{4}\left(\left|x - \frac{3}{2}\right| - |\sin(10x)| + 3\right)$$

subject to

$$g_1(x) = \frac{8}{25} - \exp(-x)\,|\sin(3\pi x)| \leq 0$$

$$g_2(x) = \exp\left(-\left|\sin\left(\frac{5}{2}\sin\left(\frac{11}{5}x\right)\right)\right|\right) - \frac{1}{2} + \frac{1}{100}x^2 \leq 0$$

The problem has 3 disjoint feasible subregions and the global optimum x^* is located on the boundary of one of the feasible subregions (see Figure 3-(5)).

Problem 6

$$\min_{x\in[0,1.5\pi]} \quad f(x) = \begin{cases} \frac{1}{3}\left(\frac{100}{9\pi^2}x^2 + \frac{1}{2}\right) & x \leq \frac{3\pi}{10} \\ \frac{5}{3}\sin\left(\frac{20}{3}x\right) + \frac{3}{2} & \frac{3\pi}{10} < x \leq \frac{9\pi}{10} \\ \frac{1}{3}\left(\frac{100}{9\pi^2}x^2 - \frac{80}{3\pi}x + \frac{33}{2}\right) & x > \frac{9\pi}{10} \end{cases}$$

subject to

$$g_1(x) = -\left|\frac{(x-\pi)^3}{100}\right| + |\cos(2(x-\pi))| - \frac{1}{2} \leq 0$$

$$g_2(x) = \frac{7}{10} - \left|\sin^3(3x) + \cos^3(x)\right| \leq 0$$

The problem has 2 disjoint feasible subregions and the global optimum x^* is located inside one of the feasible subregions (see Figure 3-(6)).

Problem 7

$$\min_{x\in[0,5.5]} f(x) = 4 - \frac{4}{3}\left(x - \frac{31}{10}\right)^2 \sin\left(\frac{1}{4}\left(x + \frac{9}{5}\right)\right).$$

$$\left(\left|\sin\left(3x+\frac{27}{5}\right)\right|-\frac{3}{10}\right)$$

subject to $g_1(x) \leq 0$, $g_2(x) \leq 0$ where

$$\begin{aligned}
g_1(x) &= -\left|\sin\left(2\left(x-\frac{3}{10}\right)\right)\right| \exp\left(\sin\left(\frac{1}{3}\left(x-\frac{3}{10}\right)\right)\right)+1 \\
g_2(x) &= \begin{cases} -\dfrac{137}{16}x+2 & x \leq 1 \\ -\left(x-\dfrac{3}{2}\right)\left(x-\dfrac{5}{2}\right)\left(x-\dfrac{7}{2}\right)\left(x-\dfrac{9}{2}\right) & 1 < x \leq 5 \\ \dfrac{137}{16}x-\dfrac{395}{8} & x > 5 \end{cases}
\end{aligned}$$

The problem has 4 disjoint feasible subregions and the global optimum x^* is located on the boundary of one of the feasible subregions (see Figure 4-(7)).

Problem 8

$$\min_{x\in[0,10]} \quad f(x) = -\cos(3x)\,|x\sin(x)|+8$$

subject to

$$\begin{aligned}
g_1(x) &= -5\left|\sin\left(\frac{1}{2}\left(x-\frac{1}{2}\right)\right)\right|+1 \leq 0 \\
g_2(x) &= -4\left|\sin\left(\frac{3}{2}x\right)\right|+2 \leq 0 \\
g_3(x) &= \begin{cases} -8x+\dfrac{9}{2} & x \leq \frac{3}{2} \\ 10\sin\left(4\left(x-\dfrac{3}{2}\right)\right)-\dfrac{15}{2} & \frac{3}{2} < x \leq 6 \\ 7x-\dfrac{99}{2}+10\sin(18) & x > 6 \end{cases} \leq 0
\end{aligned}$$

The problem has 6 disjoint feasible subregions and the global optimum x^* is located on the boundary of one of the feasible subregions (see Figure 4-(8)).

Problem 9

$$\min_{x\in[0,4]} f(x) = 3-2\exp\left(-\frac{1}{2}\left(\frac{22}{5}-x\right)\right)\left|\sin\left(\pi\left(\frac{22}{5}-x\right)\right)\right|$$

subject to

$$
\begin{aligned}
g_1(x) &= 3\left(\exp\left(-\left|\sin\left(\frac{5}{2}\sin\left(\frac{11}{5}x\right)\right)\right|\right)+\frac{1}{100}x^2-\frac{1}{2}\right)\leq 0\\
g_2(x) &= \begin{cases} 6\left(x-\frac{1}{2}\right)^2-\frac{1}{2} & x\leq\frac{1}{2}\\ \frac{1}{4}\left(x-\frac{5}{2}\right) & x>\frac{1}{2}\end{cases}\leq 0\\
g_3(x) &= \frac{4}{5}-\left(\left|\sin\left(\frac{24}{5}-x\right)\right|+\frac{6}{25}-\frac{x}{20}\right)\leq 0
\end{aligned}
$$

The problem has 3 disjoint feasible subregions and the global optimum x^* is located inside one of the feasible subregions (see Figure 4-(9)).

Problem 10

$$
\min_{x\in[0,1.5\pi]} f(x) = \begin{cases} -4x^2+\frac{89}{25} & x\leq\frac{4}{5}\\ \sin(5x-4)+x-\frac{1}{5} & x>\frac{4}{5}\end{cases}
$$

subject to

$$
\begin{aligned}
g_1(x) &= \max\left\{\left(x-\frac{3}{4}\right)\left(x-\frac{21}{5}\right),-\left(x-\frac{11}{5}\right)(x-3)\right\}\leq 0\\
g_2(x) &= -\max\left\{-\left(x-\frac{37}{10}\right)(x-2),\cos(x)\right\}\leq 0\\
g_3(x) &= \exp\left(-\frac{x}{20}\right)\left|\sin^3(x)+\cos^3(x)\right|-\frac{3}{4}\leq 0
\end{aligned}
$$

The problem has 3 disjoint feasible subregions and the global optimum x^* is located inside one of the feasible subregions (see Figure 4-(10)).

Problem 11

$$
\min_{x\in[0,4]} f(x) = \begin{cases} \frac{1}{2}\left(4-\frac{5}{2}x\right) & x\leq\frac{6}{5}\\ \frac{1}{2}\left(-\frac{50}{9}x^2+20x-15\right) & \frac{6}{5}<x\leq\frac{12}{5}\\ \frac{5}{2}\left(\frac{x}{2}-1\right) & x>\frac{12}{5}\end{cases}
$$

Table III. Differentiable functions. Numerical results obtained by the method of Pijavskii working with the penalty function (4).

Problem	XPEN	FXPEN	P^*	Iterations	Eval.
1	1.05718004	−7.61185807	15	83	166
2	1.01609254	5.46142698	90	954	1906
3	−5.99184997	−2.94292577	15	119	238
4	2.45953057	2.84080890	490	1762	5286
5	9.28468704	−1.27484673	15	765	2295
6	2.32334492	−1.68307049	15	477	1431
7	−0.77476915	−0.33007412	15	917	2751
8	−1.12719146	−6.60059658	15	821	3284
9	4.00042801	1.92220821	15	262	1048
10	4.22482084	1.47400000	15	2019	8076
Average	–	–	–	817.9	2648.1

subject to

$$\begin{aligned} g_1(x) &= \exp\left(-\left|\cos\left(\frac{13}{5}\sin\left(\frac{12}{5}\left(x+\frac{1}{5}\right)\right)\right)\right|\right)-\frac{9}{20}+ \\ &\quad \frac{1}{36}\left(x+\frac{1}{5}\right)^2 \le 0 \\ g_2(x) &= \frac{7}{10}-\exp\left(-\left(x-\frac{1}{5}\right)\right)\left|\cos\left(2\pi\left(x-\frac{1}{5}\right)\right)\right| \le 0 \end{aligned}$$

The problem is infeasible (see Figure 4-(11)).

4. Numerical experiments

In this section the method proposed by Pijavskii (see (Pijavskii, 1972; Hansen, Jaumard, and Lu, 1992a; Horst and Pardalos, 1995)) has been tested on the problems described in the previous Sections. Since the method (Pijavskii, 1972) works with problems having box constraints, in the executed experiments the constrained problems were reduced by the method of penalty functions to such a form. The same accuracy $\varepsilon = 10^{-4}(b-a)$ (where b and a represent the extrema of the optimization interval) has been used in all the experiments.

Table IV. Non-Differentiable problems. Numerical results obtained by the method of Pijavskii working with the penalty function (4).

Problem	XPEN	FXPEN	P^*	Iterations	Eval.
1	1.25810384	4.17441502	15	247	494
2	1.95953624	−0.07902265	15	241	482
3	9.40072023	−4.40072023	15	797	1594
4	0.33278550	3.34620350	15	272	819
5	0.86995489	0.74168456	20	671	2013
6	3.76944805	0.16666667	15	909	2727
7	5.20113260	0.90351752	15	199	597
8	8.02859874	4.05157770	15	365	1460
9	0.95019236	2.64804101	15	1183	4732
10	0.79988668	1.00072517	15	135	540
Average	–	–	–	501.9	1545.8

In Table III (Differentiable problems) and Table IV (Non-Differentiable problems) the results obtained by the method of Pijavskii are collected. The constrained problems were reduced to the unconstrained ones as follows

$$f_{P^*}(x) = f(x) + P^* \max\{g_1(x), g_2(x), \ldots, g_{N_v}(x), 0\}. \tag{4}$$

The coefficient P^* has been computed by the rules:

1. the coefficient P^* has been chosen equal to 15 for all the problems and it has been checked if the found solution (XPEN,FXPEN) for each problem belongs or not to the feasible subregions;

2. if it does not belong to the feasible subregions, the coefficient P^* has been iteratively increased by 10 starting from 20 until a feasible solution has been found. Particularly, this means that a feasible solution has not been found in Table III for the problem 2 when P^* is equal to 80, for the problem 4 when P^* is equal to 480, and in Table IV for the problem 5 when P^* is equal to 15.

It must be noticed that in Tables III, IV the column "Evaluations" shows the total number of evaluations of the objective function $f(x)$ and all the constraints. Thus, it is equal to

$$(N_v + 1) \times N_{iter},$$

where N_v is the number of constraints and N_{iter} is the number of iterations for each problem.

5. A brief conclusion

In this paper, 22 test problems for Lipschitz univariate constrained global optimization have been proposed. All the problems have both the objective function and constraints multiextremal. The problems have been collected in two sets. The first one contains tests with both the objective function and constraints being differentiable functions. The second set of tests contains problems with non-differentiable functions. Each series of tests consists of 3 problems with one constraint, 4 problems with 2 constraints, 3 problems with 3 constraints, and one infeasible problem with 2 constraints.

All the test problems have been studied in depth. Each problem has been provided with:

- an accurate estimate of the global solution;
- an accurate estimate of Lipschitz constants for the objective functions and constraints;
- figure showing the problem with the global solution and the feasible region with indication of the number of disjoint feasible subregions;
- indication whether the optimum was located on the boundary or inside a feasible subregion.

Numerical experiments with the introduced test problems have been executed with Pijavskii's method combined with a non-differentiable penalty function. Accurate estimates of the penalty coefficients providing the highest speed have been obtained.

References

Bomze I.M., T. Csendes, R. Horst, and P.M. Pardalos (1997) *Developments in Global Optimization*, Kluwer Academic Publishers, Dordrecht.

Floudas C.A., P.M. Pardalos, C. Adjiman, W.R. Esposito, Z.H. Gms, S.T. Harding, J.L. Klepeis, C.A. Meyer, C.A. Schweiger (1999), *Handbook of Test Problems in Local and Global Optimization*, Kluwer Academic Publishers, Dordrecht.

Floudas C.A. and P.M. Pardalos (1996), *State of the Art in Global Optimization*, Kluwer Academic Publishers, Dordrecht.

Hansen P., B. Jaumard and S.-H. Lu (1992), Global optimization of univariate Lipschitz functions: 1. Survey and properties, *Math. Programming*, **55**, 251–272.

Hansen P., B. Jaumard and S.-H. Lu (1992), Global optimization of univariate Lipschitz functions: 2. New algorithms and computational comparison, *Math. Programming*, **55**, 273–293.

Horst R. and P.M. Pardalos (1995), *Handbook of Global Optimization*, Kluwer Academic Publishers, Dordrecht.

Horst R. and H. Tuy (1993), *Global Optimization - Deterministic Approaches*, Springer–Verlag, Berlin.

Mockus, J. (1988). *Bayesian Approach to Global Optimization*, Kluwer Academic Publishers, Dordrecht.

Mockus, J., Eddy, W., Mockus, A., Mockus, L., and Reklaitis, G. (1996). *Bayesian Heuristic Approach to Discrete and Global Optimization: Algorithms, Visualization, Software, and Applications*. Kluwer Academic Publishers, Dordrecht.

Nocedal J. and S.J. Wright (1999), *Numerical Optimization* (Springer Series in Operations Research), Springer Verlag.

Pijavskii S.A. (1972), An Algorithm for Finding the Absolute Extremum of a Function, *USSR Comput. Math. and Math. Physics*, **12** 57–67.

Strongin R. G. and Ya. D. Sergeyev (2000), *Global Optimization with Non-Convex Constraints: Sequential and Parallel Algorithms*, Kluwer Academic Publishers, Dordrecht.

Sun X.L. and D. Li (1999), Value-estimation function method for constrained global optimization, *JOTA*, **102(2)**, 385–409.