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# Assignment of Reusable and Non-Reusable Frequencies*

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**Abstract**

Graph radio coloring and graph radio labelling are combinatorial models for two interesting cases of Frequency Assignment. In both problems positive integer labels (channels) must be assigned to all the vertices of a graph such that adjacent vertices get labels at distance at least two. In radio labelling all the labels must be distinct, while in radio coloring only the vertices being at distance no more than two in the input graph must be assigned distinct labels. For both problems the objective is to minimize the maximum label used.

We first prove that both radio coloring and radio labelling remain $\mathcal{NP}$-complete for graphs of diameter two, and we show that a $2\alpha$-approximation algorithm for radio coloring can be obtained from any $\alpha$-approximation algorithm for coloring squares of graphs. We also show that radio labelling is equivalent to Hamiltonian Path with distances one and two (HP(1,2)), and we present a polynomial-time algorithm for computing an optimal radio labelling, given a coloring of the input graph with constant number of colors. Thus we prove that radio labelling is in $\mathcal{P}$ for planar graphs. Additionally, we present competitive algorithms and a lower bound for on-line radio labelling.

**Keywords**: Frequency Assignment, Hamiltonian Cycle, Polynomial-Time Algorithms, On-line Algorithms.

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# 1 Introduction

The Frequency Assignment Problem (FAP) arises from the fact that radio transmitters operating at the same or closely related frequency channels have the potential to interfere with each other. FAP can be formulated as an optimization problem as follows: Given a collection of transmitters to be assigned operating channels and a set of interference constraints on transmitter pairs, find an assignment that fulfills all the interference constraints and minimizes the allocated bandwidth.

A common model for FAP is the *interference graph*. Each vertex of an interference graph represents a transmitter, while each edge represents an interference constraint between the adjacent transmitters. The frequency channels are usually assumed to be uniformly spaced in the spectrum and are labelled using positive integers. Frequency channels with adjacent integer labels are assumed adjacent in the spectrum [6, 7, 10]. Clearly, FAP is a generalization of graph coloring [7].

Instead of specifying the interference constraints for each pair of transmitters, FAP can be defined by specifying a minimum allowed spatial distance for each channel/spectral separation that is a potential source of interference [6, 7]. In [7] these are called *Frequency-Distance* (FD) constraints. FD-constraints can be given as a set of distances $\{D_0, D_1, \ldots, D_\kappa\}$, $D_0 \geq D_1 \geq \cdots \geq D_\kappa$, where the distance $D_x$, $x \in \{0, 1, \ldots, \kappa\}$, is the minimum distance between transmitters using channels at distance $x$.

The FD($\kappa$)-coloring problem has been proposed as a model for FAP in unweighted interference graphs [10, 17]. In FD($\kappa$)-coloring we seek a function $\mathrm{X}_\kappa : V \mapsto \{1, \ldots, \nu\}$ that fulfills the FD-constraints with respect to the graph distances $D_0 = \kappa + 1, D_1 = \kappa, \ldots, D_\kappa = 1$, and minimizes the largest color $\nu$ used (color span). Alternatively, $v, u \in V$ are only allowed to get colors at distance $x$, $|\mathrm{X}_\kappa(v) - \mathrm{X}_\kappa(u)| = x$, $x \in \{0, 1, \ldots, \kappa\}$, if $v$ and $u$ are at distance at least $\kappa - x + 1$ from each other in the interference graph. A polynomial-time exact algorithm for a variant of FD($\kappa$)-coloring in lattices is presented in [10]. FD(2)-coloring, which is also called *radio coloring*, is a combinatorial model for the widely used "co-channel" and "adjacent-channel" interference constraints. A problem similar to FD(2)-coloring is studied [13] in the context of mobile networks, where each vertex may require more than one colors (multicoloring). A polynomial-time approximation algorithm for triangular lattices is presented in [13].

In some practical applications the transmitters cover a local or metropolitan area. Hence, the transmitters are not allowed to operate at the same channel ($D_0 = \infty$, non-reusable frequency channels). The problem of *radio labelling* is the equivalent of radio coloring in the context of non-reusable frequency assignment. In particular, a valid radio labelling fulfills the FD-constraints with respect to $D_0 = \infty, D_1 = 2, D_2 = 1$. In radio labelling we seek an assignment of distinct integer labels to all the vertices of a graph, such that adjacent vertices get labels at distance at least two. The objective is to minimize the maximum label used (label span). The definition of radio labelling is communicated to us by [9].

Radio labelling and similar combinatorial models for non-reusable frequency assignment are used for obtaining lower bounds on the optimal values of general FAP instances [2, 17]. Since general instances of FAP are intractable, there exist many attempts to develop heuristic approximation algorithms [1]. Hence, some lower bounding techniques are necessary for assessing the quality of the solutions found by these algorithms.

### 1.1 Summary of Results

We start with proving that both radio coloring and radio labelling remain $\mathcal{NP}$-complete for graphs of diameter two. We also show that a polynomial-time $2\alpha$-approximation algorithm for radio coloring can be obtained from any polynomial-time $\alpha$-approximation algorithm for coloring squares of graphs.

We proceed to study radio labelling that is shown equivalent to Hamiltonian Path with distances one and two (HP(1,2)). Hence, radio labelling is MAX–SNP-hard and approximable in polynomial time within 7/6 [16]. Next, we present a polynomial-time algorithm for computing an optimal radio labelling, given a coloring of the graph with constant number of colors. Therefore, we prove that radio labelling is in $\mathcal{P}$ for planar graphs and graphs colorable with constant number of colors in polynomial time.

As a side effect, we show that, given a partition of the vertices of a graph into constant number of cliques, we can decide if the graph is Hamiltonian in polynomial time. We are not aware of another algorithm that exploits a partition into cliques for deciding Hamiltonicity.

Motivated by the practical applications of on-line frequency assignment in mobile networks [15], we define two on-line variations of radio labelling, and we prove that the greedy algorithm achieves a competitive ratio of two for both variations. We also obtain a lower bound of 3/2 on the competitive ratio of any on-line algorithm for the first variation.

## 2 Definitions and Techniques

Given a graph $G(V,E)$, $d(v,u)$ denotes the length of the shortest path between $v,u \in V$, and $\mathrm{diam}(G)$ denotes the diameter of $G$ defined as $\mathrm{diam}(G) = \max_{v,u \in V}\{d(v,u)\}$. The square of $G$, denoted $G^2$, is a graph on the vertex set $V$ that contains an edge between $v,u \in V$, if $v$ and $u$ are at distance at most two in $G$. The complementary graph $\overline{G}$ is a graph on the vertex set $V$ that contains an edge $\{v,u\}$, iff $\{v,u\} \notin E$.

Hamiltonian Path with distances one and two (HP(1,2)) is the problem of finding a Hamiltonian path of minimum length in a complete graph, where all the edge lengths are either one or two. A Hamiltonian path is a simple path visiting each vertex of a graph exactly once. An instance of HP(1,2) can also be defined by an unweighted graph $G(V,E)$, if the edges of $E$ are considered of length one, and the edges not in $E$

(non-edges) of length two. Therefore, HP(1,2) is a generalization of the Hamiltonian path problem. HP(1,2) is MAX–SNP-hard and approximable in polynomial time within a factor of 7/6 [16].

In the graph coloring problem, we seek an assignment of colors to the vertices of a graph such that no pair of adjacent vertices get the same color. The objective is to minimize the number of colors used. Given a graph $G$, the value of an optimal coloring is called the chromatic number of $G$ and is denoted by $\chi(G)$. It is $\mathcal{NP}$-hard even to efficiently approximate the chromatic number of general graphs [14]. However, there exist polynomial-time approximation algorithms whose performance guarantees are not so good [8, 12].

The following generalized version of graph coloring is derived by applying the FD-constraints to unweighted graphs with respect to the distances $D_0 = \kappa + 1, D_1 = \kappa, \ldots, D_\kappa = 1$.

**Definition 2.1 (FD($\kappa$)-coloring)**
INSTANCE: *A graph $G(V, E)$.*
SOLUTION: *A valid* FD($\kappa$)*-coloring, i.e. a function* $\mathrm{X}_\kappa : V \mapsto \{1, \ldots, \nu\}$ *such that, for all* $v, u \in V$, $|\mathrm{X}_\kappa(v) - \mathrm{X}_\kappa(u)| = x$, $x \in \{0, 1, \ldots, \kappa\}$, *only if* $d(v, u) \geq \kappa - x + 1$.
OBJECTIVE: *Minimize the maximum color $\nu$ used (color span).*

Clearly, the FD($\kappa$)-coloring problem allows a pair of non-adjacent vertices to be assigned the same color/channel, provided they are located far apart. In the sequel, we shall concentrate on the study of FD(2)-coloring, also called *radio coloring*, modelling the widely used "co-channel" and "adjacent-channel" interference constraints.

We remark that the problem of radio coloring a graph $G$ is not equivalent to the problem of coloring $G^2$. In particular, the objective in coloring $G^2$ is to minimize the number of different colors used, while the objective in radio coloring $G$ is to minimize the maximum color assigned to a vertex. For example, if $G$ is $K_{m,m}$ (i.e. the complete bipartite graph with $m$ vertices on each class), then $G^2$ is the complete graph on $2m$ vertices. Therefore, $\chi(G^2) = 2m$, while $X_2(G) = 2m + 1$.

A valid FD($\kappa$)-labelling fulfills the FD-constraints with respect to the graph distances $D_0 = \infty, D_1 = \kappa, \ldots, D_\kappa = 1$.

**Definition 2.2 (FD($\kappa$)-labelling)**
INSTANCE: *A graph $G(V, E)$.*
SOLUTION: *A valid* FD($\kappa$)*-labelling, i.e. a function* $\mathrm{L}_\kappa : V \mapsto \{1, \ldots, \nu\}$ *such that, for all* $v, u \in V$, $|\mathrm{L}_\kappa(v) - \mathrm{L}_\kappa(u)| = x$, $x \in \{1, \ldots, \kappa\}$, *only if* $d(v, u) \geq \kappa - x + 1$.
OBJECTIVE: *Minimize the maximum label $\nu$ used (label span).*

In the sequel, we shall concentrate on the FD(2)-labelling problem, also called *radio labelling*. Given a graph $G(V, E)$, RL($G$) denotes the value of an optimal radio labelling for $G$ and, for any $v \in V$, RL($v$) denotes the label assigned to $v$ by a valid radio labelling.

Obviously, given a coloring of $G$ with $\chi$ colors, it is easy to find a radio labelling of value at most $|V| + \chi - 1$. Therefore,

$$|V| \leq \mathrm{RL}(G) \leq |V| + \chi(G) - 1$$

It is not hard to verify that $\mathrm{RL}(G) = |V|$ for all graphs $G$ such that $\overline{G}$ contains a Hamiltonian path. On the other hand, $\mathrm{RL}(G) = |V| + \chi(G) - 1$ for any complete $r$-partite graph $G$, $r \geq 2$.

## 2.1 On-line Radio Labelling

We proceed to define the problem of on-line radio labelling (cf. Chapter 13 of [11] for the basic definitions concerning on-line computation). In the on-line setting of the problem, an induced subgraph of the interference graph appears to the on-line algorithm in a vertex-by-vertex fashion. In case of on-line radio labelling, a request is a newly arrived vertex that has to be assigned a radio label. A request is accepted if the vertex actually gets a valid label. Otherwise, the request is rejected. We define two variations of on-line radio labelling.

**Definition 2.3 (On-line Radio Labelling – Benefit Version)**
INSTANCE: *A graph $G(V, E)$ and an integer bound $\beta > 0$. A new vertex $v \in V$ is presented to the on-line algorithm at every step. The adversary may choose to request any subset $V' \subseteq V$ in any order.*
SOLUTION: *The algorithm has to decide to either accept $v$ by assigning a label no greater than $\beta$ to it, or reject $v$. At any step, the labels of the set $V_a$ of accepted vertices must form a valid radio labelling for the subgraph of $G$ induced by $V_a$.*
OBJECTIVE: *Maximize the number of accepted vertices.*

**Definition 2.4 (On-line Radio Labeling – Assignment Version)**
INSTANCE: *A graph $G(V, E)$. At every step, a new vertex $v \in V$ is presented to the on-line algorithm. The adversary may choose to request any subset $V' \subseteq V$ in any order.*
SOLUTION: *The algorithm must assign labels to all the vertices requested. At any step, the labels of the set $V_p$ of the vertices presented so far must form a valid radio labelling for the subgraph of $G$ induced by $V_p$.*
OBJECTIVE: *Minimize the maximum label used.*

## 2.2 Competitive Analysis and Lower Bounds

The usual approach of analyzing the performance of on-line algorithms is the *competitive analysis* [18]. In competitive analysis, the performance of the on-line algorithm is compared to the performance of the optimal off-line algorithm on every sequence of requests, and the worst-case ratio is considered. Let $A$ be an on-line algorithm for a benefit (maximization) problem and $\sigma$ any sequence of requests. Also, let $A(\sigma)$ denote the benefit accrued by $A$ when presented with $\sigma$, and $\mathrm{OPT}(\sigma)$ denote the benefit accrued by the optimal off-line algorithm OPT on the same sequence. We say that the on-line algorithm $A$ is *c-competitive* if there exists a constant $b$ such that on every request sequence $\sigma$,

$$c \cdot A(\sigma) + b \geq \mathrm{OPT}(\sigma)$$

The competitive ratio for cost (minimization) problems is defined similarly. In case of randomized on-line algorithms, the competitive ratio is defined with respect to the expected benefit of the algorithm.

A standard technique used in competitive analysis is to employ an *adversary* which plays against the algorithm $A$ and constructs an input which incurs a high cost for $A$ and a low cost for OPT. We say that an adversary is *oblivious* if it constructs the request sequence in advance, before $A$ starts working on the sequence. However, the oblivious adversary knows the probability distribution of the actions taken by the algorithm $A$.

A common technique for proving lower bounds on the competitive ratio of a randomized on-line algorithm against the oblivious adversary is to bound the performance of the "best" deterministic on-line algorithm on inputs generated from the "worst" probability distribution. In particular, let $\mathcal{D}$ be a probability distribution over sequences of requests $\sigma$, let $\mathrm{E}_{\mathcal{D}}[A(\sigma)]$ denote the expected value of the benefit of an algorithm $A$, and $\mathrm{E}_{\mathcal{D}}[\mathrm{OPT}(\sigma)]$ be the expected benefit of the optimal off-line algorithm on inputs generated from $\mathcal{D}$. An algorithm $A$ (for a benefit problem) is $c$-competitive against $\mathcal{D}$ if there exists a constant $b$ such that,

$$c \cdot \mathrm{E}_{\mathcal{D}}[A(\sigma)] + b \geq \mathrm{E}_{\mathcal{D}}[\mathrm{OPT}(\sigma)]$$

**Lemma 2.5 ([4])** *A real number $c$ is a lower bound on the competitive ratio of randomized on-line algorithms against the oblivious adversary if and only if there exists a probability distribution $\mathcal{D}$ such that $c$ is a lower bound on the competitive ratio of any deterministic on-line algorithm against $\mathcal{D}$.*

# 3 The Complexity of Radio Coloring and Radio Labelling

We proceed to show that both radio coloring and radio labelling remain $\mathcal{NP}$-complete, even for a very restricted class of instances, namely graphs of diameter two.

**Lemma 3.1** *Radio coloring and radio labelling restricted to graphs of diameter two are $\mathcal{NP}$-complete.*

*Proof.* Let $G(V, E)$, $|V| = n$, be any graph of diameter two. Since, for all $v, u \in V$, $d(v, u) \leq 2$, any valid radio coloring must assign distinct colors to all the vertices of

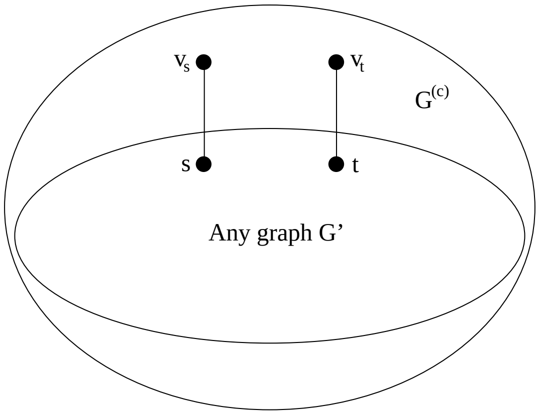


Figure 1: The complementary graph of $G^{(c)}$ has diameter two.

$G$. Moreover, if $\{v, u\} \in E$, then $|\mathrm{X}_2(v) - \mathrm{X}_2(u)| \geq 2$. Therefore, if $\mathrm{diam}(G) = 2$, the problem of radio coloring in $G$ is equivalent to the problem of radio labelling in $G$. Hence, it suffices to show that radio labelling is $\mathcal{NP}$-complete for graphs of diameter two.

Clearly, radio labelling is in $\mathcal{NP}$. Additionally, it is not hard to verify (see also Lemma 3.3) that $\mathrm{RL}(G) \leq |V|$ if and only if the complementary graph $\overline{G}$ contains a Hamiltonian path. Thus, in order to show that radio labelling is $\mathcal{NP}$-complete for graphs $G$ of diameter two, it suffices to show that the Hamiltonian Path problem remains $\mathcal{NP}$-complete for complementary graphs $\overline{G}$ such that $\mathrm{diam}(G) = 2$.

Let $G'(V', E')$ be any graph and let $s, t \in V'$ be any pair of non-adjacent vertices. The problem of deciding if $G'$ contains a Hamiltonian path starting from $s$ and ending to $t$ is $\mathcal{NP}$-complete (Hamiltonian Path between Two Vertices [5]). Let $G^{(c)}(V' \cup \{v_s, v_t\}, E' \cup \{(s, v_s), (t, v_t)\})$ be the graph obtained from $G'$ by adding two non-adjacent vertices $v_s, v_t$, and connecting $v_s$ to $s$ and $v_t$ to $t$ (Figure 1). The graph $G^{(c)}$ is the complement of a graph of diameter two. In particular, the following observations justify that $\mathrm{diam}\left(\overline{G^{(c)}}\right) = 2$, since all the vertices are at distance at most two from each other in $\overline{G^{(c)}}$.

1. The vertex pairs $(s, t), (v_s, v_t), (s, v_t)$ and $(t, v_s)$ are connected by edges in $\overline{G^{(c)}}$.

2. Any pair of vertices $u, w \in V' - \{s, t\}$ are at distance at most two from each other, because they are connected to both $v_s, v_t$.

3. Any vertex $u \in V' \cup \{v_s\} - \{s, t\}$ is at distance at most two from $s$, because both $u$ and $s$ are connected to $v_t$.

4. Any vertex $u \in V' \cup \{v_t\} - \{s, t\}$ is at distance at most two from $t$, because both $u$ and $t$ are connected to $v_s$.

Additionally, $G^{(c)}$ contains a Hamiltonian path if and only if $G'$ contains a Hamiltonian path from $s$ to $t$. Therefore, Hamiltonian path is $\mathcal{NP}$-complete for complements of graphs of diameter two. $\square$

Notice that a set $C \subseteq V$ is a valid color class of $G^2$, iff it is a valid radio color class of $G$, because any pair of vertices in $C$ is at distance at least three in $G$. Moreover, if we assign the colors $1, 3, \ldots, 2\chi(G^2) - 1$ to the color classes of $G^2$, we obtain a valid radio coloring of $G$. Therefore,

$$\chi(G^2) \leq \mathrm{X}_2(G) \leq 2\chi(G^2) - 1$$

Additionally, if $A$ is a polynomial-time $\alpha$-approximation algorithm for coloring $G^2$ and $|A(G^2)|$ denotes the number of colors used by $A$, then we can easily compute a valid radio coloring of $G$ of value no more than $2|A(G^2)| - 1$. Since

$$\mathrm{X}_2(G) \leq 2|A(G^2)| - 1 \leq 2\alpha\chi(G^2) - 1 \leq 2\alpha \mathrm{X}_2(G) - 1\ ,$$

this is a $2\alpha$-approximation for radio coloring in $G$.

**Lemma 3.2** *For any graph $G$ and real number $\alpha \geq 1$, a polynomial-time $2\alpha$-approximation algorithm for radio coloring in $G$ can be obtained from any polynomial-time $\alpha$-approximation algorithm for coloring $G^2$.*

Since radio labelling assigns distinct integer labels to all the vertices of a graph, it is a vertex arrangement problem. In particular, we show that radio labelling is equivalent to HP(1,2) in the complementary graph.

**Lemma 3.3** *Graph radio labelling and HP(1,2) are equivalent.*

*Proof.* Given an instance of radio labelling, i.e. a graph $G(V, E)$, the corresponding instance of HP(1,2) is a complete graph $\hat{G}$ on the vertex set $V$, and the distance function $\hat{d}$ is defined for all $v, u \in V$, $v \neq u$, by

$$\hat{d}(v,u) = \begin{cases} 1 & \text{if } (v,u) \notin E \\ 2 & \text{if } (v,u) \in E \end{cases}$$

Given any valid radio labelling $L$ (for $G$) of value $\mathrm{RL}(L)$, we can obtain a Hamiltonian path $H$ for $\hat{G}$ by traversing all the vertices in increasing order of their labels. Moreover, the following claim implies that the length of the Hamiltonian path $H$ is exactly $\mathrm{RL}(L) - 1$.

**Claim 1** *The length of the path up to any vertex of label $i$, $i = 1, \ldots, \mathrm{RL}(L)$, is exactly $i - 1$.*

*Proof of the Claim.* We prove the claim by induction on $i$. Clearly, it is true for the first vertex, where $i = 1$. Assume inductively that it is true for any vertex $v$ of label $i \geq 1$, and let $u$ be the next vertex in the Hamiltonian path. We proceed by case analysis:

1. If the label of $u$ is $i+1$, the edge $\{v,u\}$ is not present in $G$ and, by construction, $\hat{d}(v,u)=1$. Thus, the length of the path up to vertex $u$ is exactly $i$.

2. If the label of $u$ is $i+2$, by the construction of the Hamiltonian path, there does not exist a vertex of label $i+1$. Therefore, the edge $\{v,u\} \in E$, and $\hat{d}(v,u)=2$. Consequently, the path up to $u$ has length exactly $i+1$. □

Conversely, given an instance $(\hat{G},\hat{d})$ of HP(1,2) on the vertex set $V$, $|V|=n$, an instance $G(V,E)$ of radio labelling can be obtained by only connecting the vertex pairs that are at distance 2. Furthermore, given a Hamiltonian path $H=(v_1,v_2,\ldots,v_n)$ for $\hat{G}$ of length $l(H)$, we obtain a valid radio labelling $L$ (for $G$) as follows:

1. $\mathrm{RL}(v_1)=1$.

2. For $i=1,\ldots,n-1$,

   (a) $\mathrm{RL}(v_{i+1})=\mathrm{RL}(v_i)+1$, if $\hat{d}(v_i,v_{i+1})=1$. By the construction of $G(V,E)$, in this case $\{v_i,v_{i+1}\} \notin E$.

   (b) $\mathrm{RL}(v_{i+1})=\mathrm{RL}(v_i)+2$, if $\hat{d}(v_i,v_{i+1})=2$. By the construction of $G(V,E)$, in this case $\{v_i,v_{i+1}\} \in E$.

By construction, all the vertices get distinct labels, while, if an edge $\{v_i,v_{i+1}\}$ is present in $E$, the vertices $v_i$ and $v_{i+1}$ are assigned non-adjacent labels. Therefore, the resulting radio labelling is a valid one. Additionally, the last vertex of the path $v_n$ is assigned the label $l(H)+1$, that is the largest label used. Hence, $\mathrm{RL}(L)=l(H)+1$. □

The previous lemma implies that radio labelling is MAX–SNP-hard and approximable in polynomial time within 7/6 [16].

# 4 An Exact Algorithm for Constant Number of Colors

Since radio labelling is equivalent to HP(1,2) and a coloring corresponds to a partition into cliques of the complementary graph, we present the technical part of the proof in the context of Hamiltonian paths/cycles and partitions into cliques. In particular, we prove that, given a partition of a graph $G(V,E)$ into constant number of cliques, we can decide if $G$ is Hamiltonian in polynomial time.

## 4.1 Hamiltonian Cycles and Partitions into Cliques

**Theorem 4.1** *Given a graph $G(V,E)$, $|V|=n$, and a partition of $V$ into $\kappa>1$ cliques, there exists a deterministic algorithm that runs in time $\mathcal{O}\left(n^{\kappa(2\kappa-1)}\right)$ and decides if $G$ is Hamiltonian. If $G$ is Hamiltonian, the algorithm outputs a Hamiltonian cycle.*

*Proof.* A set of inter-clique edges $M \subseteq E$ is an HC-set if $M$ can be extended to a Hamiltonian cycle using only clique-edges, i.e. there exists a $M^{(c)} \subseteq E$ of clique edges such that $M \cup M^{(c)}$ is a Hamiltonian cycle.

We first show that, given a set of inter-clique edges $M$, we can decide if $M$ is an HC-set and construct a Hamiltonian cycle from $M$ in poly$(n, \kappa)$ time (Proposition 4.2). Then, we prove that $G$ is Hamiltonian iff there exists an HC-set of cardinality at most $\kappa(\kappa-1)$ (Lemmas 4.3 and 4.4). The algorithm exhaustively searches all the sets of inter-clique edges of cardinality at most $\kappa(\kappa-1)$ for an HC-set. Additionally, we conjecture that, if $G$ is Hamiltonian, then there exists an HC-set of cardinality at most $2(\kappa-1)$. We prove this conjecture for a special case (Lemma 4.5) and we use Lemma 4.3 to show the equivalence to Conjecture 1.

Let $\mathcal{C} = \{C_1, \ldots, C_\kappa\}$ be a partition of $V$ into $\kappa > 1$ cliques. Given a set $M \subseteq E$ of inter-clique edges, the *clique graph* $T(C, M)$ contains exactly $\kappa$ vertices, that correspond to the cliques of $\mathcal{C}$, and represents how the edges of $M$ connect the different cliques. If $M$ is an HC-set, then the corresponding clique graph $T(C, M)$ is connected and eulerian. However, the converse is not always true.

Given a set of inter-clique edges $M$, we color an edge RED, if it shares a vertex of $G$ with another edge of $M$. Otherwise, we color it BLUE. The corresponding edges of $G$ are colored with the same colors, while the remaining edges $(E - M)$ are colored BLACK. Additionally, we color RED each vertex $v \in V$, which is the common end vertex of two or more RED edges. We color BLUE each vertex $v \in V$ to which exactly one edge of $M$ (RED or BLUE) is incident. The remaining vertices of $G$ are colored BLACK (Figure 2).

Let $H$ be any Hamiltonian cycle of $G$ and let $M$ be the corresponding set of inter-clique edges. Obviously, RED vertices cannot be exploited for visiting any BLACK vertices belonging to the same clique. If $H$ visits a clique $C_i$ through a vertex $v$, and leaves $C_i$ through a vertex $u$, then $v, u \in C_i$ consist a BLUE vertex pair. A BLUE pass through a clique $C_i$ is a simple path of length at least one, that entirely consists of non-RED vertices of $C_i$. A clique $C_i$ is covered by $M$, if all the vertices of $C_i$ have degree at most two in $M$, and the existence of a non-RED vertex implies the existence of at least one BLUE vertex pair. The following proposition characterizes HC-sets.

**Proposition 4.2** *A set of inter-clique edges $M$ is an HC-set iff the corresponding clique graph $T(C, M)$ is connected, eulerian, and,*

*(a) For all $i = 1, \ldots, \kappa$, $C_i$ is covered by $M$; and*

*(b) There exists an eulerian trail $R$ for $T$ such that: For any* RED *vertex $v \in V$, $R$ passes through $v$ exactly once using the corresponding* RED *edge pair.*

*Proof.* Any HC-set corresponds to a connected, eulerian clique graph $T(C, M)$ that fulfills both (a) and (b). Conversely, we can extend $M$ into a Hamiltonian cycle $H$ following the eulerian trail $R$. All the RED vertices can be included in $H$ with degree two because of (b). Moreover, since $R$ is an eulerian trail and (a) holds for $M$, all the

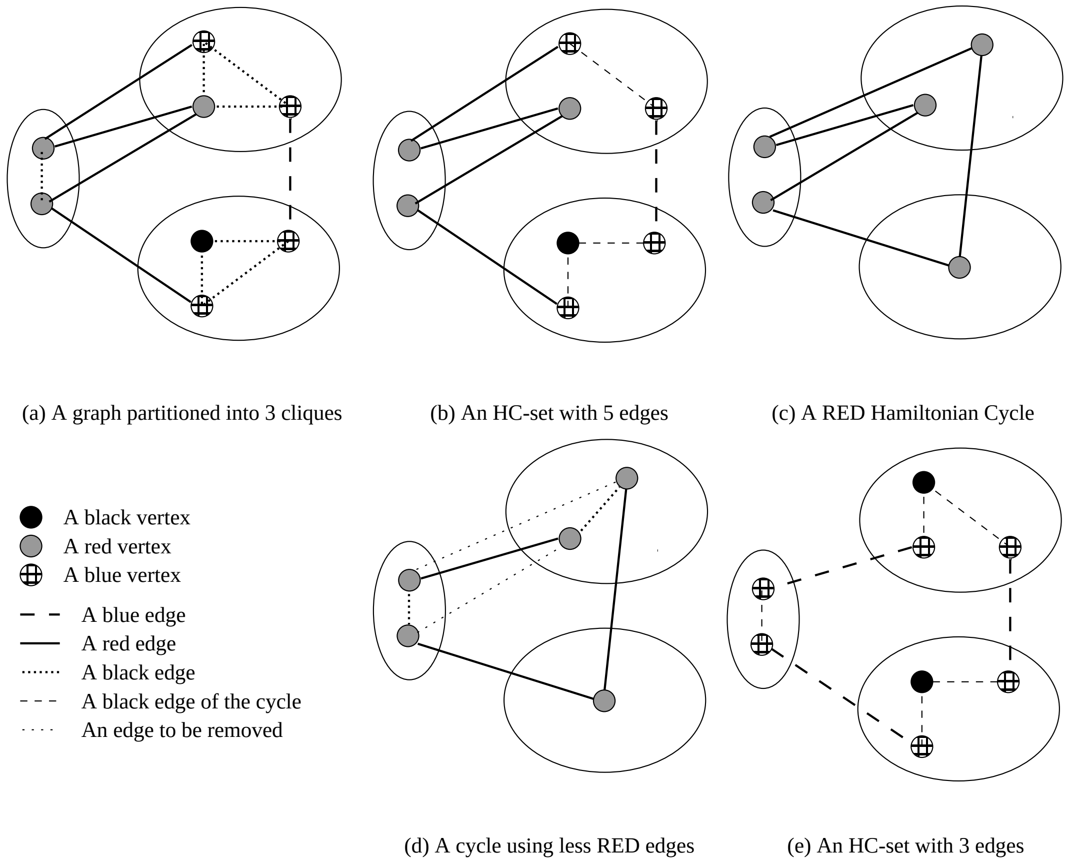


Figure 2: An application of Lemmas 4.3 and 4.4.

BLUE and the BLACK vertices can be included in $H$ with degree two. Therefore, $H$ is a Hamiltonian cycle. □

The proof of Proposition 4.2 implies a deterministic procedure for deciding if a set of inter-clique edges is an HC-set in poly$(n, \kappa)$ time. Moreover, in case that $M$ is an HC-set, this procedure outputs a Hamiltonian cycle.

**Lemma 4.3** *Let $B_\kappa \geq 2$ be some integer only depending on $\kappa$ such that, for any graph $G(V, E)$ and any partition of $V$ into $\kappa$ cliques, if $G$ is Hamiltonian and $|V| > B_\kappa$, then $G$ contains at least one Hamiltonian cycle not entirely consisting of inter-clique edges* (RED *vertices*).
*Then, for any graph $G(V, E)$ and any partition of $V$ into $\kappa$ cliques, $G$ is Hamiltonian iff it contains a Hamiltonian cycle with at most $B_\kappa$ inter-clique edges.*

*Proof.* Let $H$ be the Hamiltonian cycle of $G$ containing the minimum number of inter-clique edges and let $M$ be the corresponding set of inter-clique edges. Assume that $|M| > B_\kappa$. The hypothesis implies that $H$ cannot entirely consist of RED vertices. Therefore, $H$ should contain at least one BLUE vertex pair.

We substitute any BLUE pass of $H$ through a clique $C_i$ with a single RED super-vertex $\hat{v}$, that also belongs to the clique $C_i$. Hence, $\hat{v}$ is connected to all the remaining vertices of $C_i$ using BLACK edges. These substitutions result in a cycle $H'$ that entirely consists of RED vertices, and contains exactly the same set $M$ of inter-clique edges with $H$.

Obviously, the substitutions of all the BLUE passes of $H$ with RED super-vertices result in a graph $G'(V', E')$ that is also Hamiltonian, $|V'| > B_\kappa$, and $V'$ is partitioned into $\kappa$ cliques. Moreover, for any Hamiltonian cycle $H_{G'}$ of $G'$, the reverse substitutions of all the RED super-vertices $\hat{v}$ with the corresponding BLUE passes result in a Hamiltonian cycle of $G$ that contains exactly the same set of inter-clique edges with $H_{G'}$ (Figure 2).

Since $H'$ is a Hamiltonian cycle that entirely consists of inter-clique edges and $|V'| > B_\kappa$, the hypothesis implies that there exists another Hamiltonian cycle of $G'$ that contains strictly less inter-clique edges than $H'$. Therefore, there exists a Hamiltonian cycle of $G$ that contains less inter-clique edges than $H$. □

Lemma 4.3 implies that, in order to prove the upper bound on the cardinality of a minimum HC-set, it suffices to prove the same upper bound on the number of vertices of Hamiltonian graphs that (i) can be partitioned into $\kappa$ cliques, and (ii) only contain Hamiltonian cycles entirely consisting of inter-clique edges. It should be intuitively clear that such graphs cannot contain an arbitrarily large number of vertices.

**Lemma 4.4** *Given a graph $G(V, E)$ and a partition of $V$ into $\kappa$ cliques, $G$ is Hamiltonian iff there exists an HC-set $M$ such that $|M| \leq \kappa(\kappa - 1)$.*

*Proof.* By definition, the existence of an HC-set implies that $G$ is Hamiltonian. Conversely, let $H$ be the Hamiltonian cycle of $G$ that contains the minimum number of inter-clique edges, and let $M$ be the corresponding HC-set. If $|M| \leq \kappa(\kappa - 1)$, then we are done. Otherwise, Lemma 4.3 implies that it suffices to prove the same upper bound on $|V|$ for graphs $G(V, E)$ that only contain Hamiltonian cycles entirely consisting of inter-clique edges.

Assume that $|V| = |M|$ and the coloring of $V$ under $M$ entirely consists of RED vertices, and consider an arbitrary orientation of the Hamiltonian cycle $H$ (e.g. a traversal of the edges of $H$ in the clockwise direction). If there exist a pair of cliques $C_i$ and $C_j$ and four vertices $v_1, v_2 \in C_i$ and $u_1, u_2 \in C_j$, such that both $v_x$ are followed by $u_x$ ($x = 1, 2$) in a traversal of $H$, then the BLACK edges $\{v_1, v_2\}$ and $\{u_1, u_2\}$ can be used instead of $\{v_1, u_1\}$ and $\{v_2, u_2\}$ in order to obtain a Hamiltonian cycle containing less inter-clique edges than $H$. The previous situation can be avoided only if, for all $i = 1, \ldots, \kappa$, and $j = 1, \ldots, \kappa$, $j \neq i$, at most one vertex $v_j \in C_i$ is followed by a vertex $u \in C_j$ in any traversal of $H$. Hence, if $|V| > \kappa(\kappa - 1)$, then $G$ contains at least one Hamiltonian cycle not entirely consisting of inter-clique edges. Alternatively, any HC-set $M$ of minimum cardinality contains at most two edges between any pair of cliques $C_i$ and $C_j$. Thus, $M$ contains at most $\kappa(\kappa - 1)$ inter-clique edges. □

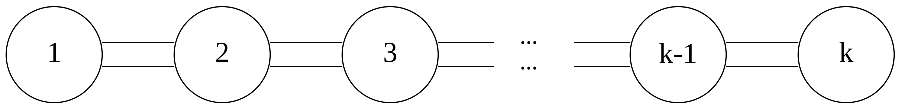


Figure 3: An HC-set containing exactly $2(\kappa-1)$ edges.

Therefore, we can decide if $G$ is Hamiltonian in time $\mathcal{O}\left(n^{\kappa(2\kappa-1)}\right)$, because the number of the different edge sets containing at most $\kappa(\kappa-1)$ inter-clique edges is at most $n^{2\kappa(\kappa-1)}$, and we can decide if a set of inter-clique edges is an HC-set in time $\mathrm{poly}(n, \kappa) = \mathcal{O}(n^{\kappa})$. $\square$

A BLUE Hamiltonian cycle is a Hamiltonian cycle that does not contain any RED vertices or edges. We can substantially improve the bound of $\kappa(\kappa-1)$ for graphs containing a BLUE Hamiltonian cycle.

**Lemma 4.5** *Given a graph $G(V, E)$ and a partition of $V$ into $\kappa$ cliques, if $G$ contains a* BLUE *Hamiltonian cycle, then there exists an HC-set $M$ entirely consisting of* BLUE *edges, such that $|M| \leq 2(\kappa-1)$.*

*Proof.* Let $H$ be the BLUE Hamiltonian cycle of $G$ that contains the minimum number of inter-clique edges and let $M$ be the corresponding HC-set. Assume that $|M| > 2(\kappa-1)$, otherwise we are done. Notice that, since RED vertices cannot be created by removing edges, any $M' \subseteq M$ that corresponds to an eulerian, connected, clique graph $T(C, M')$ is an HC-set that only contains BLUE edges.

Let $S_T(C, M_S)$ be any spanning tree of $T(C, M)$. Since $|M_S| = \kappa - 1$, the graph $T^{(S)}(C, M - M_S)$, which is obtained by removing the edges of the spanning tree from $T$, contains at least $\kappa$ edges. Therefore, $T^{(S)}$ contains a simple cycle $L$. The removal of the edges of $L$ does not affect connectivity (the edges of $L$ do not touch the spanning tree $S_T$), and subtracts two from the degrees of the involved vertices/cliques. Clearly, the clique graph $T'(C, M - L)$ is connected and eulerian, and $M - L$, $|M - L| < |M|$, is an HC-set. $\square$

Figure 3 shows an HC-set of cardinality exactly $2(\kappa-1)$ that corresponds to a Hamiltonian cycle using the minimum number of inter-clique edges. Therefore, the bound of $2(\kappa-1)$ is tight. However, we are not able to construct HC-sets that contain more than $2(\kappa-1)$ edges and correspond to Hamiltonian cycles using the minimum number of inter-clique edges. Hence, we conjecture that the bound of $2(\kappa-1)$ holds for any graph and any partition into $\kappa$ cliques. An inductive (on $|V|$) application of Lemma 4.3 suggests that this conjecture is equivalent to the following:

**Conjecture 1** *For any Hamiltonian graph $G(V, E)$ of $2\kappa - 1$ vertices and any partition of $V$ into $\kappa$ cliques, there exists at least one Hamiltonian cycle not entirely consisting of inter-clique edges.*

The previous conjecture is very similar to a theorem proved by C.A.B. Smith in 1946. This theorem states that the number of Hamiltonian cycles that contain any given edge of a cubic graph (i.e. a simple regular graph of degree 3) is even. A simple algebraic proof of this theorem can be found in [3]. Smith's Theorem implies Conjecture 1 in case that the partition $\mathcal{C}$ contains at least $k-1$ cliques of even cardinality and at most 1 clique of odd cardinality.

If we only consider Hamiltonian graphs, Smith's Theorem can be applied to graphs $G(V, E)$ consisting of a Hamiltonian cycle and a perfect matching. Therefore, Smith's Theorem is applicable to graphs that contain a RED Hamiltonian cycle, such that $V$ is partitioned into $\kappa = \frac{|V|}{2}$ cliques of two vertices (perfect matching). A more general class of graphs fulfills the hypothesis of Conjecture 1. In particular, Conjecture 1 is applicable to the graphs consisting of a RED Hamiltonian cycle and $\kappa$ cliques of arbitrary cardinalities. However, the conclusion is weaker than the conclusion of Smith's Theorem, in the sense that it only claims the existence of a Hamiltonian cycle avoiding at least one inter-clique (RED) edge.

## 4.2 A Reduction from Radio Labelling to Hamiltonian Cycle

**Lemma 4.6** *Given a graph $G(V, E)$, $|V| = n$, and a coloring of $G$ with $\kappa$ colors, an optimal radio labelling can be computed in $\mathcal{O}\Big(n^{\kappa(2\kappa+1)}\Big)$ time.*

*Proof.* Let $\overline{G}(V, \overline{E})$ be the complement of the input graph $G$. Obviously, $\mathrm{RL}(G) \leq n + \kappa - 1$. Therefore, at most $\kappa - 1$ labels remain unused by an optimal radio labelling. Hence, any optimal solution to the corresponding HP(1,2) instance (see also the proof of Lemma 3.3) contains at most $\kappa - 1$ non-edges (of $\overline{G}$), and there exists a Hamiltonian cycle containing at most $\kappa$ non-edges (of $\overline{G}$). Then, we show how to compute a Hamiltonian cycle containing the minimum number of non-edges to the complementary graph $\overline{G}$.

Let $A$ be the algorithm of Theorem 4.1. We call $A$ at most $M = \mathcal{O}(n^{2\kappa})$ times, with input the graphs $\overline{G_i}(V, \overline{E} \cup N_i)$, $i = 1, \ldots, M$. The sets $N_i$ are all possible subsets of non-edges of $\overline{G}$ with at most $\kappa$ elements, including the empty one. Let $\overline{G_i}$ be a Hamiltonian graph that corresponds to a set $N_i$ of minimum cardinality. Obviously, the Hamiltonian cycle produced by $A(\overline{G_i})$ is a Hamiltonian cycle with the minimum number of non-edges for $\overline{G}$. □

Since any planar graph can be colored with constant number of colors in polynomial time, the following theorem is an immediate consequence of Lemma 4.6.

**Theorem 4.7** *An optimal radio labelling of a planar graph can be computed in polynomial time.*

*Remark.* Conjecture 1 implies that, given a graph $G(V, E)$ and a coloring with $\kappa$ colors, an optimal radio labelling can be computed in time $n^{\mathcal{O}(\kappa)}$.

# 5 Algorithms for On-line Radio Labelling

## 5.1 On-line Radio Labeling – Benefit Version

We first analyze the performance of the greedy algorithm BGreedy. BGreedy assigns to a newly arrived request the least integer $j$, such that both $j$ has not been assigned to any previously accepted request, and the resultant radio labeling is a valid one. BGreedy accepts the request, if and only if $j$ is no more than the bound $\beta$.

**Lemma 5.1** *The competitive ratio of* BGreedy *is* $\beta\left\lceil\frac{\beta}{2}\right\rceil^{-1}$.

*Proof.* The BGreedy algorithm always accepts at least $\left\lceil\frac{\beta}{2}\right\rceil$ requests, because, for $i = 1, 2, \ldots, \left\lceil\frac{\beta}{2}\right\rceil$, BGreedy can always assign a label no more than $2i-1$ to the $i$-th request. Since the optimal off-line algorithm cannot accept more than $\beta$ requests, an upper bound of $\beta\left\lceil\frac{\beta}{2}\right\rceil^{-1}$ on the competitive ratio of BGreedy can be established.

We prove that this ratio is precisely $\beta\left\lceil\frac{\beta}{2}\right\rceil^{-1}$ by exhibiting a special sequence of requests. Let $S_m$ be any graph on $m$ vertices containing a Hamiltonian path and $H_m = \overline{S_m}$ be the complementary graph. Lemma 3.3 implies that $\mathrm{RL}(H_m) = m$. Additionally, let $L_m$ be the graph obtained from the complete graph on $\frac{m}{2}$ vertices, denoted $K_{m/2}$, and $H_{m/2}$ by connecting any vertex of $K_{m/2}$ to any vertex of $H_{m/2}$. By construction, $\mathrm{RL}(L_m) = \frac{3m}{2}$.

For some bound $\beta > 0$, let the request sequence consist of the vertices of the graph $L_{2\beta}$, such that all the $\beta$ vertices of $K_\beta$ are requested before all the vertices of $H_\beta$. Clearly, BGreedy can only accept the first $\left\lceil\frac{\beta}{2}\right\rceil$ requests, while the optimal algorithm accepts the last $\beta$ requests. This establishes the competitive ratio of BGreedy. □

Notice that BGreedy does not take into account the structure of the optimal solution on the current set of requests. The previous instance shows that unless a deterministic on-line algorithm takes into account the optimal solution on the current set of requests, it cannot achieve competitive ratio better than $\beta\left\lceil\frac{\beta}{2}\right\rceil^{-1}$. Therefore BGreedy is optimal among on-line algorithms that do not reject by choice. We next prove that no randomized on-line algorithm can achieve a competitive ratio less than $\frac{3}{2}$ against any adversary.

**Lemma 5.2** *No randomized algorithm for the benefit version of on-line radio labeling can achieve competitive ratio less than* $\frac{3}{2}$.

*Proof.* We prove the lower bound against the oblivious adversary; since the oblivious adversary is the least powerful one, this implies a lower bound of $\frac{3}{2}$ against any adversary. The proof consists of defining an appropriate probability distribution on the request sequences and applying Lemma 2.5.

Let $\{1, \ldots, \beta\}$ be the set of available labels. We consider the following probability distribution on the request sequences:

1. With probability $p$, the request sequence consists of the vertices of the graph $K_\beta$.
2. With probability $1-p$, the request sequence consists of the vertices of the graph $L_{2\beta}$, such that the vertices of $K_\beta$ precede the vertices of $H_\beta$.

If the input is $K_\beta$, the optimal off-line algorithm accepts exactly $\left\lceil \frac{\beta}{2} \right\rceil$ requests, while if the input is $L_{2\beta}$, it accepts exactly $\beta$ requests. Thus, the expected number of accepted requests for the optimal algorithm is

$$\mathrm{E}(\mathrm{OPT}) = p \left\lceil \frac{\beta}{2} \right\rceil + (1-p)\beta$$

Let $A$ be any deterministic algorithm. On input $K_\beta$, the algorithm $A$ will accept $x$ requests using labels from the set $\{1, \ldots, 2x-1\}$, where $x$ is any integer number between 1 and $\left\lceil \frac{\beta}{2} \right\rceil$. The value of $x$ is fixed after the choice of the input distribution, in order $A$ to be the deterministic on-line algorithm maximizing the expected number of accepted requests with respect to the specific probability distribution on the request sequences, i.e. $A$ to be the "best" deterministic algorithm. Thus, with probability $p$, $A$ accepts $x$ requests. Additionally, $A$ has to accept exactly $x$ requests from the graph $K_\beta$ on input $L_{2\beta}$, because $A$ is a deterministic on-line algorithm and the vertices of $K_\beta$ precede the vertices of $H_\beta$. Consequently, $A$ can accept at most $(\beta - 2x)$ requests from the subgraph $H_\beta$. Thus, the expected number of accepted requests of the best deterministic algorithm in this distribution is at most

$$\mathrm{E}(A) = px + (1-p)(\beta - x)$$

For $p = \frac{1}{2}$, this reduces to

$$\mathrm{E}(A) = \frac{\beta}{2}, \quad \text{and} \quad \mathrm{E}(\mathrm{OPT}) = \frac{1}{2}\left( \left\lceil \frac{b}{2} \right\rceil + \beta \right) \geq \frac{3\beta}{4}$$

Thus, any deterministic on-line algorithm cannot achieve a competitive ratio less than $\frac{3}{2}$ against this probability distribution. Consequently, Lemma 2.5 implies that $\frac{3}{2}$ is a lower bound on the performance of any randomized on-line algorithm against the oblivious adversary. □

In the on-line setting, we mainly face information-theoretic questions that have to do with the value of information on the computation of a minimum cost labelling. Thus, the lower bound holds for any on-line algorithm $A$, and it does not depend on the running time of $A$. On the other hand, even if we know the entire request sequence, radio labeling is $\mathcal{NP}$-complete, and, therefore, not expected to be solvable optimally in polynomial time. However, the proof of Lemma 5.2 does not take into account the $\mathcal{NP}$-completeness of radio labelling in $H_\beta$. Consequently, a polynomial-time on-line algorithm with competitive ratio $\frac{3}{2}$ is unlikely to exist.

## 5.2 On-line Radio Labeling – Assignment Version

We continue to analyze the performance of the greedy algorithm AGreedy for the assignment version of on-line radio labeling, where the algorithm is not allowed to reject requests. AGreedy assigns to a newly arrived request $i$, $i = 1, \ldots, n$, the least integer $j$ such that both $j$ has not been assigned to a previous request, and the resulting radio labeling is a valid one. Since there does not exist an upper bound on the labels used, such a label $j$ always exist.

**Lemma 5.3** *The competitive ratio $C_A$ of* AGreedy *is* $2 - \frac{3}{n} \leq C_A \leq 2 - \frac{1}{n}$.

*Proof.* For any sequence of $n$ requests, the value of a labeling computed by AGreedy is at most $2n - 1$, since the labeling $1, 3, \ldots, 2n - 1$ is always a valid one. The upper bound on the competitive ratio follows from the fact that the value of an optimal labeling cannot be less than $n$.

To show the lower bound, for any even integer $n \geq 2$, let $U_n$ be the graph on $n$ vertices obtained from $K_n$ by removing the edges of a Hamiltonian path. Hence, the complement of $U_n$ is a Hamiltonian path on $n$ vertices, and $\mathrm{RL}(U_n) = n$. Let $u_1, u_2, \ldots, u_n$ be an ordering of the vertices of $U(n)$ according to their appearance in the Hamiltonian path contained by the complementary graph; that is, for $i = 1, \ldots n - 1$, the vertices $u_i$ and $u_{i+1}$ are not adjacent in $U_n$. The adversary requests all the vertices of $U_n$ in the following order: for $i = 0, \ldots, \frac{n}{2} - 1$, it requests the vertex $u_{i+1}$ followed by the vertex $u_{n-i}$.

**Claim 2** *For $i = 0, \ldots, \frac{n}{2} - 2$,* AGreedy *assigns to the vertex $u_{i+1}$ the label $4(i+1) - 3$ and to $u_{n-i}$ the label $4(i+1) - 1$. Moreover,* AGreedy *computes a radio labelling of $U_n$ of value $2n - 3$.*

*Proof of the Claim.* We prove the first part of the claim by induction on $i$. Clearly, for $i = 0$, the vertex $u_1$ gets the label 1, and the vertex $u_n$ gets the label 3, since $u_1$ and $u_n$ are adjacent in $U_n$.

Assume inductively that the claim holds for any $i$, $0 < i < \frac{n}{2} - 2$. Then, for $i + 1$, the adversary requests the vertices $u_{i+2}$ and $u_{n-(i+1)}$. In $U_n$, the vertex $u_{i+2}$ is adjacent to all the previously requested vertices except $u_{i+1}$. Since the vertex $u_{n-i}$ has been assigned the label $4(i + 1) - 1$, AGreedy assigns to $u_{i+2}$ the label $4(i + 1) + 1 = 4(i + 2) - 3$. Similarly, since all the previously requested vertices except $u_{n-i}$ are adjacent to the vertex $u_{n-(i+1)}$, AGreedy assigns to $u_{n-(i+1)}$ the label $4(i + 2) - 1$.

Therefore, for $i = \frac{n}{2} - 2$, the AGreedy algorithm assigns to the vertex $u_{n/2-1}$ the label $2n - 7$, and to $u_{n/2+2}$ the label $2n - 5$. Then, the adversary requests the vertices $u_{n/2}$ and $u_{n/2+1}$ that get the labels $2n - 3$ and $2n - 4$ respectively. □

Since the optimal off-line algorithm can easily compute a radio labelling of value exactly $n$, we obtain a lower bound of $2 - \frac{3}{n}$ on the competitive ratio of the algorithm AGreedy. □

## 6 Open Problems

An interesting direction for further research is to obtain a polynomial-time approximation algorithm for radio coloring in planar graphs using our exact algorithm for radio labelling. One approach may be to decompose the planar graph to subgraphs, such that almost all vertices of each subgraph get distinct colors by an optimal (or near optimal) assignment. Then, our exact algorithm can be used for computing an optimal radio labelling for each of the resulting subgraphs. Obviously, the decomposition of the planar graph and the combination of the partial assignments to a near optimal solution require an appropriate planar separator theorem.

Another research direction is the conjecture that, given a graph $G(V, E)$ and a partition of $V$ into $\kappa$ cliques, $G$ is Hamiltonian iff there exists a Hamiltonian cycle containing at most $2(\kappa - 1)$ inter-clique edges. This would imply a $n^{\mathcal{O}(\kappa)}$ exact algorithm for radio labelling in the complementary graph $\overline{G}$, given a coloring of $\overline{G}$ with $\kappa$ colors. In this paper, we prove the conjectured bound for graphs and partitions that contain at least one Hamiltonian cycle $H$, such that any vertex $v \in V$ has degree at most one in the set of inter-clique edges of $H$.

Additionally, it may be possible to improve the complexity of the algorithm of Theorem 4.1 to $\mathcal{O}(f(\kappa)p(n))$, where $f(\kappa)$ is a fixed function of $\kappa$, e.g. $f(\kappa) = 2^{\kappa^2}$, and $p(n)$ is a fixed polynomial in $n$ of degree not depending on $\kappa$, e.g. $p(n) = n$.

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