Solution Methods in Non-convex Optimization: Pathfollowing and Jumps

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Extended Abstract

(A short version of Chapter 1 (Introduction) of the book manuscript [3])

We consider the following nonlinear optimization problem

$$(P) \qquad \min\{f(x)|x \in M\}$$

where

$$M := \{x \in I\!R^n | h_i(x) = 0, i \in I, g_j(x) \leq 0, j \in J\}$$

$I := \{1, \dots, m\}, m \in n, J := \{1, \dots, s\}$ and $f, h_i, g_j \in C^k(I\!R^n, I\!R), i \in I, j \in J, k \in \{2, 3\}$.

First, we introduce the well-known concept of embedding (cf. e.g. [?], [?] and the references cited there): choose a one-parametric optimization problem

$$\tilde{P}(t) \min\{f(y,t)|y \in \tilde{M}(t)\}, t \in [0,1] \tag{1}$$

where

$$\tilde{M}(t) := \{y \in I\!R^{\bar{n}} | h_i(y,t) = 0, i \in I, g_j(y,t) \leq 0, j \in \bar{J}\},$$

$n \leq \bar{n}, \bar{J}$ is a finite index set with $J \subseteq \bar{J}$ with the following properties:

(A1) A stationary point for $\tilde{P}(0)$ is known and the corresponding Lagrange multipliers are known or easy to compute.

(A2) $\tilde{P}(t)$ has a global minimizer for all $t \in [0,1]$.

(A3) A local minimizer (stationary or generalized critical point [1] (shortly g.c. point)) for (P) is easy to compute using a local minimizer (stationary or g.c. point) of $\tilde{P}(1)$.

Sometimes we weaken (A2) by

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[1] We refer to the definitions in [1],[2],[3]

(A2') $\tilde{P}(t)$ has a g.c. point for all $t \in [0,1]$

(A1) and (A2') are a minimum of properties for finding a discretization of $[0,1]$:

$$0 = t_0 < \ldots < t_k < t_{k+1} < \ldots < t_N = 1 \tag{2}$$

and corresponding local minimizers (stationary or g.c. points) $y(t_k)$ of $P(t_k), k = 1, \ldots, N$.
Of course, in $\tilde{P}(t)$ it is possible that $y = x, \bar{n} = n$ and $J = \tilde{J}$ for some embeddings, e.g. the so-called standard embedding:

$$\begin{aligned} P_s(t) : \min\{tf(x) + (1-t)\|x - x^0\|^2 \quad & |h_i(x) + (t-1)h_i(x^0) = 0, i \in I, \\ & g_j(x) + (t-1)|g_j(x^0)| \leq 0, j \in J\}, \\ & \qquad\qquad t \in [0,1] \end{aligned} \tag{3}$$

i.e., the functions from $\tilde{P}(t)$ are defined as follows

$$\begin{aligned} f(x,t) &:= tf(x) + (1-t)\|x - x^0\|^2 \\ h_i(x,t) &:= h_i(x) + (t-1)h_i(x^0), i \in I \\ g_j(x,t) &:= g_j(x) + (t-1)|g_j(x^0)|, j \in J \end{aligned}$$

where $x^0 \in I\!R^m$ is arbitrarily chosen. In our book we consider the standard embedding and some modifications (Chapter 3). Furthermore, we will investigate the penalty- (Chapter 4), exact penalty (Chapter 5), and the Lagrange multiplier methods (Chapter 6) under the same parametric approach.
We consider here the following well-known quadratic penalty embedding as an example in order to explain the difference to $P_s(t)$:

$$\begin{aligned} P_p(t) : \min\{f(x) + \left(\tfrac{t}{1-t}\right)^2 \left[\textstyle\sum_{i\in I}(h_i(x))^2 + \sum_{j\in J}(\max\{g_j(x), 0\})^2\right] x \in I\!R^n\}, \\ t \in [0,1), \end{aligned} \tag{4}$$

i.e., the objective in $P_p(t), t \in [0,1)$ is defined by

$$f(x,t) := f(x) + \left(\frac{t}{1-t)}\right)^2 \left(\sum_{i\in I}(h_i(x))^2 + \sum_{j\in J}(\max\{g_j(x), 0\}\right)^2$$

and we do not have constraints.
This embedding has the following disadvantages:

a) $f(x,t)$ is only once continuously differentiable, i.e., the main assumption for a general theory of one-parametric optimization problems $\tilde{P}(t)$ (cf. (1)). (described in Chapter 2) is not satisfied.

b) Since $P_p(1)$ is not defined, the assumption (A3) cannot be satisfied. For the embedding $P_p(t)$ this means that we have to formulate the aim instead of (2) in the following sense:
Find a sequence $\{t_k\}, k = 0, 1, \ldots$ with $t_k \in [0,1), t_k < t_{k+1}$ and $\{t_k\}$ tending to 1 as well as corresponding points $x(t_k)$ (local minimizers, stationary or g.c. points) such that $\lim\limits_{t_k \to 1} x(t_k) = \hat{x}$ and $\hat{x}$ is at least a g.c. point of (P).

The disadvantages a) and b) can be omitted if we consider the following embedding motivated by $P_p(t)$:

$$\tilde{P}_p(t) : \min\{f(x) + \|v\|^2 + \|w\|^2 | (x, v, w) \in \tilde{M}_p(t)\}, t \in [0, 1],$$

where

$$\begin{array}{ll} \tilde{M}_p(t) := (x, v, w) \in I\!R^n \times I\!R^m \times I\!R^s | & th_i(x) + (1 - t)v_i = 0, i \in I \\ & tg_j(x) + (1 - t)w_j \leq 0, j \in J\}, \end{array}$$

which is defined for $t = 1$, too, and $\tilde{P}_p(1)$ is, in a certain sense, equivalent to (P). In Chapter 4 the equivalence of $P_p(t)$ and $\tilde{P}_p(t)$ is shown for $t \in [0, 1)$.
Of course, $P_s(t)$ and $\tilde{P}_p(t)$ are special cases of the general one-parametric optimization problem $\tilde{P}(t)$, for which we describe in Chapter 2 the class of Jongen, Jonker, Twilt and the class of Kojima and Hirabayashi as well as the possibilities to follow a connected component in the set of local minimizers (stationary or g.c. points) and to jump to another connected component. Here, the four singularities (basic degeneracies) of Type i, $i \in \{2, 3, 4, 5\}$, which are included in the class $\mathfrak{F}$, play an important role. The g.c. points of Type 1 are non-degenerated critical points. The local structure is described in Chapter 2.
With respect to the standard embedding, penalty embedding, exact penalty embedding, and the Lagrange multiplier embedding we give an answer to the following questions:

(i) How can we modify these embeddings such that

   a) for $t = 0$, a non-degenerate global minimizer and the corresponding Lagrange multipliers are known as a starting point for a pathfollowing procedure (this is a stronger condition than (A1)),

   b) (A2) or at least (A2') is satisfied,

   c) the embeddings are regular in the sense of Jongen-Jonker-Twilt (briefly (JJT-regular) and in the sense of Kojima-Hirabyashi (briefly KH-regular), respectively (that means that "we are in the corresponding class").

(ii) What kind of singularities may occur?

(iii) Under which conditions does a curve of stationary points connecting the starting point $(y^0, 0)$ with a point $(\hat{y}, 1)$ exist and how restrictive is this assumption?

(iv) How restrictive are the assumptions of JJT- resp. KH-regularity?

The main result of this book is the answer to the questions (iii) and (iv): With respect to the question (iii) the so-called Enlarged Mangasarian-Fromovitz Constraint Qualification (EnMFCQ) for the feasible set M of the original problem (P) plays an essential role, which excludes " *really non-convex* " functions in the constraints. The objective function $f(x)$ can be arbitrarily chosen (not necessarily convex) in the class $C^3(I\!R^n, I\!R)$. Then we are successful with the two modified standard embeddings, a skilful penalty embedding and exact penalty embedding using an arbitrarily chosen starting point. This is the greatest class of nonlinear optimization problems that we are successful for with pathfollowing procedures only.

In Chapter 2 it is shown that this procedure is numerically stable, i.e., we do not have any numerical problems to follow this curve. Unfortunately, the Lagrange multiplier embedding is not successful in general. In Chapter 6 we present a counterexample. The reason for this phenomenon is that the starting point is not the only stationary point for the constructed embedding at $t = 0$. Hence, the curve may return to $t = 0$ and we do not achieve $t = 1$. This situation is excluded for the first three embeddings. Further, we discuss the question (iv). In particular Chapter 3 (standard embedding) and Chapter 4 (penalty embedding) include topological justification theorems (the considered two classes are also generic for these special one-parametric optimization porblems). For the general problem $\tilde{P}(t)$ (cf. (1)) this fact is well known ([4], see also Chapter 2), but for special problems it is not automatically satisfied.
Now we consider the general class (P) with "really nonconvex" constraint functions. Examples show that the following phenomenon, e.g. for the penalty embedding $\tilde{P}_1(t)$, is typical if the EnMFCQ is not satisfied and if we approach $t = 1$ on a curve of stationary points:

$$\|v(t), w(t)\| \underset{t \to \bar{t}}{\longrightarrow} +\infty \quad \text{for a certain} \quad \bar{t} \leq 1.$$

Then we can introduce an additional compactification constraint, e.g.

$$\|x\|^2 + \|v\|^2 + \|w\|^2 \leq q$$

(where q is sufficiently large). This leads to some improvement, but in the worst case we have to find all connected components in the set of all g.c. points, and this is still an open problem. We are not surprised about this fact because the problem is strongly related to the problem of global optimization (we refer to [1] , Chapter 6). In Chapter 2 JUMP II* is proposed (with several jumps to other connected components in the set Σgc of g.c. points).

We should like to mention that we can use this concept also for multiobjective optimization problems (see [4]) and linear complementarity problems)

References

[1] Guddat, J., Guerra, F. and Jongen, H.Th., Parametric optimization: singularities, pathfollowing and jumps, B.G. Teubner, Stuttgart and John Wiley, Chichester, 1990.

[2] Gómez Bofill, W., Guddat, J., Jongen, H.Th., Rückmann, J.-J, Solano, C., Curvas critica y saltos en optimizacion no lineal, electronic publication: http://www.emis.de/monographs/curvas/index.html

[3] Guddat, J., Guerra Vazquez, F., Nowack, D.,J.-J. Rückmann, A Parametric Approach for Solution Methods in Non-convex Optimization manuscript

[4] Rückmann, J. and Tammer, K., On linear quadratic pertubations in one-parametric nonlinear optimization, Systems Science, 18, 1992, 1, 37-48.

[1] Allende, S.; Guddat, J.; Nowack, D..: On a modified penalty embedding for linear complementarity problems. Investigacion Operacional 1 (2002), 37-54.

[2] Allende, S.; Guddat, J.; Nowack, D.: On a modified standard embedding for linear complementarity problems. (in preparation)

[3] Guddat, J.; Guerra, F.; Nowack, D. Rückmann, J.-J. : On a modified standard embedding for non-convex optimization problems with equality and inequality constraints, (submitted to the Journal of Optimization).

[4] Guddat, J.; Guerra, F.; Nowack, D.; Rückmann, J.-J.: New embeddings for nonlinear multiobjective optimization problems II (a standard embedding with equality and inequality constratints), submitted to Yougoslav Journal of Operations Research.