GLOBAL OPTIMIZATION OF QUADRATICALLY CONSTRAINED PROBLEMS

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OUTLINE:

Problem definition
Previous work
Tree search algorithm
Convergence properties
Computational considerations
Numerical experiments

PROBLEM (QCP)

$$z^* = \min q_0^T x + \tfrac{1}{2} x^T Q_0 x$$

subject to

$$\begin{aligned} q_k^T x + \tfrac{1}{2} x^T Q_k x &\le b_k \qquad k = 1, \ldots, m \\ l^0 \le x &\le u^0 \end{aligned}$$

$Q_0, Q_1, \ldots, Q_m$ may be indefinite

$$Q_k = \begin{pmatrix} Q_k^{11} & \cdots & Q_k^{1n} \\ \cdots & & \\ Q_k^{n1} & \cdots & Q_k^{nn} \end{pmatrix} = Q_k^T$$

PREVIOUS WORK

- Al-Khayyal & Falk (1983):
  Bilinear Programming

- Al-Khayyal (1990):
  Generalized Bilinear Programming

- Sherali & Alameddine (1990):
  Reformulation-Linearization Technique

- Visweswaran & Floudas (1990,1993):
  The GOP Algorithm

LINEAR SUBPROBLEM $LP[\omega]$

Define $\omega = [l, u]$

$$\mathbf{z}(\omega) = \min q_0^T x + \sum_{i=1}^{n} \left(\frac{1}{2} Q_0^{ii} y_{ii} + \sum_{j=i+1}^{n} Q_0^{ij} y_{ij} \right)$$

subject to

$$q_k^T x + \sum_{i=1}^{n} \left(\frac{1}{2} Q_k^{ii} y_{ii} + \sum_{j=i+1}^{n} Q_k^{ij} y_{ij} \right) \le b_k \quad \forall k$$

$$\begin{aligned} y_{ij} - l_j x_i - l_i x_j &\ge -l_i l_j & i \le j \\ y_{ij} - u_j x_i - u_i x_j &\ge -u_i u_j & i \le j \\ y_{ij} - l_j x_i - u_i x_j &\le -l_j u_i & i \le j \\ y_{ij} - u_j x_i - l_i x_j &\le -l_i u_j & i \le j \\ l \le x &\le u & \end{aligned}$$

TREE SEARCH ALGORITHM

$\Omega^0 := \left\{ \left[l^0, u^0 \right] \right\}$

$\nu := 0$

loop

Find ω^ν such that

$$\min \{ z(\omega) : \omega \in \Omega^\nu \} = z(\omega^\nu)$$

$x^\nu :=$ minimizer of $LP[\omega^\nu]$

For some i, let

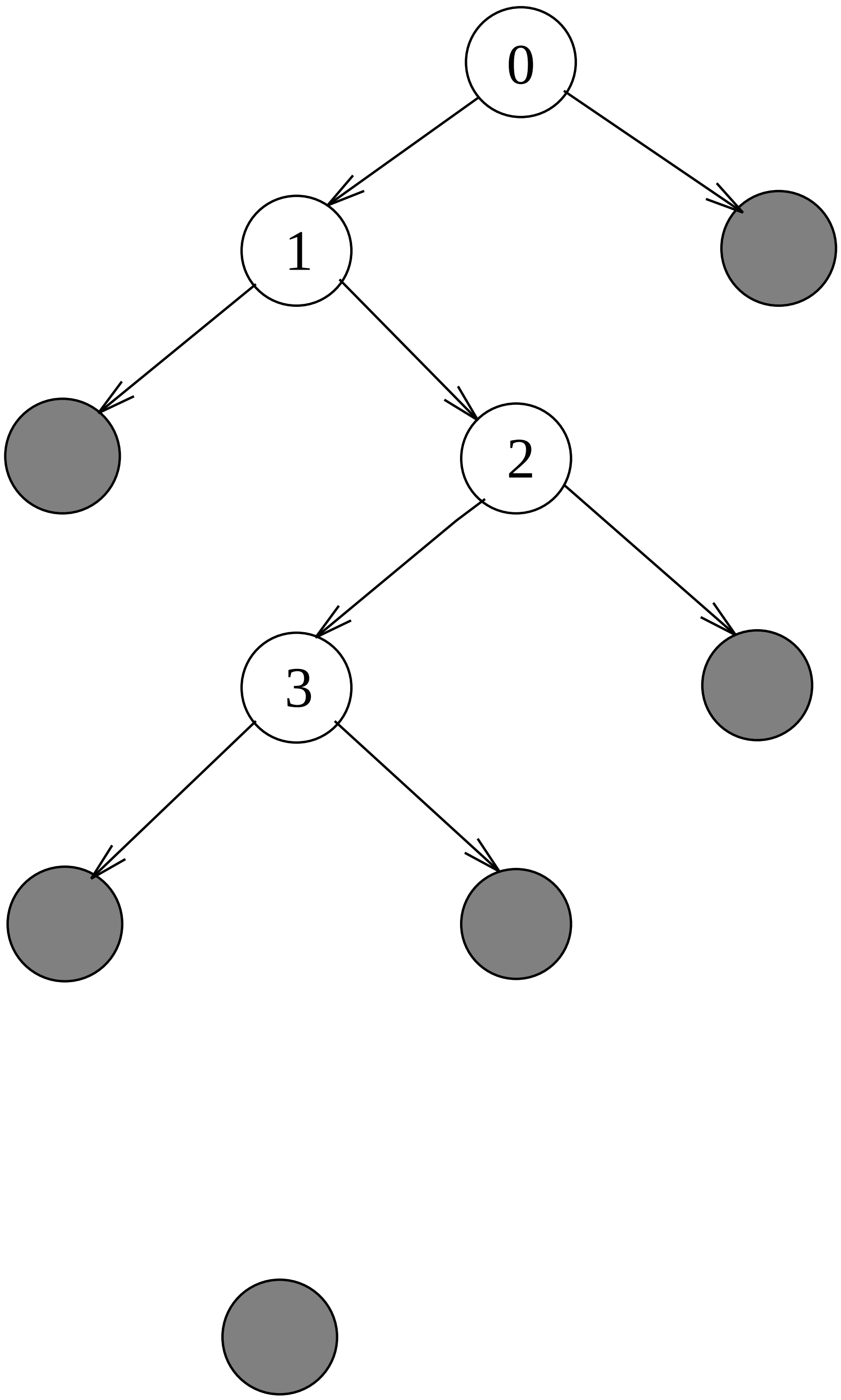
$$\Gamma^{\nu,1} := \left\{ x \in \omega^\nu : x_i \leq x_i^\nu \right\}$$

$$\Gamma^{\nu,2} := \left\{ x \in \omega^\nu : x_i \geq x_i^\nu \right\}$$

$\Omega^{\nu+1} := (\Omega^\nu \setminus \{\omega^\nu\}) \cup \left\{ \Gamma^{\nu,1}, \Gamma^{\nu,2} \right\}$

$\nu := \nu + 1$

endloop


0
1
2
3

THEOREM

$$\lim_{t \to \infty} x^{\nu^t} = x^* \text{ (global optimum)}$$
$$\text{for some } \left\{\nu^t\right\} \subseteq \{0, 1, 2, \ldots\}$$

$$z(\omega^{\nu-1}) \leq z(\omega^{\nu}) \leq z^*$$
$$\lim_{\nu \to \infty} z(\omega^{\nu}) = z^*$$

COMPUTATIONAL CONSIDERATIONS

$$\widehat{\omega} \subseteq \omega^\nu \Rightarrow z(\widehat{\omega}) \geq z(\omega^\nu)$$

If all QCP-feasible solutions in ω^ν are in $\widehat{\omega}$:

$x^\nu :=$ minimizer of $LP[\widehat{\omega}]$

Algorithm to tighten ω:

repeat
  for $i := 1, 2, \ldots, n$ **do**
    $\widehat{l}_i := \min x_i$
      subject to the constraints of $LP[\omega]$
    $\widehat{u}_i := \max x_i$
      subject to the constraints of $LP[\omega]$
  $\omega := [\widehat{l}, \widehat{u}]$
until no change in any bounds

CUTTING PLANES

Assume $Q_k = 0$
Constraint k:

$$q_k^T x \leq b_k \qquad * \qquad (x_j - l_j)$$

Resulting quadratic constraint:

$$\sum_{i=1}^{n} q_k^i x_i x_j - l_j \sum_{i=1}^{n} q_k^i x_i - b_k x_j \leq -b_k l_j$$

In $LP[\omega]$:

$$\sum_{i=1}^{n} q_k^i y_{ij} - l_j \sum_{i=1}^{n} q_k^i x_i - b_k x_j \leq -b_k l_j$$

NOTES FOR IMPLEMENTATION

- $z(\omega)$ is computed when ω enters Ω

- If $\forall k$:
$$\frac{1}{2}x^T Q_k x \le \sum_{i=1}^{n}\left(\frac{1}{2}Q_k^{ii} y_{ii} + \sum_{j=i+1}^{n} Q_k^{ij} y_{ij}\right) + \varepsilon$$
then remove from Ω^ν all ω where
$z(\omega) \ge q_0^T x + \frac{1}{2}x^T Q_0 x$

- Positive tolerance (ε) implies finite convergence

- *Binary* search tree?

- Solving $LP[\omega]$: Make warm start
$\Rightarrow$ Store optimal $LP[\omega]$-bases

- How often should ω be tightened?

NUMERICAL EXPERIMENTS

Problem source: Floudas & Pardalos (1990)

Problem				*Performance*		
Ref.	m	n	nlv	Nodes	Pivots	CPU
3.1	6	8	8	474	9E+4	5E+1
3.3	6	6	6	0	4E+1	1E−1
3.4	3	3	3	46	3E+3	2E+0
5.1	65	48	42	99	8E+5	1E+3
5.2	96	92	69	$z^* \in [1.7, 3.0]$		
5.3	26	32	24	115	2E+5	2E+2

LP-solver: CPLEX 2.1

Computer: HP 9000/700

CONCLUSIONS

- Tree search algorithm for quadratically constrained problems
- Nonexhaustive partitioning scheme
- Bilinear reformulation not required
- Converges to global optimum
- Computes a global solution to problems sized $69 \times 50(44)$
- Topics for further research
  - Experimental evaluation of algorithm variants
  - Cutting plane strategy