# Global Optimization of Rational Functions
# ( $H_2$ Optimal Model Reduction)

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## Overview

I. - Global optimization of rational functions using LMI's
    - Unconstrained optimization
    - Constrained optimization
  - Application: $H_2$ Optimal Model Reduction

II. Global optimization of polynomial functions: an exact method

## Problem

Given a polynomial (or rational) $f : \mathbf{R}^n \to \mathbf{R}$ find

$$\inf_{(x_1,x_2,\ldots,x_n)\in\mathbf{R}^n} f(x_1, x_2, \ldots, x_n)$$

OR

$$\min_{(x_1,x_2,\ldots,x_n)\in\mathbf{R}^n} f(x_1, x_2, \ldots, x_n)$$

and at least one point $x^* \in \mathbf{R}^n : \quad f(x^*) = \min_{x\in\mathbf{R}^n} f(x)$.

## Optimization of polynomial functions

Known:

- Optimization of polynomial functions is NP-hard.
- Important in applications.
- Algebraic geometry.

Methods for optimization of polynomial functions:

1. LMI relaxations ((2000) P. Parrilo, (2001) J.B. Lassere)
2. Exact method ((2001) B. Hanzon, D. Jibetean)

## Unconstrained optimization of rational functions

Given two polynomials $p,\ q \in \mathbf{R}[x_1, \ldots, x_n]$ relatively prime, find

$$\inf_{(x_1,\ldots,x_n)\in\mathbf{R}^n} \frac{p(x_1,\ldots,x_n)}{q(x_1,\ldots,x_n)} \tag{1}$$

## Examples

1. $\displaystyle\inf_{x\in\mathbf{R}} \frac{1}{x^2} = 0.$

2. $\displaystyle\inf_{(x_1,x_2)\in\mathbf{R}^2} \frac{(x_1+x_2)^2}{(x_1^2+x_2^2)} = \frac{1}{2}.$

# Main results

**Lemma 1** Let $a,\ b \in \mathbf{R}[x_1, \ldots, x_n]$ relatively prime polynomials.

$$\text{If } \frac{a(x)}{b(x)} \geq 0, \quad \forall x \in \mathbf{R}^n \text{ s.t. } b(x) \neq 0,$$

then one of the two holds:

- $a(x) \geq 0,\ b(x) \geq 0,\ \forall x \in \mathbf{R}^n.$
- $a(x) \leq 0,\ b(x) \leq 0,\ \forall x \in \mathbf{R}^n.$

**Proposition 2** Let $p(x),\ q(x)$ relatively prime.

$$\exists \hat{x}, \tilde{x} \in \mathbf{R}^n,\ q(\hat{x}) > 0,\ q(\tilde{x}) < 0 \implies \inf_{x \in \mathbf{R}^n} \frac{p(x)}{q(x)} = -\infty.$$

Converse *not* true.

**Proof** by contradiction: Let $\alpha$ be a lower bound.

$$\frac{p(x)}{q(x)} \geq \alpha, \quad \forall x \in \mathbf{R}^n \text{ s.t. } q(x) \neq 0$$

$$\iff \frac{p(x) - \alpha q(x)}{q(x)} \geq 0, \ \forall x \in \mathbf{R}^n \text{ s.t. } q(x) \neq 0$$

Use Lemma 1. $\square$

Assume w.l.g. $q(x) \geq 0, \ \forall x \in \mathbf{R}^n$. Solve

$$\begin{aligned} &\sup \quad \alpha \\ &\text{s.t.} \quad p(x) - \alpha q(x) \geq 0, \quad \forall x \in \mathbf{R}^n. \end{aligned} \tag{2}$$

**Theorem 3** The solution of (2) equals $\inf_{x \in \mathbf{R}^n} \frac{p(x)}{q(x)}$.

Approach: Check whether $q$ changes sign on $\mathbf{R}^n$.

- If YES $\Rightarrow$ $\inf_{x\in\mathbf{R}^n} p(x)/q(x) = -\infty$
- If NO and $q(x) \geq 0,\ \forall x \in \mathbf{R}^n$ $\Rightarrow$ solve (2).

How to solve (2)? We discuss here an LMI relaxation.

**Notation** $2m = \mathrm{tdeg}(p(x) - \alpha q(x))$ in $x \in \mathbf{R}^n$ ($\alpha$-parameter).

**Property** Given the polynomial $p(x) - \alpha q(x)$,

$$\exists \text{ symmetric } N(\lambda, \alpha) \in \mathbf{R}[\lambda, \alpha]^{m \times m}, \quad p(x) - \alpha q(x) = v^T N(\lambda, \alpha) v,$$

where $v^T = \begin{pmatrix} 1 & x_1 & \ldots & x_n & x_1^2 & x_1 x_2 & x_2^2 & \ldots & x_n^m \end{pmatrix}$

**Remarks**

- $N(\lambda, \alpha) = N(\lambda, \alpha)^T$.
- $N(\lambda, \alpha) = N_0 + \sum_{i=1}^{l} N_i \lambda_i + N_{l+1} \alpha, \quad N_i \in \mathbf{R}^{m \times m}$.
- $\exists \lambda^*, \quad N(\lambda^*, \alpha) \succeq 0 \Longrightarrow p(x) - \alpha q(x) \geq 0, \ \forall x \in \mathbf{R}^n$.

**Proposition** The problem

$$\begin{aligned} &\sup \quad \alpha \\ &\text{s.t.} \quad N(\lambda, \alpha) \succeq 0. \end{aligned} \tag{3}$$

where $N$ is symmetric, affine in $(\alpha, \lambda)$ is an LMI relaxation of (2),i.e.

$$\begin{aligned} &\sup \quad \alpha && \leq \quad \inf_{(x_1,\ldots,x_n)\in\mathbf{R}^n} \frac{p(x_1,\ldots,x_n)}{q(x_1,\ldots,x_n)} \\ &\text{s.t.} \quad N(\lambda, \alpha) \succeq 0. \end{aligned}$$

Equality $\iff$ $p(x) - \alpha_0 q(x) = \sum s_i(x)^2,\ s_i \in \mathbf{R}[x],\ \alpha_0$- solution of (2).

Is the relaxation exact? Find $x^* \in \mathbf{R}^n$ such that
$\inf p(x)/q(x) = p(x^*)/q(x^*)$.

Let $M$ be solution of the dual problem. If $rank(M) = 1$. Then
$M = v^* {v^*}^T$ and $v^* = v(x^*)$, where
$v^T = \begin{pmatrix} 1 & x_1 & \dots & x_n & x_1^2 & x_1 x_2 & x_2^2 & \dots & x_n^m \end{pmatrix}$.

Remarks:

- General method; no assumptions on the criterion function.
- Extension to rational functions has the same:
  - Theoretical properties.
  - Computational complexity

## Constrained optimization of rational functions

$$\inf_{x \in D \subset \mathbf{R}^n} \frac{p(x)}{q(x)}, \qquad D \text{ semi-algebraic set.}$$

Questions:

1. Do theoretical results extend (Lemma 1)?

2. How to perform computation?

Answer: 1. does not hold in general.

**Counter-example**:

$$\inf_{\{(x_1,x_2,x_3)\in\mathbf{R}^3 \mid x_2^2+x_3^2=0\}} \frac{x_1 - x_2 + x_3 + 1}{x_1 + x_2 + x_3 + 1} = 1.$$

**Lemma** Let $a,\ b \in \mathbf{R}[x_1, \ldots, x_n]$ relatively prime polynomials.

$$\text{If } \frac{a(x)}{b(x)} \geq 0, \quad \forall x \in B(0,1) \subseteq \mathbf{R}^n \text{ s.t. } b(x) \neq 0,$$

then one of the two holds:

- $a(x) \geq 0,\ b(x) \geq 0,\ \forall x \in B(0,1) \subseteq \mathbf{R}^n.$
- $a(x) \leq 0,\ b(x) \leq 0,\ \forall x \in B(0,1) \subseteq \mathbf{R}^n.$

**Proposition** Let $p(x),\ q(x)$ relatively prime.

$$\exists \hat{x}, \tilde{x} \in B(0,1) \subseteq \mathbf{R}^n,\ q(\hat{x}) > 0,\ q(\tilde{x}) < 0 \implies \inf_{x \in B(0,1) \subseteq \mathbf{R}^n} \frac{p(x)}{q(x)} = -\infty.$$

The results hold even if $B(0,1)$ is replaced by $D \subset \mathbf{R}^n$, $C \subseteq D \subseteq \bar{C}$, where $C$ is a connected, open set.

$$\begin{aligned} &\sup \quad \alpha \\ &\text{s.t.} \;\; p(x) - \alpha q(x) \geq 0, \quad \forall x \in B(0,1) \subseteq \mathbf{R}^n. \end{aligned} \tag{4}$$

Known:

$$p(x) - \alpha q(x) \geq 0, \quad \forall x \in B(0,1) \subseteq \mathbf{R}^n \Longleftrightarrow$$

$$p(x) - \alpha q(x) = \sum_{l=1}^{L} s_l^2(x) + \sum_{r=1}^{R} t_r^2(x)(1 - x_1^2 - \ldots - x_n^2).$$

$\Longrightarrow$ LMI relaxation of (4).

## Application: $H_2$ optimal model reduction

Given a mapping $\quad T(s) = C(sI_n - A)^{-1}B$, $A$-stable matrix, and $\hat{n} < n$,

Find $\quad \hat{T}(s) = \hat{C}(sI_{\hat{n}} - \hat{A})^{-1}\hat{B}$ such that

$$\min_{\hat{A}-\text{stable}} \|T(s) - \hat{T}(s)\|_2^2$$

**Example 1** Oscillatory system (see [2])

Approximate

$$T(s) = \frac{s^2 - s + 2}{s^3 + 0.5s^2 + 2s + 0.5} \qquad (n = 3)$$

$$A = \begin{pmatrix} -\frac{1}{2} & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix},$$

by a second order system ($\hat{n} = 2$).

Construct

$$\hat{A} = \begin{pmatrix} -\frac{1}{2}x_1^2 & -x_2 \\ x_2 & 0 \end{pmatrix}, \quad \hat{C} = \begin{pmatrix} x_1 & 0 \end{pmatrix}, \quad x_1 > 0,\ x_2 > 0$$

**Example 1**(continued)

$$\min_{\hat{A}-stable} \|T(s) - \hat{T}(s)\|_2^2 = \min_{x_1, x_2 > 0} \left(3 - \frac{p(x_1, x_2)}{q(x_1, x_2)}\right)$$

where

$$\begin{aligned} p(x_1, x_2) = 4x_1^2(&64 - 32x_1^2 + 20x_1^4 - 4x_1^6 + x_1^8 + 848x_2^2 + 256x_1^2x_2^2 \\ &+ 236x_1^4x_2^2 + 16x_1^6x_2^2 + 16x_1^8x_2^2 - 1616x_2^4 - 480x_1^2x_2^4 - 280x_1^4x_2^4 - 32x_1^6x_2^4 \\ &+ 1200x_2^6 + 320x_1^2x_2^6 + 80x_1^4x_2^6 - 432x_2^8 - 64x_1^2x_2^8 + 64x_2^{10}) \end{aligned}$$

$$\begin{aligned} q(x_1, x_2) = \\ &(4 + 8x_1^2 + x_1^4 + x_1^6 + 56x_2^2 - 4x_1^2x_2^2 + 8x_1^4x_2^2 - 60x_2^4 + 4x_1^2x_2^4 + 16x_2^6)^2 \end{aligned}$$

Solve: $\sup \ \alpha, s.t. \ N(\lambda, \alpha) \succeq 0.$

- $m = \text{tdeg}(p(x) - \alpha q(x))/2 = 6$
- Construct
$v^T = \begin{pmatrix} 1 & x_1^2 & x_1^4 & x_1^6 & x_2^2 & x_2^4 & x_2^6 & x_1^2 x_2^2 & x_1^4 x_2^2 & x_1^2 x_2^4 \end{pmatrix}$
- Construct the matrix $N(\lambda, \alpha) \in \mathbf{R}[\alpha, \lambda]^{10\times 10}$, $\lambda \in \mathbf{R}^{28}$.
- Compute the solution for the LMI relaxation: $\alpha = 1.2358$.
- Find a point of minimum $(x_1, x_2) = (1.1916, 0.4183)$.

The best approximant in $H_2 - norm$ is

$$\hat{A} = \begin{pmatrix} -0.7099 & -0.4183 \\ 0.4183 & 0 \end{pmatrix}, \quad \hat{C} = \begin{pmatrix} 1.1916 & 0 \end{pmatrix}, \quad \hat{B} = \begin{pmatrix} 0.2080 \\ -1.3118 \end{pmatrix}$$

Questions:

- Find *all* global minimizers.
- Find all local minimizers where the value of the criterion is close to the global minimal value.

LMI relaxations (Parrilo)

- Global lower bound (+−).
- Can be checked (under certain conditions) whether the relaxation is tight (+−).
- Matrices of size $(n+d)!/(n!d!)$ (+).

## Summary

- Formulate a criterion for deciding $\inf \frac{p(x)}{q(x)} = -\infty$.
- Formulate and solve an LMI relaxation of the rational optimization problem.
- Application to *optimal* $H_2$ model reduction.

## References


[1 ] D.Jibetean, *Global optimization of rational multivariate functions*, CWI Report PNA-R0120, 2001.

[2 ] B. Hanzon, J. Maciejowski, *Constructive algebra methods for the $L_2$-Problem for stable Linear systems*, Automatica, vol.32 no 12 pp 1645-1657.

## Global Minimization of a Polynomial using Algebraic Matrix Methods

Given a polynomial $p : \mathbf{R}^n \to \mathbf{R}$ find

$$\min_{(x_1,x_2,\ldots,x_n)\in\mathbf{R}^n} p(x_1, x_2, \ldots, x_n)$$

and at least one point $x^* \in \mathbf{R}^n : \quad p(x^*) = \min_{x\in\mathbf{R}^n} p(x)$
OR

$$\inf_{(x_1,x_2,\ldots,x_n)\in\mathbf{R}^n} p(x_1, x_2, \ldots, x_n)$$

Guaranty for finding the global optimum.

## Polynomial optimization

Let $p : \mathbf{R}^n \to \mathbf{R}$ be a polynomial function.

$$\left\{\begin{array}{l} \frac{\partial p}{\partial x_1}(x_1, \ldots, x_n) = 0 \\ \vdots \\ \frac{\partial p}{\partial x_n}(x_1, \ldots, x_n) = 0 \end{array}\right. \tag{5}$$

### Remarks

- $\inf_{x \in \mathbf{R}^n} p(x)$ may not be attained.
- $p$ may have an infinite number of critical points.

## Matrix methods for solving systems of polynomial equations

Let

$$\begin{cases} f_1(x_1,\dots,x_n)=0 \\ \vdots \\ f_s(x_1,\dots,x_n)=0 \end{cases} \qquad f_j \in \mathbf{C}[x_1,\dots,x_n], \quad j=1,\dots,s \tag{6}$$

have a finite number of solutions.

**Theorem(charact. the number of solutions)** Let (6) such that

$$\forall x_i,\ i=1,\dots,n,\ \exists f_j,\ j=1,\dots,s \text{ with } \mathrm{LM}(f_j)=x_i^{\alpha_i},$$

then the system has a finite number of solutions.

LM=Leading Monomial

**Example 1** $p(x_1, x_2) = 1/3\, {x_1}^3 - 1/2\, {x_1}^2 + x_1 x_2 + 1/3\, {x_2}^3 - 3/2\, {x_2}^2$
with the first order conditions

$$\begin{cases} f_1 = {x_1}^2 - x_1 + x_2 = 0 \\ f_2 = {x_2}^2 + x_1 - 3\, x_2 = 0 \end{cases}$$

Matrix methods (see [1], [3]) translate polynomial systems solving into linear algebra problems:
Let $I$ be a polynomial ideal and $g \in \mathbf{C}[x_1, \ldots, x_n]$

$$A_g : \mathbf{C}[x_1, \ldots, x_n]/I \to \mathbf{C}[x_1, \ldots, x_n]/I, \qquad A_g([h]) = [gh].$$

is a linear mapping $\Longrightarrow$ matrix representation.

**Example 1** (continued) $I = \langle {x_1}^2 - x_1 + x_2, {x_2}^2 + x_1 - 3\,x_2 \rangle$.

$$A_{x_1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & -3 & 1 \end{pmatrix}, \quad A_{x_2} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 3 & 0 \\ 0 & -1 & 1 & 3 \end{pmatrix}$$

**P1:** Matrices $A_{x_i},\ i = 1, \ldots, n$ are pairwise commutative. (in the example: $A_{x_1} A_{x_2} = A_{x_2} A_{x_1}$)

**P2:** $f_j(A_{x_1}, \ldots, A_{x_n}) = \mathbf{0}, \forall j = 1, \ldots, s \Rightarrow (A_{x_1}, \ldots, A_{x_n})$ is a matrix solution.

**Example 1** (continued) Indeed

$${A_{x_1}}^2 - A_{x_1} + A_{x_2} = \mathbf{0}, \quad {A_{x_2}}^2 + A_{x_1} - 3\,A_{x_2} = \mathbf{0}.$$

**P3:** Let $\xi$ satisfy $A_{x_i}\xi = \lambda_i \xi,\ \forall i = 1, \ldots, n$. Then

$$\mathbf{0} = f_j(A_{x_1}, \ldots, A_{x_n})\xi = f_j(\lambda_1, \ldots, \lambda_n)\xi$$

$\Rightarrow (\lambda_1, \ldots, \lambda_n)$ is a scalar solution.

**Example 1** (continued)

| Eigenvals $A_{x_1}$ | Eigenvectors $A_{x_1}$ | Eigenvals $A_{x_2}$ | Eigenvectors $A_{x_2}$ |
|---|---|---|---|
| 0 | (1,0,0,0) | 0 | (1,0,0,0) |
| 0.63 | (1, 0.63, 0.23, 0.14) | 0.23 | (1, 0.63, 0.23, 0.14) |
| 0.68-1.63i | (1,0.68-1.63i,2.88+0.58i,2.92-4.30i) | 2.88+0.58i | (1,0.68-1.63i,2.88+0.58i,2.92-4.30i) |
| 0.68+1.63i | (1,0.68+1.63i,2.88-0.58i,2.92+4.30i) | 2.88-0.58i | (1,0.68+1.63i,2.88-0.58i,2.92+4.30i) |

Solutions:

$$S = \{(0,0), (0.63, 0.23), (0.68 - 1.63i, 2.88 + 0.58i), (0.68 + 1.63i, 2.88 - 0.58i)\}$$

**Remark** All solutions are obtained in this way (over $\mathbf{C}^n$).

**P4:** For $g \in \mathbf{C}[x_1, \ldots, x_n]$, Eigenvalues$(A_g)= g(S)$.

**Example 1** (continued)

$p(x_1, x_2) = 1/3\, {x_1}^3 - 1/2\, {x_1}^2 + x_1 x_2 + 1/3\, {x_2}^3 - 3/2\, {x_2}^2$

$$A_p = \begin{pmatrix} 0 & 1/3 & -4/3 & 1/3 \\ 0 & 2/3 & -4/3 & -1 \\ 0 & 1 & -11/3 & 4/3 \\ 0 & 7/3 & -5 & -7/3 \end{pmatrix}$$

with eigenvalues $0, -0.4, -2.64 + 2.76i, -2.64 - 2.76i$, hence critical values of p.

**Remark** The size of $A_{x_i}$'s increases exponentially with $n$.

## General Case

Construct a new polynomial

$$p_\lambda(x_1, \ldots, x_n) = p(x_1, \ldots, x_n) + \lambda(x_1^{2m} + \ldots + x_n^{2m})$$

where $\lambda > 0,\ m \in N^*,\ 2m > TDeg(p)$.

$$\text{Note}: \quad \lim_{\lambda \downarrow 0} p_\lambda(x_1, \ldots, x_n) = p(x_1, \ldots, x_n), \quad (x_1, \ldots, x_n) \in \mathbf{R}^n.$$

**Theorem 1**

$$\lambda > 0 \text{ fixed}, \quad \inf_{x \in \mathbf{R}^n} p_\lambda(x_1, \ldots, x_n) \text{ is attained.}$$

Consider

$$\min_{(x_1,\ldots,x_n)\in\mathbf{R}^n} p_\lambda(x_1,\ldots,x_n), \quad \text{for fixed } \lambda > 0.$$

Derivatives:

$$\begin{cases} \frac{\partial p_\lambda}{\partial x_1}(x_1,\ldots,x_n) = 2m\lambda {x_1}^{2m-1} + \frac{\partial p}{\partial x_1}(x_1,\ldots,x_n) \\ \vdots \\ \frac{\partial p_\lambda}{\partial x_n}(x_1,\ldots,x_n) = 2m\lambda {x_n}^{2m-1} + \frac{\partial p}{\partial x_n}(x_1,\ldots,x_n) \end{cases}.$$

From the previous theorem $p_\lambda$ has a finite number of critical points $\Longrightarrow$ back to the previous case.

**Question:** What is the relation between the critical points of $p_\lambda$ and the critical points of $p$?

**Theorem 2**

$$\lim_{\lambda \downarrow 0} \min_{(x_1,\ldots,x_n)\in \mathbf{R}^n} p_\lambda(x_1,\ldots,x_n) = \inf_{(x_1,\ldots,x_n)\in \mathbf{R}^n} p(x_1,\ldots,x_n).$$

Define $X_\lambda = \{x \in \mathbf{R}^n \mid p_\lambda(x) = \min_{\tilde{x}\in\mathbf{R}^n} p_\lambda(\tilde{x})\}$. Denote by $L$ the limit set of $X_\lambda$:

$$L \;=\; \{x \in \mathbf{R}^n \mid \forall \varepsilon > 0 \; \exists \; \lambda_\varepsilon \;\; s.t. \; \forall \lambda, \;\; 0 < \lambda < \lambda_\varepsilon, \;\; X_\lambda \cap B(x,\varepsilon) \neq \emptyset\}$$

**Theorem 3**

- $L$ is finite.
- If $p$ has a minimum
  - $L$ is nonempty.
  - $L \subseteq \{x \in \mathbf{R}^n | p(x) = \min_{x \in \mathbf{R}^n} p(x)\}$

**Note** The method is guaranteed to find either the global minimum and at least one point where it is attained or the infimum.

Denote by $A_\lambda$ the matrix $A_{p_\lambda}$

**Question** How to compute $\lim_{\lambda \downarrow 0} z(\lambda)$, where $z(\lambda)$ is an eigenvalue of $A_\lambda$?

**Proposition** $A_\lambda$ is polynomial matrix in $1/\lambda$.

**Theorem** $z(\lambda)$ are algebraic functions and their finite limits are roots of $h_0(z)$, where

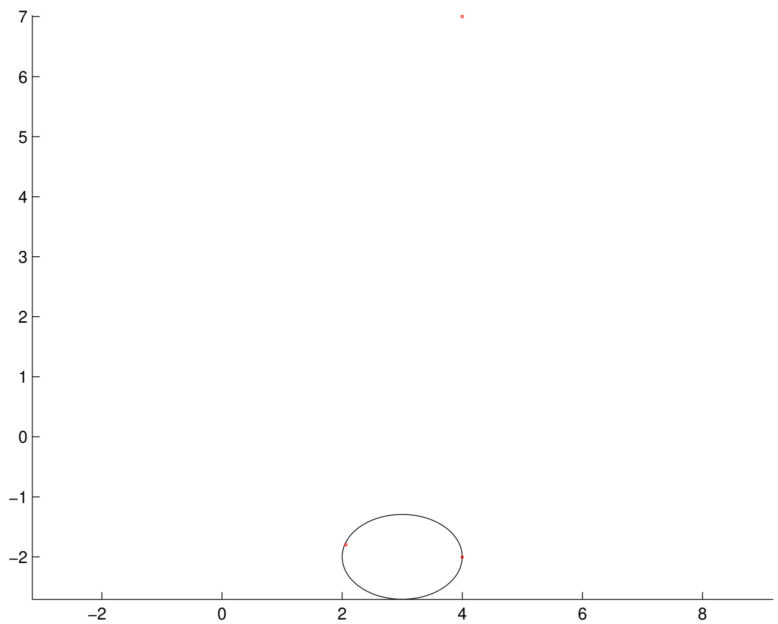
$$\det(A_\lambda - zI) = \frac{1}{\lambda^k} h_0(z) + \frac{1}{\lambda^{k-1}} h_1(z) + \ldots + h_k(z).$$

## Example 2

$$p(x_1, x_2) = [(x_1 - 3)^2 + 2(x_2 + 2)^2 - 1]^2[(x_1 - 4)^2 + (x_2 - 7)^2].$$

$$p_\lambda(x_1, x_2) = [(x_1 - 3)^2 + 2(x_2 + 2)^2 - 1]^2[(x_1 - 4)^2 + (x_2 - 7)^2] + \lambda(x_1^8 + x_2^8).$$

Square matrices of size 49. Solution: The minimum is 0 and it is attained at $\{(4, 7), (3.9999, -2.0039), (2.0406, -1.8004)\}$.



## Conclusions

- The method finds the global infimum.
- If minimum is attained, the method finds (at least) one point in each connected component of the set of minimizers.
- Arbitrary precision of the computations.
- It involves computations with large matrices, of size $(2m-1)^n$.

## References

[1 ] B.Hanzon, J.M.Maciejowski, C.T.Chou, *Model reduction in $H_2$ using matrix solutions of polynomial equations,* Cambridge University Engineering Department, Technical Report CUED/F-INFENG/TR314, 1998.

[2 ] B. Hanzon, D. Jibetean, *Global minimization of a multivariate polynomial using matrix methods*, to appear in J. Global Optimization.

[3 ] H.J.Stetter, *Matrix eigenproblems are at the heart of polynomials systems solving,* SIGSAM Bulletin, vol.30, pp.22-25, 1996.