On bounding the diameter of a distance-regular graph

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Abstract

In this note we investigate how to use an initial portion of the intersection array of a distance-regular graph to give an upper bound for the diameter of the graph. We prove three new diameter bounds. Our bounds are tight for the Hamming d-cube, doubled Odd graphs, the Heawood graph, Tutte's 8-cage and 12-cage, the generalized dodecagons of order $(1, 3)$ and $(1, 4)$, the Biggs–Smith graph, the Pappus graph, the Wells graph, and the dodecahedron.

MSC: 05E30, 05C12, 05C62

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1 Introduction

Let Γ denote a connected graph. For two vertices x, y of Γ we denote by $\partial(x, y)$ the distance between x and y in Γ. We write $\Gamma_i(x)$ for the set of vertices z with $\partial(x, z) = i$ (see Section 2 for formal definitions).

Let $d = d(\Gamma)$ denote the diameter of Γ. For any two vertices x and y of Γ at distance h, let $C_h(x, y) := \Gamma_{h-1}(x) \cap \Gamma_1(y)$, $A_h(x, y) := \Gamma_h(x) \cap \Gamma_1(y)$ and $B_h(x, y) := \Gamma_{h+1}(x) \cap \Gamma_1(y)$. A graph Γ is called *distance-regular* if there are integers b_i, c_i $(0 \leq i \leq d)$ which satisfy $c_i = |C_i(x, y)|$ and $b_i = |B_i(x, y)|$ for any two vertices x and y of Γ at distance i. The array $\{b_0, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d\}$ is called the *intersection array* of Γ. Clearly such a graph is regular of valency $k := b_0$ and $a_i := |A_i(x, y)| = k - b_i - c_i$, $(0 \leq i \leq d)$. Distance-regular graphs play an important role in algebraic combinatorics because of the relation to design theory, coding theory, finite and Euclidean geometry, and group theory (cf. [18, 13]). Bounding the diameter of a distance-regular graph in terms of some of its intersection numbers is very important in the theory of DRGs. Bang, Dubickas, Koolen and Moulton [4] proved the Bannai–Ito Conjecture (from 1984) that there are only finitely many distance-regular graphs of fixed valency greater than two. However, this still leaves open questions about the diameter given a small initial part of the intersection array. In this paper we recall old and prove some new bounds about bounding the diameter.

Only three distance-regular graphs with $d \geq 2k$ are known: Tutte's 12-cage, the Biggs–Smith graph and the Foster graph, all of valency 3. The explanation of this fact, and the construction of a tight diameter bound, is one of the main open problems about distance-regular graphs. In this paper we consider the following problem.

Problem 1.1 *Let* Γ *denote a distance-regular graph, and assume that we only know the first* $q+2$ *elements* b_i *and* c_i *of intersection array*

$$\{b_0, b_1, \ldots, b_q, b_{q+1}, \ldots; c_1, c_2, \ldots, c_q, c_{q+1}, c_{q+2}, \ldots\}, \tag{1}$$

i.e., assume that we don't know intersection intersection numbers $b_{q+2}, \ldots, b_{d-1}$ *and* $c_{q+3}, \ldots, c_d$*. Use the numbers given in the intersection array* (1) *to give an upper bound for the diameter of* Γ*.*

For example, if we know that $\{3, 2, 2, 2, 2, \ldots; 1, 1, 1, 1, 2, \ldots\}$ are the first 5 numbers of the intersection array of a distance-regular graph Γ, can we conclude that diameter bound is at most 9? An additional question is how to find the smallest such q so that these intersection numbers produce a tight diameter bound of Γ? For example, if we know that $\{k, b_1, \ldots; 1, c_2, \ldots\}$ are the first two elements of the intersection array of Γ, can we conclude that the diameter of Γ is at most k? Moreover, in that case, can we give some upper bound for the diameter, a formula that includes some linear or nonlinear combination of the numbers from the set $\{k, b_1, 1, c_2\}$? In the case where $c_2 \geq 2$ and $a_1 \leq c_2 - 1$ the answers are positive (see Theorem 1.1).

Similar problems which involve intersection numbers can be found in the dynamic survey paper by Van Dam, Koolen and Tanaka, [18, Subsection 18.9] (for example, see Problem 51 (problem raised by Bannai)).

Our main results are Theorems 1.1, 1.2 and 1.3. We prove them using a similar technique as Hiraki in [22] and our proof is mainly based on [22, Lemma 3.1]. We re-prove this lemma in the (self-contained) Section 3 using 5-point counts. This simplifies the proof and makes the paper self-contained. It demonstrates the usefulness of the t-point count technique (from [35]) in the present context. Section 3 may be skipped without impairing the remaining contents if [22, Lemma 3.1] is assumed instead.

Theorem 1.1 *Let* Γ *denote a distance-regular graph of diameter* d*, valency* $k \geq 3$ *and assume that* $c_2 \geq 2$*. If* $a_1 \leq c_2 - 1$ *then for every* c_q $(1 \leq q \leq d)$ *we have*

$$d \leq k - c_q + q. \tag{2}$$

Note that every bipartite distance-regular graph with $c_2 \geq 2$ satisfies the conditions of Theorem 1.1. For $q = 1$ inequality (2) becomes equality for the Hamming d-cube (the intersection numbers are $b_i = d - i$, $c_i = i$ $(0 \leq i \leq d)$). For $q = 4$, (2) becomes equality for all distance-regular graphs of diameter 5 and $c_2 \geq 2$ which are both antipodal and bipartite (some examples of such graphs are the doubled Gewirtz, doubled 77 and doubled Higman–Sims graphs (for the intersection numbers see [13, pg. 418])).

Theorem 1.2 *Let* Γ *denote a distance-regular graph of diameter* d*, valency* $k \geq 3$ *and let* q *be an integer with* $2 \leq q \leq d-1$*. If* $c_{q+1} > c_q$ *and* $a_q \leq c_{q+1} - c_q$ *then*

$$d \leq \left(\left\lfloor \frac{k - c_{q+1} - 1}{c_q} \right\rfloor + 2 \right) q + 1. \tag{3}$$

Note that if $c_{q+1} > c_q$ then for every bipartite distance-regular graph we have $a_q \leq c_{q+1} - c_q$. Some examples for which the inequality (3) becomes equality are doubled Odd graphs (intersection array is $b_i = m + 1 - \lfloor \frac{1}{2}(i+1) \rfloor$, $c_i = \lfloor \frac{1}{2}(i+1) \rfloor$ $(0 \leq i \leq d)$), the Heawood graph (intersection array $\{3, 2, 2; 1, 1, 3\}$), Tutte's 8-cage (intersection array $\{3, 2, 2, 2; 1, 1, 1, 3\}$) and 12-cage (intersection array $\{3, 2, 2, 2, 2, 2; 1, 1, 1, 1, 1, 3\}$), the generalized dodecagons of order $(1, 3)$ (intersection array $\{4, 3, 3, 3, 3, 3; 1, 1, 1, 1, 1, 4\}$) and of order $(1, 4)$ (intersection array $\{5, 4, 4, 4, 4, 4; 1, 1, 1, 1, 1, 5\}$), and the Biggs–Smith graph (intersection array $\{3, 2, 2, 2, 1, 1, 1; 1, 1, 1, 1, 1, 1, 3\}$). For the Foster graph, the bipartite graph with diameter 8 and intersection array $\{3, 2, 2, 2, 2, 1, 1, 1; 1, 1, 1, 1, 2, 2, 2, 3\}$, the expression on the right-hand side of (3) equals 9 for the smallest q. This suggests that a sharper bound might be possible.

Theorem 1.3 *Let Γ denote a distance-regular graph of diameter d, valency $k \geq 3$, let q be an integer with $2 \leq q \leq d-1$ and assume that $c_{q+1} > c_q$ and $a_q \leq c_{q+1} - c_q$. If $c_{q+2} > c_{q+1}$ then*

$$d \leq \left(\left\lfloor \frac{k - c_{q+1} - 2}{c_q} \right\rfloor + 2 \right) q + 2. \tag{4}$$

Some of distance-regular graphs for which inequality (4) becomes equality are the Pappus graph (intersection array is $\{3, 2, 2, 1; 1, 1, 2, 3\}$), $AG(2, 4)$ minus a parallel class (the intersection array is $\{4, 3, 3, 1; 1, 1, 3, 4\}$), the Wells graph (intersection array is $\{5, 4, 1, 1; 1, 1, 4, 5\}$), the dodecahedron (intersection array is $\{3, 2, 1, 1, 1; 1, 1, 1, 2, 3\}$), and more (cf. [13]). See also Table 1.

DRG	d	k	q	$\left(\left\lfloor \frac{k-c_{q+1}-1}{c_q} \right\rfloor + 2 \right) q + 1$	$\left(\left\lfloor \frac{k-c_{q+1}-2}{c_q} \right\rfloor + 2 \right) q + 2$
the Pappus graph	4	3	2	5	4
$AG(2, 4)$ minus a parallel class	4	4	2	5	4
the Wells graph	4	5	2	5	4
the dodecahedron	5	3	3	7	5
the Ivanov-Ivanov-Faradjev graph	8	7	6	13	8

Table 1: Some distance-regular graphs for which inequality (4) becomes equality. These graphs provide a partial positive solution of Problem 1.1, i.e., they can be characterized by the first $q + 2$ elements of their intersection arrays.

The paper is organized as follows: in Subsection 1.1 we collect several well-known diameter bounds, in Section 2 we recall notation and definitions, in Section 3 we re-prove Hiraki's inequality $c_{q+i} \geq c_q + c_i$ using 5-point counts with 3 fixed vertices, and in Section 4 we prove Theorems 1.1, 1.2 and 1.3.

Comment 1.4 After Theorems 1.1, 1.2 and 1.3, some natural questions that arise include the following:

(i) Find a tight diameter bound of a distance-regular graph for which we have $c_{q+1} > c_q$ and $a_q > c_{q+1} - c_q$.

(ii) Find a tight diameter bound of a distance-regular graph for which we have $b_q > b_{q+1}$ and $a_q \leq b_q - b_{q+1}$.

(iii) Find a tight diameter bound of a distance-regular graph for which we have $b_q > b_{q+1}$ and $a_q > b_q - b_{q+1}$.

(iv) The problem of bounding the diameter of a distance-regular graph that includes some restriction on c_q and b_{q-1} (or with some relation between these two intersection numbers) is also interesting. The simplest example is, if $c_q = 1$ and $b_{q-1} = 2$, since $c_i + b_{i-1} \geq a_1 + 2$ for any $2 \leq i \leq d-1$ (see [40], or [35, Corollary 4.7]), we have $a_1 \leq 1$ which yields $k \leq 4$. Now from [11, 14] we can conclude that $d \leq 9$.

Partial results on these four problems are given in [34].

1.1 Some known diameter bounds

Various bounds for the valency and the diameter of distance-regular graphs (with different restrictions on the structure of the graph) are known, as with different relations between intersection numbers (see [18, Chapter 7], [13, Chapter 5] and [4, 6, 7, 9, 12, 16, 20, 21, 36, 38, 39]). If we consider different restrictions on its eigenvalues, some bounds and relations between intersection numbers can be found in [3, 19, 23, 25, 26, 31], if we consider girth of a graph, some bounds and relations between intersection numbers can be found in [15, 17], and if we consider geometric distance-regular graph, some bounds and relations between intersection numbers can be found in [1, 2]. In this subsection we recall some known diameter bounds. For the case when Γ is bipartite, the diameter bounds in Theorems 1.7, 1.6 and 1.5 are well-known.

Theorem 1.5 ([41, 42]) *Let Γ denote a bipartite distance-regular graph of diameter $d \geq 3$ and valency $k \geq 3$ and girth $2q \geq 2$. Then*

$$d \leq (k-1)q + 1.$$

Theorem 1.6 ([27, Theorem 6]) *Let Γ denote a bipartite distance-regular graph of diameter d, valency $k \geq 3$ and girth $2q > 6$. If Γ is not the doubled Odd graph then*

$$d \leq (q-1)(k-1) - \left\lfloor \frac{k-3}{2} \right\rfloor .$$

Theorem 1.7 ([22, Theorem 1.2]) *Let Γ denote a bipartite distance-regular graph of diameter d, valency $k \geq 3$ and girth $2q \geq 6$. Suppose Γ is not the doubled Odd graph. Then*

$$d \leq \left\lfloor \frac{k+2}{2} \right\rfloor q + 1.$$

For the case when a graph does not need to be bipartite, the following two results are well-known.

Theorem 1.8 ([24] or [13, Theorem 5.9.8]) *Let Γ denote a distance-regular graph of diameter d, valency $k \geq 3$ and numerical girth g. Then*

$$d < g \cdot 2^{2k-3}.$$

Theorem 1.9 ([6, Corollary 1.4]) *Let Γ denote a distance-regular graph of diameter $d \geq 2$, valency $k \geq 3$, and assume that $c_{q+1} > c_q = 1$ for some fixed integer q. Then*

$$d \leq \frac{1}{2} k^{\alpha} q + 1$$

where $\alpha := \min\{x > 0 \,|\, 4^{\frac{1}{x}} - 2^{\frac{1}{x}} \leq 1\}$ (note that $1.44 < \alpha < 1.441$).

Recently, in [35], the authors proved the following.

Theorem 1.10 ([35, Theorem 10.6]) *Let Γ denote a distance-regular graph of diameter d with valency $k \geq 3$, and let q be an integer with $2 \leq q \leq d-1$. If $c_q < c_{q+1}$ and $a_q \leq c_{q+1} - c_q$ then*

$$d \leq (k + 1 - c_{q+1})q + 1.$$

2 Definitions and preliminaries

A *graph* Γ is a pair (X, R), where X is a nonempty set and R is a collection of two element subsets of X. The elements of X are called the *vertices* of Γ, and the elements of R are called the *edges* of Γ. When $xy \in R$, we say that vertices x and y are *adjacent*, or that x and y are *neighbors*. A graph is *finite* if both its vertex set and edge set are finite. By our definition for an edge it is not allowed to start and to end at the same vertex, so we can say a graph is *simple* if no two of its edges join the same pair of vertices.

Let $\Gamma = (X, R)$ be a graph. For any two vertices $x, y \in X$, a *walk* of length h from x to y is a sequence $[x_0, x_1, x_2, \ldots, x_h]$ ($x_i \in X$, $0 \leq i \leq h$) such that $x_0 = x$, $x_h = y$, and x_i is adjacent to x_{i+1} ($0 \leq i \leq h-1$). We say that Γ is *connected* if for any $x, y \in X$, there is a walk from x to y. From now on, assume that Γ is finite, simple and connected.

For any $x, y \in X$, the *distance* between x and y, denoted $\partial(x, y)$, is the length of the shortest walk from x to y. The *diameter* $d = d(\Gamma)$ is defined to be

$$d = \max\{\partial(u, v) \,|\, u, v \in X\}.$$

Let $\Gamma = (X, R)$ be a graph with diameter d. For a vertex $x \in X$ and any non-negative integer h not exceeding d, let $\Gamma_h(x)$ denote the subset of vertices in X that are at distance h from x. Let $\Gamma_{-1}(x) = \Gamma_{d+1}(x) := \emptyset$. For any two vertices x and y in X at distance h, let

$$\begin{aligned} C_h(x, y) &:= \Gamma_{h-1}(x) \cap \Gamma_1(y), \\ A_h(x, y) &:= \Gamma_h(x) \cap \Gamma_1(y), \\ B_h(x, y) &:= \Gamma_{h+1}(x) \cap \Gamma_1(y). \end{aligned}$$

We say Γ is *regular with valency* k if each vertex in Γ has exactly k neighbors. A graph Γ is called *distance-regular* if there are integers b_i, c_i ($0 \leq i \leq d$) which satisfy $c_i = |C_i(x, y)|$ and $b_i = |B_i(x, y)|$ for any two vertices x and y in X at distance i. Clearly such a graph is regular of valency $k := b_0$, $b_d = c_0 = 0$, $c_1 = 1$ and

$$a_i := |A_i(x, y)| = k - b_i - c_i \quad (0 \leq i \leq d)$$

is the number of neighbors of y in $\Gamma_i(x)$ for $x, y \in X$ ($\partial(x, y) = i$). Sometimes we will denote the intersection number a_1 by λ. From the definition of a distance-regular graph it is routine to show that Γ is distance-regular if and only if for all triples h, i, j ($0 \leq h, j, i \leq d$) and for all $x, y \in X$ with $\partial(x, y) = h$, the number

$$p^h_{ij} := |\Gamma_i(x) \cap \Gamma_j(y)|$$

is independent of choice of x and y. The numbers p^h_{ij} are called the *intersection numbers* of Γ. It is not hard to see that $a_i = p^i_{i1}$, $b_i = p^i_{i+1,1}$ and $c_i = p^i_{i-1,1}$. The array

$$i(\Gamma) = \{b_0, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d\}$$

is called the *intersection array* of Γ (also, see Figure 1). The following properties of intersection arrays are well-known.

Lemma 2.1 ([13, Proposition 4.1.6]) *Let Γ denote a distance-regular graph of diameter $d \geq 2$, valency k and intersection numbers c_i, a_i, b_i ($0 \leq i \leq d$). The following hold.*

(i) $k = b_0 > b_1 \geq b_2 \geq \cdots \geq b_d = 0$,

(ii) $1 = c_1 \leq c_2 \leq c_2 \leq \cdots \leq c_d \leq k$ *and*

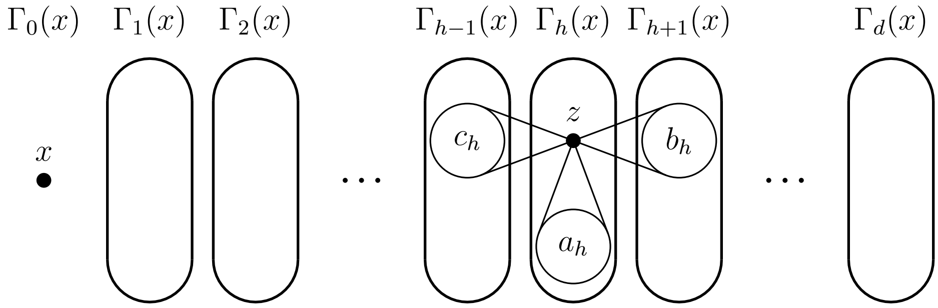


Figure 1: Intersection diagram with respect to x and illustration for intersection numbers c_h, a_h and b_h.

(iii) $c_i \le b_j \qquad$ *if* $i + j \le d$.

For vertices $x_1, x_2, \dots, x_k \in X$ and integers $i_1, i_2, \dots, i_k$ $(0 \le i_1, i_2, \dots, i_k \le d)$ we define

$$\Gamma_{i_1,i_2,\dots,i_k}(x_1, x_2, \dots, x_k) = \bigcap_{\ell=1}^{k} \Gamma_{i_\ell}(x_\ell).$$

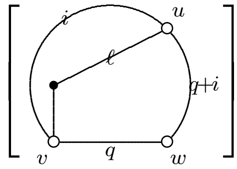
A *polygon* $p_1p_2p_3 \dots p_{m+1}$ of length m $(m \ge 3)$ (or a *circuit* of length $m \ge 3$) is a closed walk $[p_1, p_2, p_3, \dots, p_{m+1}]$ on distinct vertices, where $p_{m+1} = p_1$. A polygon of length m is called *reduced* if $m \ge 4$ and none of its proper subsets form a polygon. A shortest reduced polygon is called a *minimal polygon*. The *numerical girth* (or *girth*) of Γ, denoted by g, is the length of a minimal polygon. A graph Γ is called *bipartite* if it has no odd cycle. (If Γ is a distance-regular graph with diameter d and bipartite, then $a_1 = a_2 = \cdots = a_d = 0$.)

For more information about distance-regular graphs, definitions and notation, we refer the reader to [10, 13, 18, 37].

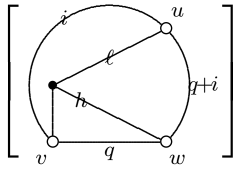
3 5-point counts with 3 fixed vertices

In [22, Lemma 3.1] Hiraki proved that if $1 \le q \le d-1$, $c_{q+1} > c_q$ and $a_q \le c_{q+1} - c_q$ then $c_q + c_i \le c_{q+1}$ for all $2 \le i \le d - q$. For completeness and clarity, and for readers who have just started to study the topic, in this section we re-prove Hiraki's inequality using the t-point counts technique from [35]. For general definition and more information about t-point counts, see [35]. A concept related to t-point counts (due to Martin in [32]) are scaffolds, providing a graph-based system for computations in Bose–Mesner algebras.

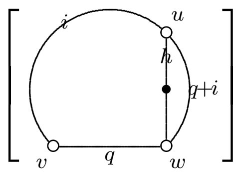
Let $\Gamma = (X, R)$ denote a distance-regular graph of diameter d, let q be an integer with $1 \le q \le d-1$ and fix i $(2 \le i \le d-q)$. Pick $u \in X$, $v \in \Gamma_i(u)$, $w \in \Gamma_{q+i,q}(u, v)$ and define



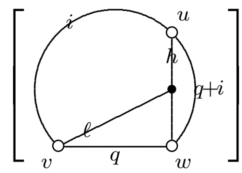
$:= |\Gamma_{\ell 1}(u, v)|$,



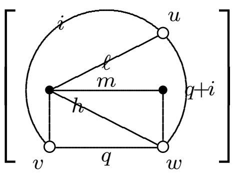
$:= |\Gamma_{\ell 1 h}(u, v, w)|$,



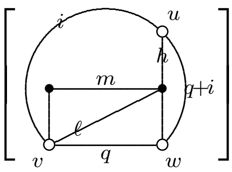
$:= |\Gamma_{h1}(u, w)|$,



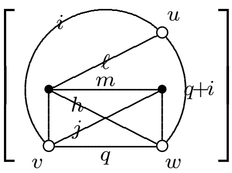
$:= |\Gamma_{h\ell 1}(u, v, w)|$,



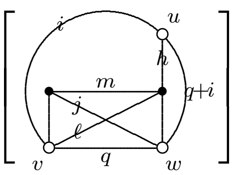
$$:= |\{(z,y) \mid z \in \Gamma_{\ell 1 h}(u,v,w),\ y \in \Gamma_{1m}(w,z)\}|$$
$$= |\{(z,y) \mid y \in \Gamma_{1}(w),\ z \in \Gamma_{\ell 1 h m}(u,v,w,y)\}|,$$



$$:= |\{(z,y) \mid y \in \Gamma_{h\ell 1}(u,v,w),\ z \in \Gamma_{1m}(v,y)\}|$$
$$= |\{(z,y) \mid z \in \Gamma_{1}(v),\ y \in \Gamma_{hm\ell 1}(u,z,v,w)\}|,$$

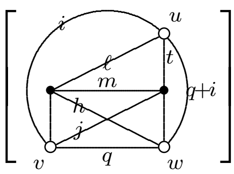


$$:= |\{(z,y) \mid z \in \Gamma_{\ell 1 h}(u,v,w),\ y \in \Gamma_{1jm}(w,v,z)\}|$$
$$= |\{(z,y) \mid y \in \Gamma_{j1}(v,w),\ z \in \Gamma_{\ell 1 h m}(u,v,w,y)\}|,$$



$$:= |\{(z,y) \mid y \in \Gamma_{h\ell 1}(u,v,w),\ z \in \Gamma_{1jm}(v,w,y)\}|$$
$$= |\{(z,y) \mid z \in \Gamma_{1j}(v,w),\ y \in \Gamma_{hm\ell 1}(u,z,v,w)\}|,$$

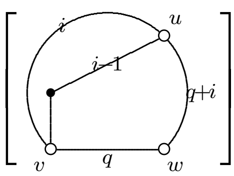
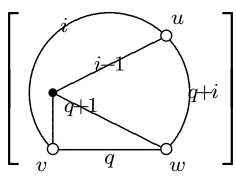
and

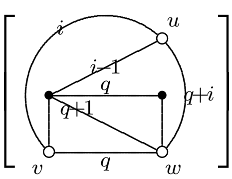
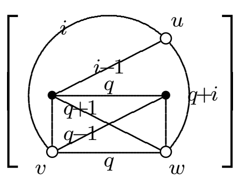


$$:= |\{(z,y) \mid z \in \Gamma_{\ell 1 h}(u,v,w),\ y \in \Gamma_{1jmt}(w,v,z,u)\}|$$
$$= |\{(z,y) \mid y \in \Gamma_{tj1}(u,v,w),\ z \in \Gamma_{\ell 1 h m}(u,v,w,y)\}|.$$

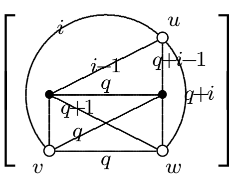
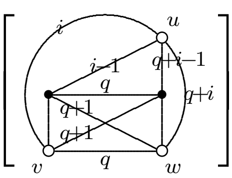
In the above notation, for the reason of simplicity, we have split the definition of a 5-point count with 3 fixed vertices into several parts.

Now, note that from

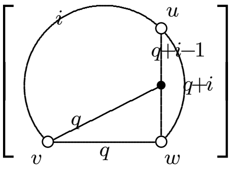
$c_i = |C_i(u,v)| =$  $=$  ,

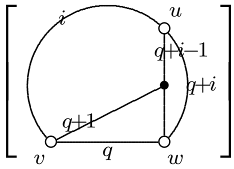
$c_i c_{q+1} =$  and $c_i c_q =$ 

we have

$c_i(c_{q+1} - c_q) =$  $+$  . (5)

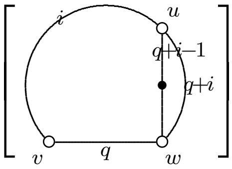
Next, if we define numbers $|Z_A|$ and $|Z_B|$ in the following way:

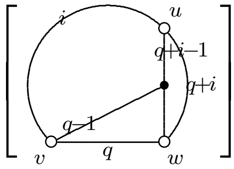
$|Z_A| :=$ 

, $|Z_B| :=$ 

,

since

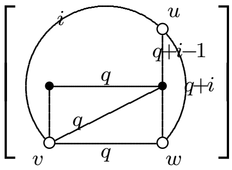
$c_{q+i} = |C_{q+i}(u,w)| =$ 

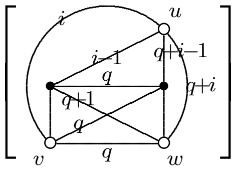
and $c_q = |C_q(v,w)| =$ 

we have

$$|Z_A| + |Z_B| = c_{q+i} - c_q \tag{6}$$

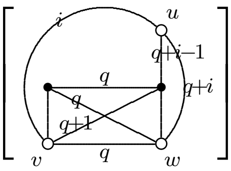
(for $c_{q+1} > c_q$ we have $|Z_A| + |Z_B| > 0$). Also, note that

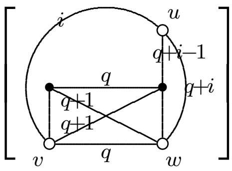
$|Z_A|a_q =$ 

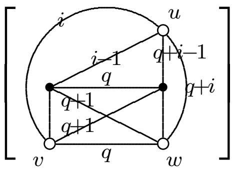
$\geq$ 

(7)

and

$|Z_B|(c_{q+1} - c_q) =$ 

$+$ 

$\geq$ 

. (8)

By (5), (7) and (8) we have

$$c_i(c_{q+1} - c_q) \leq |Z_A|a_q + |Z_B|(c_{q+1} - c_q). \tag{9}$$

If $c_{q+1} > c_q$ and $a_q \leq c_{q+1} - c_q$ then, by (6) and (9), we have

$$c_i(c_{q+1} - c_q) \leq (|Z_A| + |Z_B|)(c_{q+1} - c_q) = (c_{q+i} - c_q)(c_{q+1} - c_q),$$

and with that we have proven Lemma 3.1 (note that $c_q < c_{q+1}$ yields $c_{q+1} \geq c_q + c_1$).

Lemma 3.1 ([22, Lemma 3.1]) *Let Γ denote a distance-regular graph of diameter d, and let q be an integer with $1 \leq q \leq d-1$. Suppose $c_q < c_{q+1}$. If $a_q \leq c_{q+1} - c_q$ then*

$$c_{q+i} \geq c_i + c_q \qquad \textit{for all } i \ (1 \leq i \leq d-q).$$

Remark 3.2 Studying the proof of Lemma 3.1, if we fix i $(1 \leq i \leq d-q)$, the reader can figure out what structure the graph will have, if we assume that $a_q = c_{q+1} - c_q$ and $c_{q+i} = c_i + c_q$ (also see [22] and [35, Lemma 9.1]).

4 Bounding the diameter

In this section we prove Theorems 1.1, 1.2 and 1.3. For the proof of Theorem 1.1 we need Lemma 4.1.

Lemma 4.1 *Let Γ denote a distance-regular graph of diameter d, valency $k \geq 3$ and assume that $c_2 \geq 2$. If $a_1 \leq c_2 - 1$ then for every t $(1 \leq t \leq d)$,*

$$c_t \geq t.$$

Proof. By assumption $c_2 > c_1 = 1$, $a_1 \leq c_2 - c_1$, so from Lemma 3.1 we have $c_{i+1} \geq c_i + c_1$ for all i $(1 \leq i \leq d-1)$.

We prove the inequality by induction on t. For the induction basis, note that for $t = 1$ we have $c_t = c_1 \geq 1 = t$, while for $t = 2$ we have $c_t = c_2 \geq 2 = t$. For the induction step, assume that $c_s \geq s$ for every $2 \leq s \leq t-1$, and prove that inequality also holds for t. Indeed, $c_t = c_{(t-1)+1} \geq c_{t-1} + c_1 \geq t - 1 + 1 = t$ and the result follows. ∎

Our proof of Theorem 1.1 goes along the same lines as the proof of [22, Theorem 1.2].

Proof of Theorem 1.1. Pick c_q $(1 \leq q \leq d)$. If $d \leq k - c_q$, the proof is over. Assume that $k - c_q + 1 \leq d$ and let $t := k - c_q + 1$. We claim $d \leq t + q - 1$. To derive a contradiction, suppose $t + q \leq d$. Then $t \leq d - q$. Thus, because of Lemma 2.1, we have $b_t \geq b_{d-q} \geq c_q$, and $c_q \geq q$ by Lemma 4.1. It follows that

$$t + c_q \leq c_t + b_t \leq k = t + c_q - 1.$$

This contradiction proves the theorem. ∎

More results on bounding the diameter with different restrictions on intersection numbers c_2 and a_1 (and with different restrictions on the combinatorial structure of Γ) can be found in [5, 8, 12, 28, 29, 30, 31, 33, 43].

For the proof of Theorem 1.2 we need Lemma 4.2.

Lemma 4.2 *Let Γ denote a distance-regular graph of diameter d and let q be an integer with $2 \leq q \leq d-1$. If $c_{q+1} > c_q$ and $a_q \leq c_{q+1} - c_q$ then for every t $(1 \leq tq + 1 \leq d)$,*

$$c_{tq+1} \geq tc_q + 1.$$

Proof. Note that by Lemma 3.1 we have $c_{q+i} \geq c_i + c_q$ for all i $(1 \leq i \leq d-q)$.

We prove the inequality by induction on t. For the induction basis, note that for $t = 0$ we have $c_{tq+1} = c_1 \geq 1 = tc_q + 1$, while for $t = 1$ we have $c_{tq+1} = c_{q+1} \geq c_q + 1 = tc_q + 1$. For the induction step, assume that $c_{sq+1} \geq sc_q + 1$ for every $1 \leq s \leq t-1$, and prove that inequality also holds for t. Indeed, $c_{tq+1} = c_{(t-1)q+1+q} \geq c_{q(t-1)+1} + c_q \geq (t-1)c_q + 1 + c_q = tc_q + 1$ and the result follows. ∎

Our proof of Theorem 1.2 goes along the same lines as the proof of [22, Theorem 1.2].

Proof of Theorem 1.2. Let $t := \left\lfloor \frac{k - c_{q+1} - 1}{c_q} \right\rfloor + 1$ (note that $t \geq 0$). By the definition of t we have $t > \frac{k - c_{q+1} - 1}{c_q}$ which yields $tc_q > k - c_{q+1} - 1$ and with that $k \leq tc_q + c_{q+1}$. If $d \leq tq + 1$, the proof is over. Assume that $tq + 2 \leq d$. We claim $d \leq (t+1)q + 1$. To derive a contradiction, suppose $(t+1)q + 2 \leq d$. Then $tq + 1 \leq d - q - 1$. Thus, because of Lemma 2.1, we have $b_{tq+1} \geq b_{d-q-1} \geq c_{q+1}$, and $c_{tq+1} \geq tc_q + 1$ by Lemma 4.2. It follows that

$$tc_q + 1 + c_{q+1} \leq c_{tq+1} + b_{tq+1} \leq k \leq tc_q + c_{q+1}.$$

This contradiction proves the theorem. ∎

For the proof of Theorem 1.3 we need Lemma 4.3.

Lemma 4.3 *Let Γ denote a distance-regular graph of diameter d, let q be an integer with $2 \le q \le d-1$ and assume that $c_{q+2} > c_{q+1} > c_q$ and $a_q \le c_{q+1} - c_q$. Then for every t $(q+2 \le tq+2 \le d)$,*

$$c_{tq+2} \ge tc_q + 2.$$

Proof. Note that by Lemma 3.1 we have $c_{q+i} \ge c_i + c_q$ for all i $(1 \le i \le d-1)$.

We prove the inequality by induction on t. For the induction basis, note that for $t = 1$ we have $c_{tq+2} = c_{q+2} \ge c_{q+1} + 1 \ge c_q + 2 = tc_q + 2$, while for $t = 2$ we have $c_{tq+2} = c_{2q+2} \ge c_{q+2} + c_q \ge 2c_q + 2 = tc_q + 2$. For the induction step, assume that $c_{sq+2} \ge sc_q + 2$ for every $1 \le s \le t-1$, and prove that inequality also holds for t. Indeed, $c_{tq+2} = c_{(t-1)q+2+q} \ge c_{q(t-1)+2} + c_q \ge (t-1)c_q + 2 + c_q = tc_q + 2$ and the result follows. ∎

Our proof of Theorem 1.3 goes along the same lines as the proof of [22, Theorem 1.2].

Proof of Theorem 1.3. Let $t := \left\lfloor \frac{k-c_{q+1}-2}{c_q} \right\rfloor + 1$. By assumption $c_{q+2} \ge c_{q+1}+1$, which implies that $c_{q+1} \le k-1$, and with that $t \ge 0$. If $t = 0$ then $c_{q+1} = k-1$, which yields $c_{q+2} = k$, and with that $d = q+2$. So for the case $t = 0$, the theorem is proven.

Now consider the case when $t \ge 1$. By the definition of t we have $t > \frac{k-c_{q+1}-2}{c_q}$ which yields $tc_q > k - c_{q+1} - 2$ and with that $k \le tc_q + c_{q+1} + 1$. If $d \le tq+2$, the proof is over. Assume that $tq+3 \le d$. We claim $d \le (t+1)q+2$. To derive a contradiction, suppose $(t+1)q+3 \le d$. Then $tq+2 \le d-q-1$. Thus, because of Lemma 2.1, we have $b_{tq+2} \ge b_{d-q-1} \ge c_{q+1}$, and $c_{tq+2} \ge tc_q + 2$ by Lemma 4.2. It follows that

$$tc_q + 2 + c_{q+1} \le c_{tq+2} + b_{tq+2} \le k \le tc_q + c_{q+1} + 1.$$

This contradiction proves the theorem. ∎

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