On graphs whose smallest eigenvalue is at least $-1-\sqrt{2}$

by

Renee Woo

11 Kilduff Place
Mairangi Bay, Auckland, New Zealand

and

Arnold Neumaier

Institut für Mathematik
Universität Wien
Strudlhofgasse 4
A-1090 Wien, Austria
email: neum@cma.univie.ac.at

Dedicated to Prof. Jaap Seidel

Abstract

Main result: If the smallest eigenvalue of a graph H exceeds a fixed number larger than the smallest root (≈ -2.4812) of the polynomial x^3+2x^2-2x-2, and if every vertex of H has sufficiently large valency, then the smallest eigenvalue of H is at least $-1-\sqrt{2}$ and the structure of H is completely characterized through a new generalization of line graphs.

September 1994

Keywords: smallest eigenvalue, graph representation, Ramsey theorem, generalized line graph, Hoffman graph, quasiclique

1991 MSC Classification: 05C50, 05C75

1 Introduction

The second author recalls with pleasure the time when (in 1980) he worked with Jaap Seidel in Eindhoven on eigenvalues of graphs. The elegant line graph theorem of Cameron, Goethals, Seidel & Shult [3], characterizing generalized line graphs as the graphs with smallest eigenvalue ≥ -2 (apart from finitely many exceptions related to the exceptional root systems E_6, E_7, E_8) was still new at that time, and among other things, we were wondering what happened for smallest eigenvalues < -2. (See chapter 3 of Brouwer, Cohen & Neumaier [1] and Bussemaker & Neumaier [2] for the current state of affairs.)

A first step in this direction had been done by Hoffman in the paper [5] where he noted that for large minimum valency, the smallest eigenvalue $-1-\sqrt{2}$ was a natural threshold where new structure beyond the generalized line graph structure would be expected to appear.

In this paper we show that this is indeed the case, and we determine completely the structure of graphs with smallest eigenvalue $\geq -1-\sqrt{2}$ in case that the minimum valency is sufficiently large. Since we use brute force (Ramsey type arguments) to keep the proof short, the bound on the minimum valency implicit in our treatment is probably much larger than necessary. A number of open problems remain.

Hoffman's analysis was based on the properties of simple subgraphs involving large cliques, which he depicted by fat vertices. We shall idealize this situation as follows.

A *Hoffman graph* is a graph H with a distinguished coclique Δ. Vertices in Δ are called *fat*, other vertices *slim*. We draw Hoffman graphs by depicting vertices as large (small) black dots if they are fat (slim).

The *fatness* $\varphi(H)$ of a Hoffman graph H is the minimal valency of slim vertices without a fat neighbor, if such vertices exist. If not, we call H *fat* and define $\varphi(H) = \infty$.

(Indeed, we essentially treat fat vertices as infinite cliques whose vertices are adjacent to the neighbors of the fat vertex; ordinary *slim* graphs have no fat vertices, and their fatness is just the minimum valency.)

In the following, all graphs considered are Hoffman graphs (even when this is not mentioned explicitly), and all subgraphs considered are induced subgraphs. A graph is *embedded* in a graph H if it is isomorphic to a subgraph of H. Notation is as in [1].

2 A new generalization of line graphs

In this section we show that line graphs and traditional generalized line graphs are special cases of a new generalization of the line graph concept.

Let $\mathcal{H}$ be a family of Hoffman graphs. An *$\mathcal{H}$-line graph* is a subgraph of a graph H with the following property:

(L) There is a family of (induced) subgraphs H^l ($l = 1, \dots, s$) such that

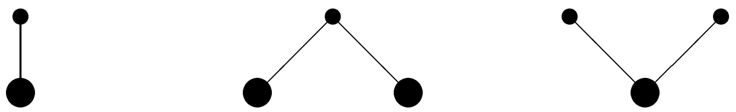
(i) each slim vertex of H is in exactly one H^l,

(ii) two slim vertices in different H^l have at most one common fat neighbor, and they have one iff they are adjacent,

(iii) each H^l is isomorphic to a graph in $\mathcal{H}$,

(iv) Each vertex of H is one of the H^l.

An *$\mathcal{H}$-quasiclique* is a graph which is a subgraph of an $\mathcal{H}$-line graph which has a fat vertex adjacent to all other vertices.



H_1 H_2 H_3

2.1. Examples. If $\mathcal{H}$ consists of the graph H_2 only, or if $\mathcal{H} = \{H_1, H_2\}$ (which makes no difference since $H_1 \subset H_2$) then $\mathcal{H}$-line graphs without fat vertices are just line graphs, and $\mathcal{H}$-quasicliques are just the cliques (possibly including a fat vertex). If $\mathcal{H} = \{H_1, H_2, H_3\}$ then $\mathcal{H}$-line graphs without fat vertices are just the generalized line graphs, and $\mathcal{H}$-quasicliques are the complements of a disjoint union of isolated vertices and edges (where a fat vertex must be isolated in the complement), i.e., the subgraphs of so-called cocktail party graphs.

We say a graph H is *blocked* if a family of subgraphs called *blocks* are distinguished in such a way that every vertex is in some block. For a blocked graph, we call edges which are not in a block, and nonedges which are in a block *special.*

For example, a fat graph H can be blocked by taking as blocks all sets consisting of a fat vertex and all its neighbors. We call the blocks defined in this way *fat* blocks. The special pairs are edges without a fat neighbor and nonedges having a fat neighbor. The graph on the slim vertices whose edges are the special pairs is called the *special graph* of H.

We call two slim vertices of a blocked graph H *related* if they form a special pair or if they are in more than one block. A *piece* of a blocked graph is the Hoffman graph induced on an equivalence class of slim vertices (with respect to the transitive closure of the relation 'related') together with the fat vertices adjacent to any of them.

2.2. Theorem. *Let $\mathcal{H}$ be a family of fat graphs. A graph is an $\mathcal{H}$-line graph iff it can be blocked in such a way that all blocks are $\mathcal{H}$-quasicliques and all pieces are isomorphic to some subgraph of a graph in $\mathcal{H}$.*

Proof. If $\mathcal{H}$ is a family of fat graphs then any $\mathcal{H}$-line graph can be blocked (possibly in several ways) by taking as blocks the restrictions of the fat blocks of the graph H in the definition. With this blocking, pieces are subgraphs of the H^l in condition (L). Conversely, taking the H^l as the extensions of the pieces of H to graphs in $\mathcal{H}$ (using as extra vertices different vertices in each piece), condition (L) is satisfied. □

2.3. Remark. For a fat $\mathcal{H}$-line graph blocked by the fat blocks, the pieces are not necessarily isomorphic to some subgraph of a Hoffman graph in $\mathcal{H}$. (Consider a triangle and three fat vertices adjacent to one slim vertex each.)

3 Representations of Hoffman graphs

In this section we relate certain representations of Hoffman graphs in a Euclidean space to the smallest eigenvalue of a related matrix, and show that the concept of $\mathcal{H}$-line graphs is naturally adapted to the study of such representations.

A mapping which assigns to each vertex γ of a Hoffman graph H a vector $\bar{\gamma} \in \mathbb{R}^n$ such that

$$(\bar{\gamma}, \bar{\delta}) = \begin{cases} m & \text{if } \bar{\gamma} = \bar{\delta} \text{ is a slim vertex,} \\ 1 & \text{if } \bar{\gamma} = \bar{\delta} \text{ is a fat vertex,} \\ 1 & \text{if } \bar{\gamma} \text{ and } \bar{\delta} \text{ are adjacent,} \\ 0 & \text{otherwise,} \end{cases}$$

is called a *representation of norm m*.

3.1. Example. Consider the Hoffman graph formed from a line graph $L(P)$ of a graph P by taking $L(P)$ as the subgraph of slim vertices, and adding the vertices (but not the edges) of P as fat vertices, with each fat vertex adjacent to all slim vertices (i.e., edges of P) which emanate from the fat vertex (as a vertex in P). If we pick an orthonormal basis of unit vectors e^γ indexed by the fat vertices, we get a representation of norm 2 by representing fat vertices γ by e^γ and slim vertices given as edge $\overline{\gamma\delta}$ in P by $e^\gamma + e^\delta$.

3.2. Theorem. *A Hoffman graph H has a representation of norm m iff the smallest eigenvalue λ_{min} of the matrix*

$$B_H := A - CC^T \tag{1}$$

satisfies $\lambda_{min} \geq -m$, where $\begin{pmatrix} A & C \\ C^T & 0 \end{pmatrix}$ is the adjacency matrix of the graph, in a labeling in which the fat vertices come last.

We call λ_{min} the *smallest eigenvalue* of H.

Proof. (i) In a representation of norm m, the vectors

$$x^\gamma := \bar{\gamma} - \sum_{\delta \text{ fat}} c_{\gamma\delta}\bar{\delta} \quad (\gamma \text{ slim})$$

have Gram matrix $A - CC^T + mI$. Hence this matrix is positive semidefinite, and all its eigenvalues are $\geq -m$. Thus $\lambda_{min} \geq -m$.

(ii) Conversely, if $\lambda_{min} \geq -m$ then $A - CC^T + mI$ is positive semidefinite, hence has a Cholesky factorization LL^T with square L. Now it is easy to check that (with standard unit vectors e^γ)

$$\bar{\gamma} := \begin{cases} e^\gamma & \text{if } \gamma \text{ is a fat vertex,} \\ \sum\limits_{\delta \text{ slim}} L_{\gamma\delta}e^\delta + \sum\limits_{\delta \text{ fat}} C_{\gamma\delta}e^\delta & \text{otherwise,} \end{cases}$$

defines a representation of norm m. $\square$

3.3. Corollary. *The smallest eigenvalue of a subgraph of H is not less than the smallest eigenvalue of H itself.*

Proof. The corresponding representation of H of norm m (where $-m$ is the smallest eigenvalue of H) restricts to such a representation of H'. $\square$

3.4. Lemma. *The matrix B_H defined in* (1) *has diagonal entries*

$$B_{\gamma\gamma} = - \text{ number of fat neighbors of } \gamma,$$

and off-diagonal entries

$$B_{\gamma\delta} = A_{\gamma\delta} - \text{ number of common fat neighbors of } \gamma \text{ and } \delta. \quad \square$$

The next result can be interpreted intuitively as "pulling finite cliques out of the infinite cliques represented by fat vertices".

3.5. Proposition. *Let H' be the graph obtained from a Hoffman graph H by adjoining to one or more nonisolated fat vertices slim cliques in such a way that the vertices of a new clique are adjacent precisely to each other and to one corresponding fat vertex and its neighbors. Then the smallest eigenvalue of H' equals the smallest eigenvalue of H.*

Proof. It is easy to see that $B_{H'} = \begin{pmatrix} B_H & 0 \\ 0 & -I \end{pmatrix}$. Thus the smallest eigenvalue of H' equals that of H, unless the smallest eigenvalue of H is larger than -1. But in the latter case, Corollary 3.3 shows that H contains no paths of length two (involving slim and/or fat vertices). Thus H is a disjoint union of cliques, and if there is a nonisolated fat vertex, the smallest eigenvalue equals -1, contradiction. $\square$

3.6. Proposition. *Let H be a Hoffman graph such that each slim vertex has a unique fat neighbor. Then the smallest eigenvalue of H is uniquely determined by the special graph of H.*

Proof. Since every slim vertex has exactly one fat neighbor, we conclude from Lemma 3.4 that

$$B_{\gamma\delta} = \begin{cases} +1 & \text{if } \{\gamma,\delta\} \text{ is a special edge,} \\ -1 & \text{if } \gamma=\delta \text{ or } \{\gamma,\delta\} \text{ is a special nonedge,} \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Clearly, this matrix, and hence the smallest eigenvalue of H, is determined by the special graph of H. □

3.7. Theorem. *Let $\mathcal{H}$ be a family of graphs with a representation of norm m. Then every $\mathcal{H}$-line graph has a representation of norm m.*

Proof. Construct representations of norm m of the H^l from condition (L) in mutually orthogonal Euclidean spaces E_l, and identify in the direct sum of the E_l the unit vectors representing identical fat vertices (by factoring out their differences). This yields a representation of norm m of the graph H from condition (L) and by restriction a representation of norm m for arbitrary $\mathcal{H}$-line graphs. □

As convenient abbreviations of certain numerical values for smallest eigenvalues, we denote in the following by α_j ($j = 0,\dots,4$) the smallest roots of the polynomials $p_j(x)$, where

$$\begin{aligned}
&p_0(x) = x^2+2x-1, && \alpha_0 = -1-\sqrt{2} \approx -2.4142,\\
&p_1(x) = x^3+2x^2-2x-2, && \alpha_1 \approx -2.4812,\\
&p_2(x) = x^2+x-4, && \alpha_2 = \tfrac{1}{2}(-1-\sqrt{17}) \approx -2.5616,\\
&p_3(x) = x^2+3x+1, && \alpha_3 = \tfrac{1}{2}(-3-\sqrt{13}) \approx -2.6180,\\
&p_4(x) = x^2+2x-2, && \alpha_4 = -1-\sqrt{3} \approx -2.7321.
\end{aligned}$$

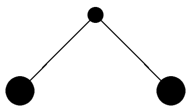
Note that $\alpha_i < \alpha_0 = -1-\sqrt{2}$ for $i > 0$.

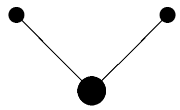
4 A forbidden subgraph theorem

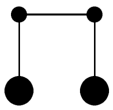
In this section we give a complete characterization of *fat* Hoffman graphs with smallest eigenvalue of H at least $-1-\sqrt{2}$.

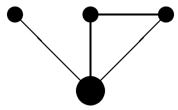
We shall need explicitly the Hoffman graphs from the following list. With each graph we give a graph name H_i and the smallest eigenvalue, as defined in Section 3.

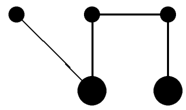

H_1, $\lambda_{min} = -1$

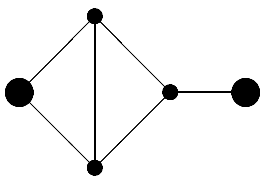

H_2, $\lambda_{min} = -2$

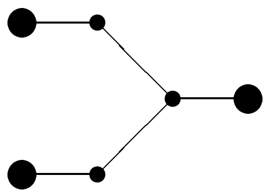

H_3, $\lambda_{min} = -2$

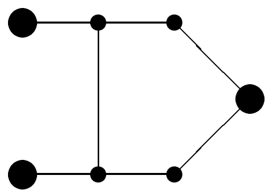

H_4, $\lambda_{min} = -2$

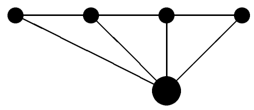

H_5, $\lambda_{min} = \alpha_0$

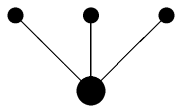

H_6, $\lambda_{min} = \alpha_0$

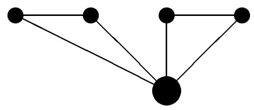

H_7, $\lambda_{min} = \alpha_0$

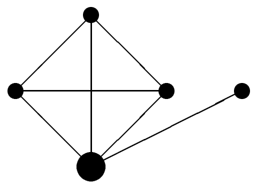

H_8, $\lambda_{min} = \alpha_0$

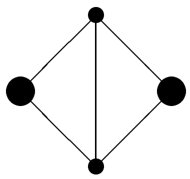

H_9, $\lambda_{min} = \alpha_0$

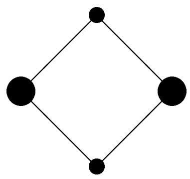

H_{10}, $\lambda_{min} = \alpha_3$

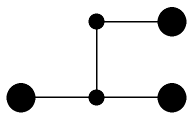

H_{11}, $\lambda_{min} = -3$

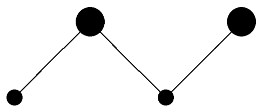

H_{12}, $\lambda_{min} = -3$

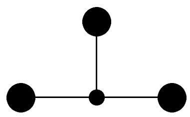

H_{13}, $\lambda_{min} = \alpha_4$


H_{14}, $\lambda_{min} = -3$

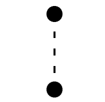

H_{15}, $\lambda_{min} = -4$


H_{16}, $\lambda_{min} = \alpha_3$

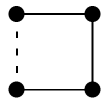

H_{17}, $\lambda_{min} = \alpha_3$


H_{18}, $\lambda_{min} = -3$

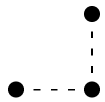
We shall also need a list of special graphs S_i (drawn by depicting special edges as full lines and special nonedges as dashed lines), together with the smallest eigenvalues of the corresponding Hoffman graphs satisfying the hypothesis of Proposition 3.6. (It is easy to see that a special graphs gives rise to only few such Hoffman graphs.)



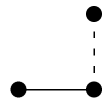
S_1, $\lambda_{min} = -2$



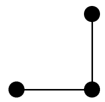
S_2, $\lambda_{min} = \alpha_0$



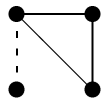
S_3, $\lambda_{min} = \alpha_0$



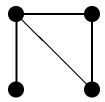
S_4, $\lambda_{min} = \alpha_0$



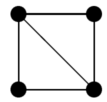
S_5, $\lambda_{min} = \alpha_0$



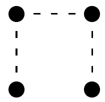
S_6, $\lambda_{min} = \alpha_1$



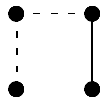
S_7, $\lambda_{min} = \alpha_1$



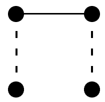
S_8, $\lambda_{min} = \alpha_2$



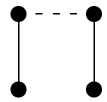
S_9, $\lambda_{min} = \alpha_3$



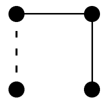
S_{10}, $\lambda_{min} = \alpha_3$



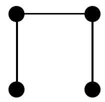
S_{11}, $\lambda_{min} = \alpha_3$



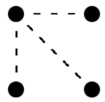
S_{12}, $\lambda_{min} = \alpha_3$



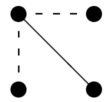
S_{13}, $\lambda_{min} = \alpha_3$



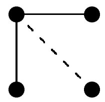
S_{14}, $\lambda_{min} = \alpha_3$



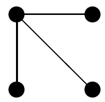
S_{15}, $\lambda_{min} = \alpha_4$



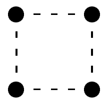
S_{16}, $\lambda_{min} = \alpha_4$



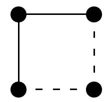
S_{17}, $\lambda_{min} = \alpha_4$



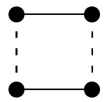
S_{18}, $\lambda_{min} = \alpha_4$



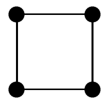
S_{19}, $\lambda_{min} = -3$



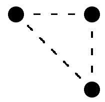
S_{20}, $\lambda_{min} = -3$



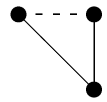
S_{21}, $\lambda_{min} = -3$



S_{22}, $\lambda_{min} = -3$



S_{23}, $\lambda_{min} = -3$



S_{24}, $\lambda_{min} = -3$

Let $\mathcal{H} = \{H_2, H_5, H_7, H_9\}$ and $\mathcal{H}' = \{H_i \mid i = 1, \ldots, 9\}$. Because of the inclusions $H_1, H_3 \subset H_5$ and $H_4, H_6, H_8 \subset H_9$, $\mathcal{H}$-line graphs and $\mathcal{H}'$-line graphs are the same.

4.1. Proposition. *A graph is an $\mathcal{H}$-quasiclique iff its complement is a disjoint union of paths of length at most two, and a fat vertex (if any exists) is isolated in the complement.*

Proof. The neighborhoods of the fat vertices in graphs from $\mathcal{H}$ are precisely the complements of paths of length at most two. $\square$

We now define

$$\mathcal{M} := \mathcal{M}_1 \cup \mathcal{M}_2, \tag{3}$$

where $\mathcal{M}_1 = \{H_i \mid i = 10, \ldots, 18\}$, and $\mathcal{M}_2$ is the set of graphs such that each slim vertex has exactly one fat neighbor and the special graph is one of the graphs S_i $(i = 6, \ldots, 24)$.

4.2. Theorem. *For a fat graph H, the following conditions are equivalent:*

(i) H has a representation of norm $-1-\sqrt{2}$.

(ii) The smallest eigenvalue of H is at least $-1-\sqrt{2}$.

(iii) H contains no subgraph isomorphic to some graph in $\mathcal{M}$.

(iv) H is an $\mathcal{H}$-line graph.

Proof. Since all graphs in $\mathcal{H}$ have smallest eigenvalue $\geq -1-\sqrt{2}$, they have a representation of norm $-1-\sqrt{2}$ by Theorem 3.2. Theorem 3.7 then shows that (iv) implies (i). Directly by Theorem 3.2, (i) implies (ii). Since all graphs in $\mathcal{M}$ have smallest eigenvalue $< -1-\sqrt{2}$, (ii) implies (iii) by Corollary 3.3.

The difficult part is to show that (iii) implies (iv). Assume that H is a fat graph not containing a subgraph isomorphic to some graph in $\mathcal{M}$. We consider the blocking defined by the fat blocks, and note that by assumption, every slim vertex has a fat neighbor. To show that H is an $\mathcal{H}$-line graph we proceed in a number of steps.

Step 1. *The complement of a block is a disjoint union of paths of length at most two. In particular, blocks are $\mathcal{H}$-quasicliques.* Indeed, paths of length 3, triangles, quadrangles, and 3-claws are forbidden in the complement since they give rise to subgraphs H_{10}, H_{11}, H_{12} and H_{13}, respectively. This leaves as connected components in the complement only paths of length at most 2. Each such component, together with the fat vertex in the block, defines a subgraph of H_5; it is easy to see that with these subgraphs, the block becomes an $\mathcal{H}$-line graph which is an $\mathcal{H}$-quasiclique.

Step 2. *Two fat vertices have at most one common neighbor. In particular, two slim vertices are related iff they form a special pair.* For two common neighbors are forbidden since they would produce a subgraph H_{14} or H_{15}.

Step 3. *Any subgraph H_2 is a piece.* For if not, the piece containing H_2 has a further related pair containing the slim vertex γ of H_2, which would lead to subgraphs H_{16} or H_{17}, or γ has another fat neighbor, which would lead to H_{18}.

STEP 4. We now consider pieces with more than one slim vertex. By the previous steps, each slim vertex has exactly one fat neighbor. We look at the special graph of such a piece. By construction of $\mathcal{M}$, *the special graph has none of the special graphs* S_i $(i = 6, \ldots, 24)$ *as a subgraph.*

STEP 5. *If the piece contains a special nonedge,* a simple extension process pruned with the list of S_i shows that the only possibilities are the special graphs S_i $(i = 1, \ldots, 4)$. Thus, *the pieces must be isomorphic to one of the graphs* H_3, H_9, H_5, *or* H_6.

STEP 6. *If the piece contains no special nonedge,* its special edges form a connected graph S. If S is not a clique it contains a path of length 2. Extension of this path would lead to one of the S_i $(i = 6, \ldots, 24)$, hence the path of length 2 is already all of S, i.e., $(S = S_5)$. This special graph is realized by two fat graphs, H_7 and H_8. And if this S is a clique we can add to H a fat vertex adjacent to all points of this clique, and in this extended graph, the piece splits into pieces of the form H_2. Thus, in either case, *the pieces (of the extended graph) must be isomorphic to one of the graphs* H_7, H_8, *or* H_2.

STEP 7. Thus, after possible extension by some fat vertices, H has only pieces in $\mathcal{H}$. By Theorem 2.2, it is an $\mathcal{H}$-line graph. $\square$

5 Slim graphs of large valency

In this section, we use Ramsey type arguments to extend the results of the previous section to nonfat graphs of sufficiently large fatness, and in particular to ordinary (slim) graphs of sufficiently large minimum valency.

5.1. Theorem. *Let* λ *be a number larger than* $\alpha_1 \approx -2.4812$, *the smallest root of the polynomial* $x^3 + 2x^2 - 2x - 2$. *Then there exists a number* k_λ *such that every Hoffman graph* H *with fatness* $\varphi(H) \geq k_\lambda$ *whose smallest eigenvalue* λ_{min} *is at least* λ *is an* $\{H_2, H_5, H_7, H_9\}$*-line graph. In particular,* $\lambda_{min} \geq -1 - \sqrt{2}$.

To prove this theorem, we construct a larger fat graph to which we can apply the results of the previous section. The delicate part consists in showing that this extended graph cannot contain subgraphs from $\mathcal{M}$.

This is enabled by two key observations due to HOFFMAN [5]. Hoffman's first observation is a Ramsey type argument which shows that each vertex is in at least one large clique. Recall that the Ramsey number $R(a, b)$ is the smallest number v such that every graph with at least v vertices contains either an a-coclique or a b-clique. This number is always finite (see, e.g., GRAHAM et al. [4]).

5.2. Proposition. *Let* H *be a graph with smallest eigenvalue* $> \alpha_1$. *Then* H *contains no* 7*-claw, and every vertex of valency* $\geq R(7, c)$ *is contained in a* $(c+1)$*-clique.*

Proof. An s-claw has smallest eigenvalue $-\sqrt{s}$; since $\alpha_0 > -\sqrt{7}$, H cannot contain a 7-claw. The neighborhood of a vertex γ of valency $\geq R(7,c)$ therefore cannot contain a 7-coclique, and by definition of Ramsey numbers, the neighborhood contains a c-clique C. Now $C \cup \{\gamma\}$ is the required $(c+1)$-clique. □

This observation can be used to find approximate fat Hoffman subgraphs. The second observation of Hoffman is an approximation result for fat vertices by large finite cliques.

5.3. Proposition. *Let H be a Hoffman graph, and denote by H^n the clique extension obtained by replacing all fat vertices by slim n-cliques, all of whose vertices are adjacent to the neighbors of the corresponding fat vertices. Then*

$$\lim_{n\to\infty} \lambda_{min}(H^n) = \lambda_{min}(H).$$

Proof. See HOFFMAN [5]. □

5.4. Corollary. *If $\mathcal{M}$ is a finite set of Hoffman graphs then there is an integer n such that*

$$\lambda_{min}(H^n) < \lambda \text{ for all } H \in \mathcal{M}. \quad \square \tag{4}$$

Now let $n = n_\lambda$ be an integer such that (4) holds for the set $\mathcal{M}$ of Hoffman graphs defined in (3). ($\mathcal{M}$ is finite since every graph in $\mathcal{M}$ has at most 4 slim vertices and at most 4 fat vertices.) We shall show that Theorem 5.1 holds with

$$k_\lambda := R(7, c_\lambda),$$

where

$$c_\lambda = (3n_\lambda + 16)(n_\lambda + 1). \tag{5}$$

Proof of Theorem 5.1. Let H be a Hoffman graph with fatness $\varphi(H) \geq k_\lambda$ whose smallest eigenvalue λ_{min} is at least λ. By pulling out cliques (Proposition 3.5) for the vertices with a fat neighbor, we may assume without loss of generality that H has minimum valency at least k_λ. By Proposition 5.2, every slim vertex is in a $(c_\lambda+1)$-clique, and hence in a slim c_λ-clique (since the fat vertices form a coclique, a clique contains at most one fat vertex). Since these cliques can be extended to maximal quasicliques (in the following short for $\{H_2, H_5, H_7, H_9\}$-quasicliques), H becomes blocked by calling blocks the maximal quasicliques which contain a slim c_λ-clique.

Now let B be a block containing a slim c_λ-clique C, and let γ be a vertex not in B. We claim that *γ is adjacent to at most $m := n_\lambda + 5$ vertices of C.*

Indeed, since a block is a maximal quasiclique, $B \cup \{\gamma\}$ is not a quasiclique, hence contains the complement of a path of length 3, a triangle, a quadrangle or a 3-claw D. Since B is a quasiclique, D is not a subgraph of B, so that γ is one of the vertices of D. Now suppose that the neighbors of γ in C form an s-clique N. A vertex in N has no nonneighbor in N since N is a clique, and a vertex in $B \setminus N$ has at most two nonneighbors in N since B is a quasiclique. After removing the vertices of $D \setminus \{\gamma\}$ and their nonneighbors from N we are therefore left with a clique of size $r \geq s - 6$. The subgraph induced on the union of D and this clique is a graph H_i^r with $i \in \{10, \ldots, 13\}$, and by Corollary 5.4, we must have $r < n_\lambda$. Thus $s \leq n_\lambda + 5$.

If a block contains a fat vertex, this vertex must be adjacent to all other vertices of the block (since the block is a quasiclique). For every block not containing a fat vertex, we adjoin a new fat vertex adjacent precisely to all vertices in that block. If we can show that the resulting Hoffman graph H' contains no subgraph isomorphic to a graph in $\mathcal{M}$, Theorem 4.2 implies that H' (and hence its subgraph H) is an $\mathcal{H}$-line graph. Theorems 3.7 and 3.2 then imply that the smallest eigenvalue of H is at least $-1 - \sqrt{2}$.

Thus assume by way of contradiction that H' contains a subgraph isomorphic to a graph $H_0 \in \mathcal{M}$. H_0 has at most 4 slim vertices and at most 4 fat vertices, and each fat vertex has at least one slim neighbor. By the preceding results, every fat vertex γ of H_0 is contained in a maximal quasiclique of H' containing a slim c_λ-clique C_γ^0.

We now remove from C_γ^0 the neighbors of γ in H_0 and the neighbors of the nonneighbors of γ in H_0, a total of at most $3m + 1$ vertices. Pick n_λ of these vertices to form the clique C_γ. Now we repeat the procedure with another fat vertex γ' of H_0 (if one exists) and a corresponding slim c_λ-clique $C_{\gamma'}^0$, but also remove all neighbors of vertices in C_γ, a total of at most $3m + 1 + mn_\lambda$ vertices, before picking an n_λ-clique $C_{\gamma'}$. We can do the analogous procedure for a third and fourth fat vertex γ'' and γ''' to construct n_λ-cliques $C_{\gamma''}$ and $C_{\gamma'''}$ such that none of its vertices has a neighbor in an earlier $C_{\gamma^{(l)}}$. (Since $c_\lambda - (3m+1) = 4mn_\lambda$, there are always enough vertices left to choose.)

Now the graph induced on the slim vertices of H_0 and the n_λ-cliques C_γ (one for each fat vertex γ of H_0) is contained in H. But it is a clique extension $H_0^{n_\lambda}$, which is forbidden as a subgraph of H by Proposition 5.4, contradiction.

Therefore, H' contains no subgraph isomorphic to a graph $H_0 \in \mathcal{M}$, and the theorem is proved. □

As a special case we find the following neat result.

5.5. Theorem. *Every $\{H_2, H_5, H_7, H_9\}$-line graph has smallest eigenvalue $\geq -1 - \sqrt{2}$. Conversely, every graph with smallest eigenvalue $\geq -1 - \sqrt{2}$ and sufficiently large minimum valency is an $\{H_2, H_5, H_7, H_9\}$-line graph.* □

5.6. Remark. In the same way, but with much less detailed work, we can get a new proof of Hoffman's result in [5] that *for $\lambda > -1 - \sqrt{2}$, there exists a number h_λ such that every graph H with minimum valency at least h_λ whose* *smallest eigenvalue λ_{min} is at least λ is a generalized line graph. In particular, $\lambda_{min} \geq -2$.*

6 Open problems

In this final section we mention some open problems suggested by our approach.

1. It would be interesting to determine the smallest value of the minimum valency for which Theorem 5.5 still holds. (Our Ramsey type arguments can certainly be improved a lot by making use of height techniques developed in NEUMAIER [7].)

2. Regular generalized line graphs, and regular graphs with smallest eigenvalue ≥ -2 have been completely determined; see [2] for a summary of the results. When we replace -2 by $-1-\sqrt{2}$, a natural conjecture is:

Conjecture. (i) A regular $\{H_2, H_5, H_7, H_9\}$-line graph is a line graph or a cocktail party graph.
(ii) A regular graph with smallest eigenvalue $\geq -1-\sqrt{2}$ and sufficiently large valency is a line graph or a cocktail party graph.
Clearly, (ii) is a consequence of (i) and the present results; but it may hold even when (i) turns out to be false (e.g., if there are only finitely many counterexamples to (i)).

3. It might be possible to find a complete list of minimal forbidden subgraphs for slim (or arbitrary) $\{H_2, H_5, H_7, H_9\}$-line graphs. This would, however, probably be a much larger list than our list $\mathcal{M}$ in the fat case.

4. It seems feasible to classify all fat graphs with smallest eigenvalue ≥ -3. Indeed, root lattice techniques apply since it is not difficult to show that the image of such a graph H under a representation of norm 3 generates a root lattice (see [1] for definitions and properties of root lattices). Since all root lattices are known, the structure of fat graphs with smallest eigenvalue ≥ -3 appears limited and their determination tractable.

5. For graphs in which every edge has a large number of common neighbors, Ramsey theory shows that every edge is in a large clique. This is a much more restrictive condition which may be idealized by looking at Hoffman graphs where any two adjacent slim vertices have a common fat neighbor. It might be possible to classify the corresponding graphs even for larger negative eigenvalues. The only strongly regular graphs surviving are line graphs and complete multipartite

graphs; see [6]. For distance-regular (or only edge-regular) graphs, it is likely that the same holds, but nothing is known.

Acknowledgment. Part of this work was done while the authors were at the Institut für Angewandte Mathematik, Universität Freiburg (Germany).

References

1. A.E. Brouwer, A.M. Cohen and A. Neumaier, Distance-Regular Graphs, Springer, Berlin 1989.

2. F.C. Bussemaker and A. Neumaier, Exceptional graphs with smallest eigenvalue -2 and related topics, Math. Comput. 59 (1992), 583-608.

3. P. Cameron, J.-M. Goethals, J.J. Seidel and E.E. Shult, Line graphs, root systems and elliptic geometry, J. Algebra 43 (1976), 305-327.

4. R.L. Graham, B.L. Rothschild and J.H. Spencer, Ramsey Theory, 2nd ed., Wiley, New York 1990.

5. A. Hoffman, On graphs whose least eigenvalue exceeds $-1-\sqrt{2}$, Linear Algebra Appl. 16 (1977), 153-165.

6. A. Neumaier, Strongly regular graphs with smallest eigenvalue $-m$, Arch. Math. 33 (1979), 392-400.

7. A. Neumaier, Graph representations, two-distance sets and equiangular lines, Linear Algebra Appl. 114/115 (1989), 141-156.