

A UNIFIED VIEW OF INEQUALITIES FOR DISTANCE-REGULAR GRAPHS, PART I

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Abstract

In this paper, we introduce the language of a configuration and of t -point counts for distance-regular graphs (DRGs). Every t -point count can be written as a sum of $(t - 1)$ -point counts. This leads to a system of linear equations and inequalities for the t -point counts in terms of the intersection numbers, i.e., a linear constraint satisfaction problem (CSP). This language is a very useful tool for a better understanding of the combinatorial structure of distance-regular graphs. Among others we prove a new diameter bound for DRGs that is tight for the Biggs–Smith graph. We also obtain various old and new inequalities for the parameters of DRGs, including the diameter bounds by Terwilliger.

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1 Configurations and t -point counts

In this paper we introduce and apply the language of configurations and t -point counts, a pictorial way of representing the number of ordered t -tuples of vertices (configurations of vertices of prescribed type, e.g., 4-cycles, induced quadrangles, etc.). Every t -point count can be written as a sum of $(t - 1)$ -point counts. Using this we obtain a system of linear equations and inequalities, i.e., a linear constraint satisfaction problem (CSP) whose variables are the t -point counts. All our graphs are undirected, without loops.

Definition 1.1 Let Γ denote a graph with vertex set X and diameter d . A **configuration** is a finite ordered list $Z = (z_1, z_2, \dots, z_t)$ (i.e., a t -tuple) of vertices of Γ . A t -point **type** is an undirected graph Δ with t nodes $1, 2, \dots, t$ whose edges are labelled by subsets of $\{0, 1, \dots, d\}$; we fix one drawing of Δ and a fixed labelling of the vertices of Δ as first, second, etc. A configuration Z is of type Δ if

$$\text{for every edge } ij \text{ of } \Delta \text{ we have } \partial_\Gamma(z_i, z_j) \in \Delta_{ij} \quad (1 \leq i, j \leq t),$$

where Δ_{ij} denotes the label of the edge ij . In drawings, missing labels are taken as having the value 1, edges labeled by i represent the set $\{i\}$, and edges labelled by constraints $\geq i$ (resp. $\neq i$, and $\leq i$) represent the set $\{i, i+1, \dots, d\}$ (resp. $\{0, \dots, i-1, i+1, \dots, d\}$, and $\{0, 1, \dots, i-1, i\}$). A missing edge represents the set $\{0, 1, \dots, d\}$. Let $V\Delta$ and $E\Delta$ denote the vertex set and edge set, respectively, of a graph Δ . We define the set $\{\Delta\}$ as the set of all configurations of type Δ , and we define the integer $[\Delta]$ as the number of configurations of type Δ , that is,

$$\{\Delta\} = \left\{ Z \in X^{V\Delta} \mid \text{if } uv \in E\Delta \text{ then } \partial_\Gamma(Z(u), Z(v)) \in \Delta_{uv} \right\},$$

and

$$[\Delta] = |\{\Delta\}|.$$

A t -point type is called **complete** if any two nodes are joined by a labelled edge. A **t -point count** is a number $[\Delta]$, where Δ is a complete t -point type. In drawing, vertices of diagrams Δ are shown with black circles. Later, in Subsection 1.1 we will define t -point counts with fixed vertices – these diagrams will be marked by white circles with a label.

For the moment write EL_t for the set of 2-element subsets of a t -element set L_t of vertices. Then a t -point count $[\Delta]$ is the number of t -tuples (z_1, z_2, \dots, z_t) of elements of X for which $\partial(z_i, z_j) \in \Delta_{ij}$ for all edges ij of EL_t . For example, consider the graph from Figure 1(a). Directly from the definition it follows that

$$\begin{aligned} \left[\begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} \right] &= 5, & \left[\begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} \right] &= 10, & \left[\begin{array}{c} \overset{2}{\bullet \text{---} \bullet} \\ \diagdown \quad \diagup \\ \bullet \end{array} \right] &= 10, & \left[\begin{array}{c} \overset{3}{\bullet \text{---} \bullet} \\ \diagdown \quad \diagup \\ \bullet \end{array} \right] &= 0, \\ \left[\begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} \right] &= 6, & \left[\begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} \right] &= 10, & \left[\begin{array}{c} \overset{2}{\bullet \text{---} \bullet} \\ \diagdown \quad \diagup \\ \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} \right] &= 4, & \left[\begin{array}{c} \overset{2}{\bullet \text{---} \bullet} \\ \diagdown \quad \diagup \\ \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} \right] &= 12, \\ & & \left[\begin{array}{c} \overset{\{0,2\}}{\bullet \text{---} \bullet} \\ \diagdown \quad \diagup \\ \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} \right] &= \left[\begin{array}{c} \overset{0}{\bullet \text{---} \bullet} \\ \diagdown \quad \diagup \\ \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} \right] + \left[\begin{array}{c} \overset{2}{\bullet \text{---} \bullet} \\ \diagdown \quad \diagup \\ \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} \right] & & & (1) \end{aligned}$$

and

$$\begin{array}{c} 0 \\ \diagdown \quad \diagup \\ \{1,2\} \quad \{1,3\} \\ \diagdown \quad \diagup \\ \{2,3\} \end{array} = \left[\begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} \right]. \quad (2)$$

For a second example, consider the graph from Figure 1(b). The configuration (u, x, w, v) is both of type $\begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array}$ and of type $\begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array}$. The configuration (z, w, y, v) is both of type $\begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array}$ and of type $\begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array}$. Thus an induced quadrangle is both of type $\begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array}$ and of type $\begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array}$, but the second type also comprises 4-cycles that induce K_{211} or K_4 . Using Definition 1.1, for the graph from Figure 1(b), it can be computed that

$$\begin{aligned} \left[\begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} \right] &= 24, & \left[\begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} \right] &= 12, & \left[\begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} \right] &= 22, \\ \left[\begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} \right] &= \left[\begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} \right], & \left[\begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} \right] &= 24, & \left[\begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} \right] &= 20, \\ \left[\begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} \right] &= \left[\begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} \right], & \left[\begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} \right] &= 0, & \left[\begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} \right] &= \left[\begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} \right], \end{aligned}$$

$$\text{and} \quad \left[\begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} \right] = \left[\begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} \right] + \left[\begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} \right] + \left[\begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} \right] = 66 + 36 + 58 = 160.$$

Clearly,

$$[\Delta] \geq 0 \quad \text{for all types } \Delta. \quad (3)$$

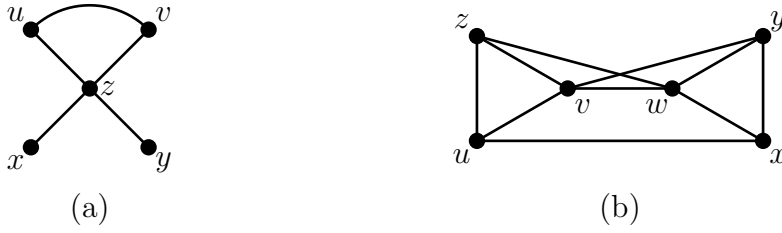


Figure 1: (a) Simple connected non-regular graph with vertex set $X = \{u, v, x, y, z\}$ and edge set $R = \{\{u, v\}, \{u, z\}, \{v, z\}, \{x, z\}, \{y, z\}\}$.
(b) Simple connected non-regular graph with vertex set $X = \{u, v, w, x, y, z\}$ and edge set $R = \{\{u, x\}, \{u, v\}, \{u, z\}, \{x, w\}, \{x, y\}, \{v, z\}, \{v, w\}, \{v, y\}, \{w, z\}, \{w, y\}\}$.

From now on, $\Gamma = (X, R)$ denotes a distance-regular graph of diameter d with $v_\Gamma := |X|$ vertices and intersection array $i(\Gamma) = \{b_0, \dots, b_{d-1}; c_1, \dots, c_d\}$, as defined, e.g., in BROUWER et al. [3]. We write

$$n := b_0, \quad a_i := n - b_i - c_i, \quad \lambda := a_1.$$

Since we will often use k as an index, the valency (conventionally denoted by k) is denoted by n . More background information is given in Section 3.

In a distance-regular graph, t -point counts with $t \leq 3$ are determined by the intersection array; in particular,

$$\begin{aligned}
\left[\begin{array}{c} \bullet \\ \bullet \end{array} \right] &= v_\Gamma, & \left[\begin{array}{c} \bullet \xrightarrow{i} \bullet \\ \bullet \end{array} \right] &= v_\Gamma \cdot n_i, \\
\left[\begin{array}{c} \bullet \xrightarrow{i} \bullet \\ \bullet \xrightarrow{i} \bullet \\ \bullet \end{array} \right] &= \left[\begin{array}{c} \bullet \xrightarrow{i} \bullet \\ \bullet \xrightarrow{\neq 0} \bullet \\ \bullet \end{array} \right] = v_\Gamma \cdot n_i \cdot (n_i - 1), & \left[\begin{array}{c} \bullet \xrightarrow{i} \bullet \\ \bullet \xrightarrow{0} \bullet \\ \bullet \end{array} \right] &= v_\Gamma n_i, \\
\left[\begin{array}{c} \bullet \xrightarrow{i} \bullet \\ \bullet \xrightarrow{0} \bullet \\ \bullet \end{array} \right] &= 0, \quad \text{if } i \neq j, & \left[\begin{array}{c} \bullet \xrightarrow{i} \bullet \\ \bullet \xrightarrow{0} \bullet \\ \bullet \end{array} \right] &= \left[\begin{array}{c} \bullet \xrightarrow{i} \bullet \\ \bullet \xrightarrow{0} \bullet \\ \bullet \end{array} \right] + \left[\begin{array}{c} \bullet \xrightarrow{i} \bullet \\ \bullet \xrightarrow{\neq 0} \bullet \\ \bullet \end{array} \right], \\
\left[\begin{array}{c} \bullet \xrightarrow{i} \bullet \\ \bullet \xrightarrow{j} \bullet \\ \bullet \end{array} \right] &= v_\Gamma n_k p_{ij}^k = v_\Gamma n_i p_{jk}^i = v_\Gamma n_j p_{ik}^j, & & (4)
\end{aligned}$$

and the three equal values in (4) come from the different ways of labelling the vertices of the type.

Section 4 contains many specific examples of t -point counts with $t \geq 4$; we show how to compute 4-point counts for a given graph and in Subsection 4.1 we consider examples where we fix three vertices in a given 4-point type. The corresponding counts give the triple intersection numbers of COOLSAET & JURISÍĆ [5].

If an intersection array $i(\Gamma)$ determines Γ up to isomorphism, then all t -point counts are determined by the intersection array. It is conceivable that in many concrete cases, the t -point counts with small $t > 3$ are already determined by equations and inequalities between counts valid for arbitrary distance-regular graphs.

Indeed, there are many such relations between different counts. The simplest ones are obtained by summing and elimination according to the following rules.

Edges labeled by subsets of cardinality ≥ 2 . Fix an edge $uv \in E\Delta$. If $|\Delta_{uv}| \geq 2$ then

$$[\Delta] = \sum_{i \in \Delta_{uv}} [\Delta(uv \rightarrow \{i\})] = \sum_{\substack{i \in \Delta_{uv} \\ \Delta' = \Delta(uv \rightarrow \{i\})}} [\Delta'],$$

where $\Delta' = \Delta(uv \rightarrow \{i\})$ is an undirected graph with the same vertex and edge set as Δ , in which the edge uv is labeled by the set $\{i\}$, and all the other edges have the same label as in Δ (for example, see (1)).

Sum over a distance, e.g.,

$$\sum_h \left[\begin{array}{c} i \quad \ell \\ \diagdown \quad \diagup \\ h \quad k \\ \diagup \quad \diagdown \\ j \quad m \end{array} \right] = \left[\begin{array}{c} i \quad \ell \\ \diagdown \quad \diagup \\ j \quad k \\ \diagup \quad \diagdown \\ j \quad m \end{array} \right], \quad \sum_j \left[\begin{array}{c} i \\ \diagdown \quad \diagup \\ j \quad k \end{array} \right] = \left[\begin{array}{c} i \\ \diagdown \quad \diagup \\ \cdot \quad k \end{array} \right], \quad (5)$$

$$\sum_{j \geq 1} \left[\begin{array}{c} i \\ \diagdown \quad \diagup \\ j \quad k \end{array} \right] = v_\Gamma n_i n_k \quad \text{if } i \neq k, \quad \text{and} \quad \sum_{j \geq 1} \left[\begin{array}{c} i \\ \diagdown \quad \diagup \\ j \quad k \end{array} \right] = v_\Gamma n_i (n_i - 1) \quad \text{if } i = k. \quad (6)$$

Elimination of nodes of valency 2, e.g.,

$$\left[\begin{array}{c} i \quad \ell \\ \diagdown \quad \diagup \\ \cdot \quad k \\ \diagup \quad \diagdown \\ \cdot \quad m \end{array} \right] = p_{\ell m}^k \left[\begin{array}{c} i \\ \diagdown \quad \diagup \\ j \quad k \end{array} \right] = v_\Gamma n_k p_{ij}^k p_{\ell m}^k. \quad (7)$$

Elimination of distance 0, e.g.,

$$\left[\begin{array}{c} i \quad \ell \\ \diagdown \quad \diagup \\ \theta \quad k \\ \diagup \quad \diagdown \\ j \quad m \end{array} \right] = \delta_{i\ell} \delta_{jm} \left[\begin{array}{c} i \\ \diagdown \quad \diagup \\ j \quad k \end{array} \right] = \delta_{i\ell} \delta_{jm} v_\Gamma n_k p_{ij}^k, \quad (8)$$

where δ_{ik} is the Kronecker symbol. For the general case, let $\Delta = (V\Delta, E\Delta)$ and fix an edge $uv \in E\Delta$. If $\Delta_{uv} = \{0\}$ then

$$[\Delta] = [\Delta'] = [(\Delta - v)(uw \rightarrow \Delta_{uw} \cap \Delta_{vw} \mid w \in V\Delta \setminus \{v\})]$$

where $\Delta' = (\Delta - v)(uw \rightarrow \Delta_{uw} \cap \Delta_{vw} \mid w \in V\Delta \setminus \{v\})$ is an undirected graph with the vertex set $V\Delta \setminus \{v\}$, the edge set $E\Delta \setminus \{\{z, v\} \mid z \in V\Delta, \{z, v\} \in E\Delta\}$, in which the edge uw ($w \in V\Delta \setminus \{v\}$) has label $\Delta_{uw} \cap \Delta_{vw}$, and all the other edges (which do not have v as an endpoint) have the same label as in Δ (for example, see (2)).

Moreover, isomorphic types clearly have the same count. As in the special case (5), every $[\Delta]$ can be written as a sum of the t -point counts corresponding to the various completions of the type Δ . Thus we obtain a system of linear equations and inequalities, i.e., a linear **constraint satisfaction problem (CSP)**, whose variables are the t -point counts. We refer to this as the **t -point CSP** of a given intersection array.

Using more general configuration algebra techniques, further nonlinear equations or inequalities can be made. In this paper, we shall only consider the simplest of these, obtained by multiplication and by summing over products according to the following rules.

Multiplication, e.g.,

$$\left[\begin{array}{c} i \\ \diagdown \quad \diagup \\ h \quad \ell \end{array} \right] \cdot \left[\begin{array}{c} i \\ \diagdown \quad \diagup \\ j \quad k \end{array} \right] = v_\Gamma n_i \left[\begin{array}{c} k \quad h \\ \diagdown \quad \diagup \\ j \quad i \\ \diagup \quad \diagdown \\ \cdot \quad \ell \end{array} \right] = \left[\begin{array}{c} i \\ \cdot \end{array} \right] \cdot \left[\begin{array}{c} k \quad h \\ \diagdown \quad \diagup \\ j \quad i \\ \diagup \quad \diagdown \\ \cdot \quad \ell \end{array} \right]. \quad (9)$$

Let $\Gamma_i(x)$ denote the set of vertices in X that are at distance i from a vertex x , and for vertices $x, y, z \in X$, let $\Gamma_{ij}(x, y) = \Gamma_i(x) \cap \Gamma_j(y)$ and $\Gamma_{ijk}(x, y, z) = \Gamma_i(x) \cap \Gamma_j(y) \cap \Gamma_k(z)$. If the sizes of $|\Gamma_{\ell mk}(x, y, z)|$ and $|\Gamma_{spt}(x, y, z)|$ do not depend on $x \in X$, $y \in \Gamma_h(x)$ and $z \in \Gamma_{ij}(x, y)$ then

$$\left[\begin{array}{c} i \quad \ell \\ \diagdown \quad \diagup \\ m \quad h \\ \diagup \quad \diagdown \\ j \quad k \end{array} \right] \cdot \left[\begin{array}{c} i \quad s \\ \diagdown \quad \diagup \\ t \quad h \\ \diagup \quad \diagdown \\ j \quad p \end{array} \right] = v_\Gamma n_h p_{ij}^h \left[\begin{array}{c} j \\ \diagdown \quad \diagup \\ \ell \quad h \\ \diagup \quad \diagdown \\ k \quad m \\ \diagdown \quad \diagup \\ \cdot \quad s \\ \diagup \quad \diagdown \\ \cdot \quad p \end{array} \right] = \left[\begin{array}{c} i \\ \diagdown \quad \diagup \\ j \quad h \end{array} \right] \left[\begin{array}{c} i \quad j \\ \diagdown \quad \diagup \\ \ell \quad h \\ \diagup \quad \diagdown \\ k \quad m \\ \diagdown \quad \diagup \\ \cdot \quad s \\ \diagup \quad \diagdown \\ \cdot \quad p \end{array} \right]. \quad (10)$$

Looking at (9) and (10), the reader will immediately note that, for the general case, perhaps we have something like

$$[\Delta] \cdot [\Delta'] = [\Delta \cup \Delta'] \cdot [\Delta \cap \Delta'], \quad (11)$$

where $\Delta \cup \Delta'$ is a type with $V(\Delta \cup \Delta') = V\Delta \cup V\Delta'$, $(\Delta \cup \Delta')_{uv} = \Delta_{uv}$ for all $u, v \in V\Delta$, $(\Delta \cup \Delta')_{uv} = \Delta'_{uv}$ for all $u, v \in V\Delta'$, and $(\Delta \cup \Delta')_{uv} = \Delta_{uv} = \Delta'_{uv}$ for all $u, v \in V\Delta \cap V\Delta'$. On the other hand $\Delta \cap \Delta'$ is a subtype of Δ and Δ' , i.e., $V(\Delta \cap \Delta') = V\Delta \cap V\Delta'$ and $(\Delta \cap \Delta')_{uv} = \Delta_{uv} = \Delta'_{uv}$ for all $u, v \in V\Delta \cap V\Delta'$. The claim (9) does not hold in general, for example, for $K_{1,2,1}$ we have

$$\left[\begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \right] \cdot \left[\begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \right] = 12 \cdot 12 \neq 10 \cdot 4 = \left[\bullet \text{---} \bullet \right] \cdot \left[\begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \right].$$

Also, note that the multiplication formula (10) does not hold in general for distance-regular graphs, for example, let Γ be the Petersen graphs. Then equation (10) with all labels equal to $\{2\}$ does not hold, as

$$\left[\begin{array}{c} 2 \quad \bullet \quad 2 \\ \diagup \quad \diagdown \\ 2 \quad \bullet \quad 2 \end{array} \right] \cdot \left[\begin{array}{c} 2 \quad \bullet \quad 2 \\ \diagup \quad \diagdown \\ 2 \quad \bullet \quad 2 \end{array} \right] = 120 \cdot 120 \neq 180 \cdot 120 = \left[\begin{array}{c} 2 \quad \bullet \quad 2 \\ \diagup \quad \diagdown \\ 2 \quad \bullet \quad 2 \end{array} \right] \left[\begin{array}{c} 2 \quad \bullet \quad 2 \\ \diagup \quad \diagdown \\ 2 \quad \bullet \quad 2 \end{array} \right].$$

For a version of the equation when the vertices of $\Delta \cap \Delta'$ are fixed, see Subsection 1.1.

To get a felling when (11) holds in general, for the moment let Δ and Δ' denote t -point counts with $V\Delta = \{1, 2, \dots, t\}$ and $V\Delta' = \{p, q\}$. We have

$$\begin{aligned} [\Delta] &= |\{(Z(1), Z(2), \dots, Z(t)) \in X^{V\Delta} \mid Z \in \{\Delta\}\}| \\ &= |\{(y, W) \mid y \in X, W \in \{\Delta \setminus \{1\}\}, (1j \in E\Delta \implies \partial_\Gamma(y, W(j)) \in \Delta_{1j})\}|, \\ [\Delta'] &= |\{Z' \in X^{V\Delta'} \mid Z' \in \{\Delta'\}\}| \\ &= |\{(w, z) \in X \times X \mid pq \in E\Delta' \implies \partial_\Gamma(w, z) \in \Delta_{pq}\}|. \end{aligned}$$

Now assume that Δ' is isomorphic to K_2 with the edge labeled by a set A , assume that Γ is a graph with vertex set X for which $|\Gamma_i(y)|$ does not depend on $y \in X$, for all i ($0 \leq i \leq d$). Pick $y \in X$, abbreviate $n_A = \sum_{i \in A} |\Gamma_i(y)|$ and assume $V\Delta' = \{1, s\}$ ($s \notin \{1, \dots, t\}$). Under these assumptions, since $V\Delta \cap V\Delta' = \{1\}$, we have

$$\begin{aligned} [\Delta] \cdot [\Delta'] &= [\Delta] \cdot v_\Gamma \cdot n_A, \\ [\Delta \cup \Delta'] &= |\{(y, z, W) \mid y, z \in X, W \in \{\Delta \setminus \{1\}\}, \\ &\quad \partial_\Gamma(y, z) \in \Delta'_{1s} \text{ and } (1j \in E\Delta \implies \partial_\Gamma(y, W(j)) \in \Delta_{1j})\}|, \\ &= [\Delta] \cdot n_A \\ [\Delta \cap \Delta'] &= v_\Gamma. \end{aligned}$$

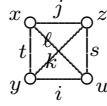
Therefore, under above assumptions, (11) holds.

Similarly, the formula (11) holds for all choices of Δ and Δ' such that Δ' isomorphic to (a subgraph of) K_3 and $\Delta \cap \Delta'$ isomorphic to (a subgraph of) K_2 if the graph is distance-regular. We may generalize this further – the formula is true for all choices of Δ and Δ' such that Δ' isomorphic to (a subgraph of) K_4 and $\Delta \cap \Delta'$ isomorphic to (a subgraph of) K_3 when the graph is triply regular, and so on.

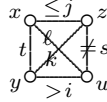
If a t -point CSP for a given intersection array has no feasible solution then no graph with the assumed intersection array exists. If a feasible solution exists, it might happen in some extremal cases that certain t -point counts are forced to vanish for *every* feasible solution. This implies a geometric statement that Γ contains no configurations of certain types. This information often allows further analysis which might lead to uniqueness or nonexistence of a graph with given properties.

1.1 Diagrams with fixed vertices

A diagram



in which we have white circles with a label, represents a fixed 4-tuple (x, y, u, z) , for which we have $y \in \Gamma_t(x)$, $z \in \Gamma_{jk}(x, y)$ and $u \in \Gamma_{lis}(x, y, z)$ ($0 \leq i, j, k, \ell, s, t \leq d$). In the case when in this kind of diagram we have constraints, like



then this diagram represents a fixed 4-tuple (x, y, u, z) , for which $y \in \Gamma_t(x)$, $z \in \Gamma_k(y)$, $u \in \Gamma_\ell(x)$, $\partial(x, z) \leq j$, $\partial(y, u) > i$ and $\partial(z, u) \neq s$ (note that we don't know the exact distance between x and z , we only know that distance is $\leq j$, similarly for the distance between y and u , and the distance between z and u).

Our goal in Definition 1.2 is to define diagrams in which both kind of vertices are allowed, "free vertices" marked as black circles, and fixed vertices marked as white circles with a label. We also want the Definition 1.1 to be a special case of this definition.

Definition 1.2 Let Γ denote a graph with vertex set X and diameter d . Let Δ denote an undirected graph with vertex set $Y = Y_\circ \cup Y_\bullet$ ($Y_\circ \subseteq X$, $Y_\bullet = \{1, 2, \dots, t\}$, $t \geq 1$) whose edges are labelled by subsets of $\{0, 1, \dots, d\}$, such that

$$\text{if } \{u, v\} \subseteq Y_\circ \text{ and } uv \in E\Delta \quad \text{then} \quad \partial_\Gamma(u, v) \in \Delta_{uv}$$

where $E\Delta$ denotes the edge set of Δ and Δ_{uv} denotes the label of the edge uv . In drawings, vertices of diagrams Δ which are from Y_\circ are marked as white circles with a label and vertices of diagrams Δ which are from Y_\bullet are shown as black circles without a label. Furthermore, missing labels are taken as having the value 1, edges labeled by a number i represent the set $\{i\}$, and edges labeled by constraints $\leq i$ (resp. $\neq i$) represent the set $\{0, 1, \dots, i\}$ (resp. $\{0, \dots, i-1, i+1, \dots, d\}$). Let EY_\bullet denote the set of 2-element subsets ij of a t -element set Y_\bullet such that the edge ij exists in $E\Delta$, i.e.,

$$EY_\bullet = \{ij \mid 1 \leq i < j \leq t, ij \in E\Delta\}$$

(the graph Δ does not need to be a complete graph). We define the set $\{\Delta\}$ as the set of t -tuples (z_1, z_2, \dots, z_t) of elements of X for which $\partial_\Gamma(z_i, z_j) \in \Delta_{ij}$ for all i, j of EY_\bullet , and $\partial_\Gamma(u, z_j) \in \Delta_{uj}$ for all $u \in Y_\circ$ and $j \in Y_\bullet$ for which uj is an edge of $E\Delta$. In other words,

$$\{\Delta\} = \left\{ Z \in X^{Y_\bullet} \mid \text{if } i, j \in Y_\bullet \text{ and } ij \in E\Delta \text{ then } \partial_\Gamma(Z(i), Z(j)) \in \Delta_{ij} \right. \\ \left. \text{and } \text{if } u \in Y_\circ, j \in Y_\bullet \text{ and } uj \in E\Delta \text{ then } \partial_\Gamma(u, Z(j)) \in \Delta_{uj} \right\}.$$

For a such Δ we also define the integer $[\Delta]$ as the number of elements of the set $\{\Delta\}$, that is,

$$[\Delta] = |\{\Delta\}|.$$

For example, consider the graph from Figure 1(b). Using Definition 1.2 it can be computed that

$$\left[\begin{array}{c} \bullet \\ \circ \\ \bullet \\ \circ \end{array} \right] = \left[\begin{array}{c} \circ \\ \bullet \\ \circ \\ \bullet \end{array} \right] = 4, \quad \left[\begin{array}{c} \bullet \\ \circ \\ \bullet \\ \circ \end{array} \right] = \left[\begin{array}{c} \circ \\ \bullet \\ \circ \\ \bullet \end{array} \right] = 2, \quad \left[\begin{array}{c} \bullet \\ \circ \\ \bullet \\ \circ \end{array} \right] = \left[\begin{array}{c} \circ \\ \bullet \\ \circ \\ \bullet \end{array} \right] = 2,$$

$$\begin{aligned}
\left[\begin{array}{c} u \\ \text{---} \\ \text{---} \\ w \end{array} \right] &= \left[\begin{array}{c} x \\ \text{---} \\ \text{---} \\ v \end{array} \right] = 2, & \left[\begin{array}{c} u \\ \text{---} \\ \text{---} \\ y \end{array} \right] &= \left[\begin{array}{c} x \\ \text{---} \\ \text{---} \\ z \end{array} \right] = 0, & \left[\begin{array}{c} z \\ \text{---} \\ \text{---} \\ x \end{array} \right] &= \left[\begin{array}{c} y \\ \text{---} \\ \text{---} \\ u \end{array} \right] = 0, \\
\left[\begin{array}{c} u \\ \text{---} \\ \text{---} \\ u \end{array} \right] &= \left[\begin{array}{c} x \\ \text{---} \\ \text{---} \\ x \end{array} \right] = 2, & \left[\begin{array}{c} z \\ \text{---} \\ \text{---} \\ z \end{array} \right] &= \left[\begin{array}{c} y \\ \text{---} \\ \text{---} \\ y \end{array} \right] = 4, & \left[\begin{array}{c} v \\ \text{---} \\ \text{---} \\ v \end{array} \right] &= \left[\begin{array}{c} w \\ \text{---} \\ \text{---} \\ w \end{array} \right] = 6,
\end{aligned}$$

and so on.

For any Γ with a vertex set X , note that

$$\begin{aligned}
\left[\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \right] &= \sum_{x \in X} \sum_{y \in \Gamma_2(x)} \left[\begin{array}{c} x \\ \text{---} \\ \text{---} \\ y \end{array} \right] = \sum_{z \in X} \left[\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ z \end{array} \right] \\
&= \sum_{u \in X} \sum_{v \in \Gamma_1(u)} \sum_{w \in \Gamma_{21}(u,v)} \underbrace{\left[\begin{array}{c} v \\ \text{---} \\ \text{---} \\ u \end{array} \right]}_{=|\Gamma_{121}(u,v,w)|} = \sum_{u \in X} \sum_{v \in \Gamma_1(u)} \sum_{w \in \Gamma_{21}(u,v)} \sum_{y \in \Gamma_{121}(u,v,w)} 1,
\end{aligned}$$

where, for example,

$$\left[\begin{array}{c} x \\ \text{---} \\ \text{---} \\ y \end{array} \right] = |\{(u, z) \in X \times X \mid u \in \Gamma_{11}(x, y), z \in \Gamma_{11}(x, y), \partial(u, z) = 2\}|.$$

To obtain a more general identity, let Δ' be a subtype of Δ (i.e., $V\Delta' \subseteq V\Delta$ and $\Delta'_{uv} = \Delta_{uv}$ for all $u, v \in V\Delta'$). For the moment assume that $V\Delta' = Y'_\circ \cup Y'_\bullet$, and note that if $W \in \{\Delta'\}$ then the domain of W is precisely Y'_\bullet . We have

$$[\Delta] = \sum_{W \in \{\Delta'\}} [\Delta \mid W],$$

where $\Delta \mid W$ is the type defined by $V(\Delta \mid W) = V(\Delta \mid W)_\circ \cup V(\Delta \mid W)_\bullet$, $V(\Delta \mid W)_\circ = V\Delta_\circ \cup W(V\Delta'_\bullet)$ and $V(\Delta \mid W)_\bullet = V\Delta_\bullet \setminus V\Delta'_\bullet$.

Now assume that for the types $[\Delta']$ and $[\Delta'']$ we have $V\Delta' = V\Delta'_\circ \cup V\Delta'_\bullet$, $V\Delta'' = V\Delta''_\circ \cup V\Delta''_\bullet$, $V\Delta'_\circ = V\Delta''_\circ$, that $V\Delta'_\bullet$ and $V\Delta''_\bullet$ are disjoint, and consider a type $\Delta = \Delta' \cup \Delta''$. Recall $V\Delta = V\Delta' \cup V\Delta''$ and $\Delta_{uv} = \Delta'_{uv} = \Delta''_{uv}$ for all $u, v \in V\Delta' \cap V\Delta''$. Then

$$\begin{aligned}
[\Delta'] \cdot [\Delta''] &= |\{Z' \in X^{V\Delta'} \mid Z' \in \{\Delta'\}\}| \cdot |\{Z'' \in X^{V\Delta''} \mid Z'' \in \{\Delta''\}\}| \\
&= |\{(Z', Z'') \mid Z' \in X^{V\Delta'}, Z' \in \{\Delta'\}, Z'' \in X^{V\Delta''}, Z'' \in \{\Delta''\}\}| \\
&= |\{Z \in \{\Delta\} \mid Z \in X^{V\Delta'_\bullet \cup V\Delta''_\bullet}\}| \\
&= [\Delta] = [\Delta' \cup \Delta''].
\end{aligned}$$

Thus, when $V\Delta'_\circ = V\Delta''_\circ$, and $V\Delta'_\bullet$ and $V\Delta''_\bullet$ are disjoint, then

$$[\Delta'] \cdot [\Delta''] = [\Delta' \cup \Delta''].$$

1.2 Scaffolds

The first author discovered the usefulness of the t -count technique in 1989 while spending a year with Paul Terwilliger in Madison. He gave a lecture there and at Eindhoven University, but didn't publish anything since his interests changed to applied mathematics and physics. In 2017, the second author indicated interest in working with the first author on DRGs, and we jointly extended this old work.

At the time when this project ended, the manuscript [13] by MARTIN on “scaffolds” appeared. Our t -point counts are equivalent to Martin’s scaffolds of order zero labeled with adjacency matrices corresponding to a union of distances. For the cases of t -point counts whose labels are singletons i , corresponding to adjacency matrix labels A_i , this follows easily from the following result.

Lemma 1.3 *Suppose we are given*

- (i) *A finite simple graph $\Gamma = (X, R)$ of diameter d and with distance- i matrices A_i ($0 \leq i \leq d$) ($A_i \in \text{Mat}_X(\mathbb{C})$ is the matrix for which (x, y) -entry is equal to 1 if $\partial(x, y) = i$, and 0 otherwise, for all $x, y \in X$).*
- (ii) *A finite simple graph $\Delta = (V\Delta, E\Delta)$ (the diagram of the scaffold).*
- (iii) *A map from edges of Δ to the set $\{A_0, A_1, \dots, A_d\}$: $w : E\Delta \rightarrow \{A_i\}_{i=0}^d$ (edge weights). If $e \in E\Delta$ and $w(e) = A_i$, we label the edge e by i (so that Δ is a t -point type whose edges are labelled with integers $\in \{0, 1, \dots, d\}$).*

Then t -point count $[\Delta]$ can be computed as

$$[\Delta] = \sum_{\varphi: V\Delta \rightarrow X} w(\varphi)$$

where $w(\varphi) = \prod_{\substack{e \in E\Delta \\ e = \{a, b\}}} w(e)_{\varphi(a), \varphi(b)}$.

Proof. There are $|X|^{|V\Delta|}$ different maps φ from the set $V\Delta$ to the set X . Fix one such φ , and note that $w(\varphi) \in \{0, 1\}$. If there exists an edge $e \in E\Delta$ ($e = \{a, b\}$) such that $\partial_{\Gamma}(\varphi(a), \varphi(b)) \neq i$, where $A_i = w(e)$, we have $w(\varphi) = 0$. We have $w(\varphi) = 1$ if for every $e \in E\Delta$ ($e = \{a, b\}$) $\partial_{\Gamma}(\varphi(a), \varphi(b)) = \Delta_{ab}$ holds, where Δ_{ab} denotes the label of the edge $\{a, b\}$. In other words, $\sum_{\varphi: V\Delta \rightarrow X} w(\varphi)$ is the number of configurations of type Δ . ■

2 Main results

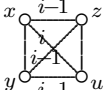
We now summarize our main results.

One of the main purposes of this paper is to present the language of configurations and t -point counts – we begin that in Section 4; here we re-prove some well-known inequalities using the notation of this language. Recall that in [5], COOLSAET and JURISÍĆ introduced the notation of triple intersection numbers, and used them to prove nonexistence of certain distance-regular graphs. In our notation their triple intersection numbers are in fact our 4-point counts with three vertices fixed (see Subsection 4.1). In Subsection 4.2 we have 5-point counts with four vertices fixed, i.e., we have quadruple intersection numbers. In Section 5 we re-prove Koolen’s inequalities, if $2 \leq e \leq d$ (where d is the diameter of the graph) then

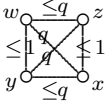
$$c_e > c_{e-1} \quad \Rightarrow \quad c_e \geq c_i + c_{e-i} \quad (0 \leq i \leq e-1)$$

and

$$b_e > b_{e+1} \quad \Rightarrow \quad b_e \geq c_i + b_{e+i} \quad (0 \leq i \leq d-e).$$

A quadruple $xyzu$ of vertices is called a parallelogram of length i if $xyzu$ is consistent with . The graph Γ is called m -parallelogram-free for some $m = 2, 3, \dots, d$ if Γ does

not contain any parallelogram of length at most m . In [15], SUZUKI studied strongly closed subgraphs of diameter 2 in a parallelogram-free distance-regular graph Γ of diameter $d \geq 4$ such that $b_1 > b_2$ and $a_2 \neq 0$. A quadruple $wxyz$ of vertices is called a root of size q if $wxyz$

is consistent with . Related conditions called $(CR)_q$ and $(SS)_q$ are studied by HIRAKI [8, 9]. Recall that these two conditions are defined as

$$(CR)_q : \quad \{z\} \cup C(x, z) \cup A(x, z) = \{z\} \cup C(y, z) \cup A(y, z)$$

(see Section 3 for definition of $C(\cdot, \cdot)$ and $A(\cdot, \cdot)$) for any triple xyz of vertices with $\partial(x, z) = \partial(y, z) = q$ for which there exist three sequences of vertices $(x_0, x_1, \dots, x_m = x)$, $(y_0, y_1, \dots, y_m = y)$ and $(z_0, z_1, \dots, z_m = z)$ such that $\partial(x_0, y_0) \leq 1$, $x_{i-1}z_{i-1}x_i z_i$ and $y_{i-1}z_{i-1}y_i z_i$ are roots of size q for all $1 \leq i \leq m$; and

$$(SS)_r : \quad \{z\} \cup C(x, z) \cup A(x, z) = \{z\} \cup C(y, z) \cup A(y, z)$$

for any triple of vertices xyz with $\partial(x, z) = \partial(y, z) = q$ and $\partial(x, y) \leq 1$. In certain cases these conditions are necessary and sufficient for the existence of a strongly closed $(c_{r+1} + a_{r+1})$ -regular subgraph of diameter $r + 1$ (where $r = r(\Gamma) = \max\{j \mid (c_j, a_j, b_j) = (c_1, a_1, b_1)\}$). In Section 6 we introduce 4-point counts containing two disjoint edges, which also includes a parallelogram of length m and a root of size q . Using them we derive some inequalities and we explain their combinatorial meaning for distance-regular graphs. In Section 7 we generalize 4-point counts from Section 6, and we derive some inequalities when $\left[\begin{array}{c} z \\ \text{---} \\ i \end{array} \right] > 0$ for some i ($1 \leq i \leq d$). In Section 8 we prove that if $1 \leq i < j \leq d$, then

$$a_j > a_i \implies b_i > b_j$$

and

$$a_j < a_i \implies c_j > c_i.$$

In Section 9 we re-prove Hiraki's inequality

$$c_q < c_{q+1} \quad \text{and} \quad a_q \leq c_{q+1} - c_q \quad \implies \quad c_{q+i} \geq c_i + c_q \quad (2 \leq i \leq d - q).$$

Finally, in Section 10, using a similar technique as HIRAKI in [10], we prove that if $c_q < c_{q+1}$ and $a_q \leq c_{q+1} - c_q$ (where $q \geq 2$), then

$$d \leq (k + 1 - c_{q+1})q + 1.$$

This diameter bound is tight for the Biggs–Smith graph (the intersection array of this graph is $\{3, 2, 2, 2, 1, 1, 1, 1, 1, 1, 1, 1, 3\}$).

3 Preliminaries

A *graph* Γ is a pair (X, R) , where X is a nonempty set and R is a collection of two element subsets of X . The elements of X are called the *vertices* of Γ , and the elements of R are called the *edges* of Γ . When $xy \in R$, we say that vertices x and y are *adjacent*, or that x and y are *neighbors*. Adjacency between vertices x and y will be denoted by $x \sim y$. A subset $C \subseteq X$ is called a *clique* if every pair of distinct vertices $x, y \in C$ are neighbors. A graph is *finite* if both its vertex set and edge set are finite. If we allow for an edge to start and to end at the same vertex, then an edge with identical ends is called a *loop*, and a graph is *simple* if it has no loops and no two of its edges join the same pair of vertices.

Let $\Gamma = (X, R)$ be a graph. For any two vertices $x, y \in X$, a walk of length h from x to y is a sequence $[x_0, x_1, x_2, \dots, x_h]$ ($x_i \in X$, $0 \leq i \leq h$) such that $x_0 = x$, $x_h = y$, and x_i is adjacent to x_{i+1} ($0 \leq i \leq h-1$). We say that Γ is connected if for any $x, y \in X$, there is a walk from x to y . From now on, assume that Γ is finite, simple and connected.

For any $x, y \in X$, the *distance* between x and y , denoted $\partial(x, y)$, is the length of the shortest walk from x to y . The *diameter* $d = d(\Gamma)$ is defined to be

$$d = \max\{\partial(u, v) \mid u, v \in X\}.$$

A walk in Γ is said to be *closed* if it starts and ends at the same vertex.

A *polygon* $p_1 p_2 p_3 \dots p_{m+1}$ of length m ($m \geq 3$) (or a *circuit* of length $m \geq 3$) is a closed walk $[p_1, p_2, p_3, \dots, p_{m+1}]$ on distinct vertices, where $p_{m+1} = p_1$. A polygon of length m is called *reduced* if $m \geq 4$ and none of its proper subsets form a polygon. A shortest reduced polygon is called a *minimal polygon*. An *induced polygon* $p_1 p_2 p_3 \dots p_m p_1$ is a reduced polygon such that if $i \leq j$ then $\partial(p_i, p_j) = \min\{|i - j|, |i + m - j|\}$ is implied.

Two graphs $\Gamma_1 = (X_1, R_1)$ and $\Gamma_2 = (X_2, R_2)$ are said to be *isomorphic* if there are bijections $\varphi : X_1 \rightarrow X_2$ and $\psi : R_1 \rightarrow R_2$ such that $e = uv$ if and only if $\psi(e) = \varphi(u)\varphi(v)$. Such a pair (φ, ψ) of mappings is called an *isomorphism* between Γ_1 and Γ_2 .

A simple graph in which each pair of distinct vertices is joined by an edge is called a *complete graph*. Up to isomorphism, there is just one complete graph on n vertices; it is denoted by K_n . A *bipartite* graph is one whose vertex set can be partitioned into two subsets X and Y , so that each edge has one end in X and one end in Y ; such a partition (X, Y) is called a bipartition of the graph. A *complete bipartite graph* is a simple bipartite graph with bipartition (X, Y) in which each vertex of X is joined to each vertex of Y ; if $|X| = m$ and $|Y| = n$, such a graph is denoted by $K_{m,n}$. A *k-partite graph* is a graph whose vertices can be decomposed into k disjoint sets such that no two vertices within the same set are adjacent; if there are p_1, p_2, \dots, p_k vertices in the k sets, the complete k -partite graph is denoted by K_{p_1, p_2, \dots, p_k} – for example $K_{2,1,1}$ is a graph with 4 vertices and with five edges.

Suppose that Y is a nonempty subset of X . The subgraph of $\Gamma = (X, R)$ whose vertex set is Y and whose edge set S is the set of those edges of Γ that have both ends in Y is called the subgraph of Γ *induced* by Y ; we say that $\Delta = (Y, S)$ is an *induced subgraph* of $\Gamma = (X, R)$.

Let $\Gamma = (X, R)$ be a graph with diameter d . For a vertex $x \in X$ and any non-negative integer h not exceeding d , let $\Gamma_h(x)$ denote the subset of vertices in X that are at distance h from x . Let $\Gamma_{-1}(x) = \Gamma_{d+1}(x) := \emptyset$. For any two vertices x and y in X at distance h , let

$$\begin{aligned} C(x, y) &= C_h(x, y) := \Gamma_{h-1}(x) \cap \Gamma_1(y), \\ A(x, y) &= A_h(x, y) := \Gamma_h(x) \cap \Gamma_1(y), \\ B(x, y) &= B_h(x, y) := \Gamma_{h+1}(x) \cap \Gamma_1(y). \end{aligned}$$

We say Γ is regular with valency n if each vertex in Γ has exactly n neighbours. A graph Γ is called *distance-regular* if there are integers b_i, c_i ($0 \leq i \leq d$) which satisfy $c_i = |C_i(x, y)|$ and $b_i = |B_i(x, y)|$ for any two vertices x and y in X at distance i . Clearly such a graph is regular of valency $n := b_0$ and

$$a_i := |A_i(x, y)| = n - b_i - c_i \quad (0 \leq i \leq d)$$

is the number of neighbours of y in $\Gamma_i(x)$ for $x, y \in X$ ($\partial(x, y) = i$). From the definition of distance-regular graph it is routine to show that Γ is distance-regular if and only if for all triples h, i, j ($0 \leq h, j, i \leq D$) and for all $x, y \in X$ with $\partial(x, y) = h$, the number

$$p_{ij}^h := |\Gamma_i(x) \cap \Gamma_j(y)|$$

is independent of choice of x and y . The numbers p_{ij}^h are called the *intersection numbers* of Γ . It is not hard to see that $a_i = p_{i1}^i$, $b_i = p_{i+1,1}^i$ and $c_i = p_{i-1,1}^i$. The array

$$i(\Gamma) = \{b_0, b_1, \dots, b_{d-1}; c_1, c_2, \dots, c_d\}$$

is called the *intersection array* of Γ .

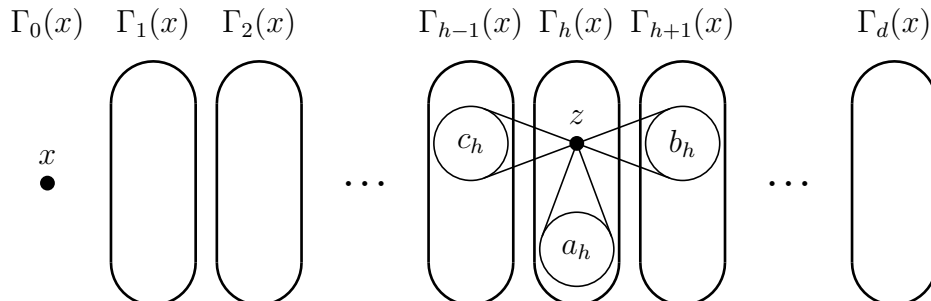


Figure 2: Intersection diagram (of rank 0) with respect to x and illustration for intersection numbers c_h , a_h and b_h .

For vertices $x_1, x_2, \dots, x_k \in X$ and integers i_1, i_2, \dots, i_k ($0 \leq i_1, i_2, \dots, i_k \leq d$) we define

$$\Gamma_{i_1, i_2, \dots, i_k}(x_1, x_2, \dots, x_k) = \bigcap_{\ell=1}^k \Gamma_{i_\ell}(x_\ell).$$

A distance-regular graph with no parallelogram of length two (i.e., with $\left[\begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \right] = 0$) is called a *geometric distance-regular graph of order (s, t)* , where $s = a_1 + 1$ and $t = b_1/s$, because the incidence geometry defined on the set of vertices and the set of maximal cliques has the property that each line contains $s + 1$ points and each point is on $t + 1$ lines. For a distance-regular graph of order (s, t) , its *geometric girth* is defined to be the length of the shortest reduced polygon. If a distance-regular graph Γ satisfies the conditions $b_1 > b_2$ and $c_2 = 1$, then it is of order (s, t) and the geometric girth is five. The *numerical girth* of Γ , denoted by g , is the length of a minimal polygon.

Definition 3.1 Throughout the rest of this paper, $\Gamma = (X, R)$ shall refer to a fixed distance-regular graph with diameter d , valency n , intersection numbers p_{ij}^h ($0 \leq h, i, j \leq d$) and intersection array $\{b_0, \dots, b_{d-1}; c_1, \dots, c_d\}$. We will also denote the intersection number a_1 by λ , the intersection number c_2 by μ and we define $v_\Gamma := |X|$. For an edge-labelled graph Δ on t -vertices, $[\Delta]$ will denote the number of configurations of type Δ and will therefore be a non-negative integer. For $x \in X$, $y \in \Gamma_h(x)$, we use the abbreviations $C(x, y) = C_h(x, y) = \Gamma_{h-1,1}(x, y) := \Gamma_{h-1}(x) \cap \Gamma_1(y)$, $A(x, y) = A_h(x, y) = \Gamma_{h,1}(x, y) := \Gamma_h(x) \cap \Gamma_1(y)$ and $B(x, y) = B_h(x, y) = \Gamma_{h+1,1}(x, y) := \Gamma_{h+1}(x) \cap \Gamma_1(y)$.

4 Simple examples

In this section we give examples illustrating how to compute 4-point counts for a given graph, and we prove some equations that we use in the rest of the paper. Let us begin with the simplest one.

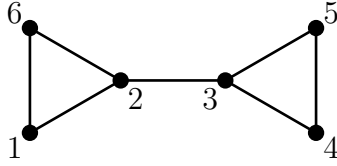


Figure 3: Graph with vertex set $X = \{1, \dots, 6\}$ and edge set $R = \{12, 16, 23, 26, 34, 35, 45\}$.


Example 4.1 Let $\Gamma = (X, R)$ denote a non-regular graph with vertex set $X = \{1, \dots, 6\}$ and edge set $R = \{12, 16, 23, 26, 34, 35, 45\}$ (see Figure 3).

If we fix a drawing of K_4 (as a square with diagonals) and fix the labelling of the vertices as, for example, first (vertex α – left upper corner), second (vertex β – left bottom corner), third (vertex γ – right bottom corner) and fourth (vertex δ – right upper corner)



then the configurations 1212 and 2121 are of the type . Note that

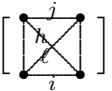
$$\left[\begin{array}{c} \alpha \\ \beta \\ \gamma \\ \delta \end{array} \right] = \left[\begin{array}{c} \bullet \\ \bullet \end{array} \right] = 2 + 3 + 3 + 2 + 2 + 2 = 14.$$

Alternatively, since every edge has two different configurations of type , we can compute

$$\left[\begin{array}{c} \alpha \\ \beta \\ \gamma \\ \delta \end{array} \right] = 7 \cdot 2 = 14.$$

Note that we count the number of $z_1 z_2 z_3 z_4$ with $z_1 = z_3$, $z_2 = z_4$ and adjacent $z_1 \sim z_2$, hence we count the number of ordered edges $z_1 z_2$.

Note that the family of graphs that we have in the next example is interesting for study on its own. This family is nonempty; for example, every hypercube satisfies the given condition.

Example 4.2 Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter d , valency n and assume that for all i, j and h ($0 \leq i, j, h \leq d$), the sizes $|\Gamma_{ijh}| = |\Gamma_{ijh}(x, y, z)| = |\Gamma_i(x) \cap \Gamma_j(y) \cap \Gamma_h(z)|$ do not depend on the choice of vertices $x, y, z \in X$ but only on their distances $\partial(x, y)$, $\partial(x, z)$ and $\partial(y, z)$. (Note the similarity between this condition and the definition of a distance-regular graph: $\Gamma = (X, R)$ is a distance-regular graph if and only if for any i, j ($0 \leq i, j \leq d$), $|\Gamma_{ij}| = |\Gamma_{ij}(x, y)| = |\Gamma_i(x) \cap \Gamma_j(y)|$ does not depend on the choice of vertices $x, y \in X$ but only on the value of $\partial(x, y)$.) Then we can calculate the 4-point count 

in $4! = 24$ different ways. Four of them are the following, and in each of these examples we use names of vertices from the following drawing:



(i) Pick $x \in X$, $y \in \Gamma_1(x)$ and $z \in \Gamma_{hi}(x, y)$ and let $\Gamma_{j\ell 1} = \Gamma_{j\ell 1}(x, y, z)$. By identifying $\alpha = x$, $\beta = y$ and $\gamma = z$, we derive

$$\left[\begin{array}{c} \alpha \\ \beta \\ \gamma \\ \delta \end{array} \right] = v_\Gamma \cdot n \cdot p_{hi}^1 \cdot |\Gamma_{j\ell 1}|.$$

(ii) Pick $x \in X$, $y \in \Gamma_i(x)$ and $z \in \Gamma_{\ell 1}(x, y)$ and let $\Gamma_{1hj} = \Gamma_{1hj}(x, y, z)$. By identifying $\beta = x$, $\gamma = y$ and $\delta = z$, we derive

$$\left[\begin{array}{c} j \\ \text{---} \\ \text{---} \\ \text{---} \\ i \end{array} \right] = v_\Gamma \cdot n_i \cdot p_{\ell 1}^i \cdot |\Gamma_{1hj}|.$$

(iii) Pick $x \in X$, $y \in \Gamma_1(x)$ and $z \in \Gamma_{hj}(x, y)$ and let $\Gamma_{i\ell 1} = \Gamma_{i\ell 1}(x, y, z)$. By identifying $\gamma = x$, $\delta = y$ and $\alpha = z$, we derive

$$\left[\begin{array}{c} j \\ \text{---} \\ \text{---} \\ \text{---} \\ i \end{array} \right] = v_\Gamma \cdot n \cdot p_{hj}^1 \cdot |\Gamma_{i\ell 1}|.$$

(iv) Pick $x \in X$, $y \in \Gamma_j(x)$ and $z \in \Gamma_{\ell 1}(x, y)$ and let $\Gamma_{1hi} = \Gamma_{1hi}(x, y, z)$. By identifying $\delta = x$, $\alpha = y$ and $\beta = z$, we derive

$$\left[\begin{array}{c} j \\ \text{---} \\ \text{---} \\ \text{---} \\ i \end{array} \right] = v_\Gamma \cdot n_j \cdot p_{\ell 1}^j \cdot |\Gamma_{1hi}|.$$

■

The results in Lemma 4.3 are well known (see [3, Chapter 4] or [6, Section 2.4]). We re-prove them using t -point counts.

Lemma 4.3 *The following hold.*

(i) *Assume $i + j \leq d$. Then $c_i \leq b_j$ and*

$$c_i = b_j \quad \text{if and only if} \quad \left[\begin{array}{c} i+j \quad j+1 \\ \text{---} \quad \text{---} \\ \text{---} \quad \text{---} \\ \text{---} \quad \text{---} \\ i \end{array} \right] = 0 = \left[\begin{array}{c} i+j \quad j+1 \\ \text{---} \quad \text{---} \\ \text{---} \quad \text{---} \\ \text{---} \quad \text{---} \\ i \end{array} \right].$$

(ii)

$$p_{ij}^{h+1} = \frac{1}{b_h} \left(b_{j-1} p_{i,j-1}^h + (a_j - a_h) p_{ij}^h + c_{j+1} p_{i,j+1}^h - c_h p_{ij}^{h-1} \right).$$

(iii)

$$p_{j+1,h}^i = \frac{1}{c_{j+1}} \left(p_{j,h-1}^i b_{h-1} + (a_h - a_j) p_{jh}^i + p_{j,h+1}^i c_{h+1} - b_{j-1} p_{j-1,h}^i \right).$$

Proof. (i) Note that

$$v_\Gamma n_i p_{j,i+j}^i c_i = \left[\begin{array}{c} i+j \\ \text{---} \\ \text{---} \\ \text{---} \\ i \end{array} \right] = \underbrace{\left[\begin{array}{c} i+j \quad j-1 \\ \text{---} \quad \text{---} \\ \text{---} \quad \text{---} \\ \text{---} \quad \text{---} \\ i \end{array} \right]}_{=0 \text{ (tr. ineq.)}} + \underbrace{\left[\begin{array}{c} i+j \quad j \\ \text{---} \quad \text{---} \\ \text{---} \quad \text{---} \\ \text{---} \quad \text{---} \\ i \end{array} \right]}_{=0 \text{ (tr. ineq.)}} + \left[\begin{array}{c} i+j \quad j+1 \\ \text{---} \quad \text{---} \\ \text{---} \quad \text{---} \\ \text{---} \quad \text{---} \\ i \end{array} \right]$$

and

$$v_\Gamma n_i p_{j,i+j}^i b_j = \left[\begin{array}{c} i+j \quad j+1 \\ \text{---} \quad \text{---} \\ \text{---} \quad \text{---} \\ \text{---} \quad \text{---} \\ i \end{array} \right] = \left[\begin{array}{c} i+j \quad j+1 \\ \text{---} \quad \text{---} \\ \text{---} \quad \text{---} \\ \text{---} \quad \text{---} \\ i \end{array} \right] + \left[\begin{array}{c} i+j \quad j+1 \\ \text{---} \quad \text{---} \\ \text{---} \quad \text{---} \\ \text{---} \quad \text{---} \\ i \end{array} \right] + \left[\begin{array}{c} i+j \quad j+1 \\ \text{---} \quad \text{---} \\ \text{---} \quad \text{---} \\ \text{---} \quad \text{---} \\ i \end{array} \right] \geq v_\Gamma n_i p_{j,i+j}^i c_i.$$

(ii) Since

$$\left[\begin{array}{c} i \quad h \\ \text{---} \quad \text{---} \\ \text{---} \quad \text{---} \\ \text{---} \quad \text{---} \\ j \quad k \end{array} \right] = \sum_{\ell=0}^d \left[\begin{array}{c} i \quad h \\ \text{---} \quad \text{---} \\ \text{---} \quad \text{---} \\ \text{---} \quad \text{---} \\ j \quad k \end{array} \right] = \sum_{\ell=0}^d v_\Gamma \cdot n_h \cdot p_{\ell k}^h \cdot p_{ij}^\ell$$

and

$$\left[\begin{array}{cc} i & j \\ \swarrow & \searrow \\ h & k \end{array} \right] = \sum_{r=0}^d \left[\begin{array}{ccc} i & & j \\ & \swarrow & \searrow \\ & r & \\ & \swarrow & \searrow \\ h & & k \end{array} \right] = \sum_{r=0}^d v_{\Gamma} \cdot n_h \cdot p_{ir}^h \cdot p_{jk}^r$$

we have

$$\sum_{\ell=0}^d p_{ij}^{\ell} p_{\ell k}^h = \sum_{r=0}^d p_{ir}^h p_{jk}^r.$$

Setting $k = 1$ gives $\sum_{\ell=0}^d p_{ij}^{\ell} p_{\ell 1}^h = \sum_{r=0}^d p_{ir}^h p_{j1}^r$, which yields

$$c_h p_{ij}^{h-1} + a_h p_{ij}^h + b_h p_{ij}^{h+1} = b_{j-1} p_{i,j-1}^h + a_j p_{ij}^h + c_{j+1} p_{i,j+1}^h$$

and with that

$$p_{ij}^{h+1} = \frac{1}{b_h} \left(b_{j-1} p_{i,j-1}^h + (a_j - a_h) p_{ij}^h + c_{j+1} p_{i,j+1}^h - c_h p_{ij}^{h-1} \right).$$

(iii) Similarly as before, we derive

$$\left[\begin{array}{ccc} & i & h \\ & \swarrow & \searrow \\ & j & k \end{array} \right] = v_{\Gamma} n_i \sum_{\ell=0}^d p_{h\ell}^i p_{jk}^{\ell} = v_{\Gamma} n_i \sum_{\ell=0}^d p_{j\ell}^i p_{hk}^{\ell},$$

which yields

$$\sum_{\ell=0}^d p_{h\ell}^i p_{jk}^{\ell} = \sum_{\ell=0}^d p_{j\ell}^i p_{hk}^{\ell}.$$

Setting $k = 1$ gives $\sum_{r=0}^d p_{hr}^i p_{j1}^r = \sum_{\ell=0}^d p_{j\ell}^i p_{h1}^{\ell}$, which yields

$$c_{j+1} p_{j+1,h}^i + a_j p_{jh}^i + b_{j-1} p_{j-1,h}^i = p_{j,h-1}^i b_{h-1} + p_{jh}^i a_h + p_{j,h+1}^i c_{h+1}$$

and with that

$$p_{j+1,h}^i = \frac{1}{c_{j+1}} \left(p_{j,h-1}^i b_{h-1} + (a_h - a_j) p_{jh}^i + p_{j,h+1}^i c_{h+1} - b_{j-1} p_{j-1,h}^i \right).$$

■

Lemma 4.4 ([3, pg. 134]) *We have*

$$p_{i,i-1}^1 = \frac{c_i n_i}{n} = \frac{b_{i-1} n_{i-1}}{n}, \quad p_{ii}^1 = \frac{a_i n_i}{n} \quad \text{and} \quad p_{i,i+1}^1 = \frac{b_i n_i}{n} \quad (1 \leq i \leq d);$$

$$p_{i+1,i-1}^2 = p_{i-1,i+1}^2 = \frac{b_2 b_3 \cdots b_i}{c_1 c_2 \cdots c_{i-1}} = \frac{n_i c_i b_i}{n b_1} \quad \text{and}$$

$$p_{i,i+1}^2 = p_{i+1,i}^2 = \frac{b_2 b_3 \cdots b_i}{c_1 c_2 \cdots c_i} (a_i + a_{i+1} - \lambda) = \frac{n_i b_i}{n b_1} (a_i + a_{i+1} - \lambda) \quad (2 \leq i \leq d-1);$$

$$p_{22}^2 = \frac{1}{c_2} (c_2 b_1 + a_2^2 + c_3 b_2 - n - \lambda a_2) = \frac{1}{c_2} (c_2 (b_1 - 1) + b_2 (c_3 - 1) + a_2 (a_2 - \lambda - 1)) \quad \text{and}$$

$$p_{ii}^2 = \frac{b_2 b_3 \cdots b_{i-1}}{c_1 c_2 \cdots c_i} (c_i b_{i-1} + a_i^2 + c_{i+1} b_i - n - \lambda a_i) \quad (3 \leq i \leq d-1);$$

$$p_{ij}^{i+j} = \frac{c_{i+1} \cdots c_{i+j}}{c_1 \cdots c_j}, \quad p_{i,j+1}^{i+j} = p_{ij}^{i+j} \frac{a_i + \cdots + a_{i+j} - a_1 - \cdots - a_j}{c_{j+1}},$$

$$p_{ij}^{i-j} = \frac{b_{i-1} \cdots b_{i-j}}{c_1 \cdots c_j}, \quad \text{and} \quad p_{i,j+1}^{i-j} = p_{ij}^{i-j} \frac{a_i + \cdots + a_{i-j} - a_1 - \cdots - a_j}{c_{j+1}}.$$

Also,

$$p_{2i}^i = \frac{1}{c_2} (c_i b_{i-1} + a_i (a_i - \lambda) + b_i c_{i+1} - n) \quad (2 \leq i \leq d-1).$$

Proof. By definition, $p_{0h}^i = \delta_{ih}$, $p_{j0}^i = \delta_{ij}$, $p_{j,d+1}^i = 0$, $p_{1,i-1}^i = c_i$, $p_{1i}^i = a_i$, $p_{1,i+1}^i = b_i$ and $p_{1h}^i = 0$ if $h \leq i-2$ or $h \geq i+2$. Now use Lemma 4.3(ii, iii) and induction on i . ■

Remark 4.5 In [5], COOLSAET and JURIŠIĆ introduced the notation of triple intersection numbers and used them to prove nonexistence of certain distance-regular graphs. In our notation their triple intersection numbers are in fact our 4-point counts with three vertices fixed (see Subsection 4.1). In Subsection 4.2, we have 5-point counts with four vertices fixed, i.e., we have quadruple intersection numbers.

4.1 Induced path of length 2

Pick $z_1 \in X$, $z_2 \in \Gamma_1(z_1)$ and $z_3 \in \Gamma_{21}(z_1, z_2)$ and abbreviate

$$\left[\begin{array}{c} \bullet \\ \swarrow \quad \searrow \\ \circ \quad \circ \\ \swarrow \quad \searrow \\ \circ \quad \circ \\ \hline z_1 \quad z_2 \quad z_3 \\ \hline 2 \end{array} \right] := |\Gamma_{ijk}(z_1, z_2, z_3)|.$$

Lemma 4.6 ([20, Section 2]) *For $z_1 \in X$, $z_2 \in \Gamma_1(z_1)$, $z_3 \in \Gamma_{21}(z_1, z_2)$ we have*

$$\left[\begin{array}{c} \bullet \\ \swarrow \quad \searrow \\ \circ \quad \circ \\ \swarrow \quad \searrow \\ \circ \quad \circ \\ \hline z_1 \quad z_2 \quad z_3 \\ \hline 2 \end{array} \right] = \left[\begin{array}{c} \bullet \\ \swarrow \quad \searrow \\ \circ \quad \circ \\ \swarrow \quad \searrow \\ \circ \quad \circ \\ \hline z_1 \quad z_2 \quad z_3 \\ \hline 2 \end{array} \right], \quad (12)$$

$$\left[\begin{array}{c} \bullet \\ \swarrow \quad \searrow \\ \circ \quad \circ \\ \swarrow \quad \searrow \\ \circ \quad \circ \\ \hline z_1 \quad z_2 \quad z_3 \\ \hline 2 \end{array} \right] = \left[\begin{array}{c} \bullet \\ \swarrow \quad \searrow \\ \circ \quad \circ \\ \swarrow \quad \searrow \\ \circ \quad \circ \\ \hline z_1 \quad z_2 \quad z_3 \\ \hline 2 \end{array} \right] \quad (13)$$

and

$$\left[\begin{array}{c} \bullet \\ \swarrow \quad \searrow \\ \circ \quad \circ \\ \swarrow \quad \searrow \\ \circ \quad \circ \\ \hline z_1 \quad z_2 \quad z_3 \\ \hline 2 \end{array} \right] + \left[\begin{array}{c} \bullet \\ \swarrow \quad \searrow \\ \circ \quad \circ \\ \swarrow \quad \searrow \\ \circ \quad \circ \\ \hline z_1 \quad z_2 \quad z_3 \\ \hline 2 \end{array} \right] = 2p_{i-1,i}^1 - p_{i-1,i+1}^2 - p_{i,i-2}^2 - p_{i,i-1}^2 \quad (14)$$

for all i ($2 \leq i \leq d-1$).

Proof. Let

$$\begin{aligned} D &= \left[\begin{array}{c} \bullet \\ \swarrow \quad \searrow \\ \circ \quad \circ \\ \swarrow \quad \searrow \\ \circ \quad \circ \\ \hline z_1 \quad z_2 \quad z_3 \\ \hline 2 \end{array} \right], & E &= \left[\begin{array}{c} \bullet \\ \swarrow \quad \searrow \\ \circ \quad \circ \\ \swarrow \quad \searrow \\ \circ \quad \circ \\ \hline z_1 \quad z_2 \quad z_3 \\ \hline 2 \end{array} \right], & F &= \left[\begin{array}{c} \bullet \\ \swarrow \quad \searrow \\ \circ \quad \circ \\ \swarrow \quad \searrow \\ \circ \quad \circ \\ \hline z_1 \quad z_2 \quad z_3 \\ \hline 2 \end{array} \right], \\ P &= \left[\begin{array}{c} \bullet \\ \swarrow \quad \searrow \\ \circ \quad \circ \\ \swarrow \quad \searrow \\ \circ \quad \circ \\ \hline z_1 \quad z_2 \quad z_3 \\ \hline 2 \end{array} \right], & Q &= \left[\begin{array}{c} \bullet \\ \swarrow \quad \searrow \\ \circ \quad \circ \\ \swarrow \quad \searrow \\ \circ \quad \circ \\ \hline z_1 \quad z_2 \quad z_3 \\ \hline 2 \end{array} \right] & \text{and} & R = \left[\begin{array}{c} \bullet \\ \swarrow \quad \searrow \\ \circ \quad \circ \\ \swarrow \quad \searrow \\ \circ \quad \circ \\ \hline z_1 \quad z_2 \quad z_3 \\ \hline 2 \end{array} \right]. \end{aligned}$$

Note that

$$p_{i-1,i}^1 = \left[\begin{array}{c} \bullet \\ \swarrow \quad \searrow \\ \circ \quad \circ \\ \swarrow \quad \searrow \\ \circ \quad \circ \\ \hline z_1 \quad z_2 \quad z_3 \\ \hline 2 \end{array} \right] = D + E + F, \quad p_{i-1,i}^1 = \left[\begin{array}{c} \bullet \\ \swarrow \quad \searrow \\ \circ \quad \circ \\ \swarrow \quad \searrow \\ \circ \quad \circ \\ \hline z_1 \quad z_2 \quad z_3 \\ \hline 2 \end{array} \right] = P + Q + R,$$

$$F = p_{i-1,i+1}^2, \quad P = p_{i,i-2}^2,$$

yielding

$$D + E = p_{i,i-1}^1 - p_{i-1,i+1}^2 \quad \text{and} \quad Q + R = p_{i,i-1}^1 - p_{i,i-2}^2. \quad (15)$$

Since

$$p_{i,i-1}^2 = \left[\begin{array}{c} \bullet \\ \swarrow \quad \searrow \\ \circ \quad \circ \\ \swarrow \quad \searrow \\ \circ \quad \circ \\ \hline z_1 \quad z_2 \quad z_3 \\ \hline 2 \end{array} \right] = E + Q,$$

from (15) we have

$$D + R = 2p_{i-1,i}^1 - p_{i-1,i+1}^2 - p_{i,i-2}^2 - p_{i,i-1}^2,$$

and thus (14) follows. Similarly, if we abbreviate

$$M = \left[\begin{array}{c} \bullet \\ \swarrow \quad \searrow \\ \circ \quad \circ \\ \swarrow \quad \searrow \\ \circ \quad \circ \\ \hline z_1 \quad z_2 \quad z_3 \\ \hline 2 \end{array} \right], N = \left[\begin{array}{c} \bullet \\ \swarrow \quad \searrow \\ \circ \quad \circ \\ \swarrow \quad \searrow \\ \circ \quad \circ \\ \hline z_1 \quad z_2 \quad z_3 \\ \hline 2 \end{array} \right], S = \left[\begin{array}{c} \bullet \\ \swarrow \quad \searrow \\ \circ \quad \circ \\ \swarrow \quad \searrow \\ \circ \quad \circ \\ \hline z_1 \quad z_2 \quad z_3 \\ \hline 2 \end{array} \right], T = \left[\begin{array}{c} \bullet \\ \swarrow \quad \searrow \\ \circ \quad \circ \\ \swarrow \quad \searrow \\ \circ \quad \circ \\ \hline z_1 \quad z_2 \quad z_3 \\ \hline 2 \end{array} \right],$$

then

$$p_{i-1,i}^1 = \left[\begin{array}{c} \bullet \\ \swarrow \quad \searrow \\ \circ \quad \circ \\ \swarrow \quad \searrow \\ \circ \quad \circ \\ \hline z_1 \quad z_2 \quad z_3 \\ \hline 2 \end{array} \right] = S + M + R, \quad p_{i-1,i}^1 = \left[\begin{array}{c} \bullet \\ \swarrow \quad \searrow \\ \circ \quad \circ \\ \swarrow \quad \searrow \\ \circ \quad \circ \\ \hline z_1 \quad z_2 \quad z_3 \\ \hline 2 \end{array} \right] = D + N + T,$$

$$T = p_{i-1,i+1}^2, \quad S = p_{i,i-2}^2,$$

yielding

$$D + N = p_{i,i-1}^1 - p_{i-1,i+1}^2 \quad \text{and} \quad M + R = p_{i,i-1}^1 - p_{i,i-2}^2. \quad (16)$$

By (15) and (16), we have $E = N$ and $M = Q$, so (12) and (13) follow. \blacksquare

Corollary 4.7 *We have*

$$c_i + b_{i-1} \geq \lambda + 2 \quad (17)$$

for all i ($2 \leq i \leq d-1$).

Proof. From Lemma 4.4 and the fact that $a_i + b_i = n - c_i$, we have

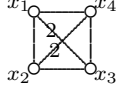
$$\begin{aligned} & 2p_{i-1,i}^1 - p_{i-1,i+1}^2 - p_{i,i-2}^2 - p_{i,i-1}^2 = \\ &= \frac{2n_{i-1}b_{i-1}}{n} - \frac{b_2b_3 \dots b_i}{c_1c_2 \dots c_{i-1}} - \frac{b_2b_3 \dots b_{i-1}}{c_1c_2 \dots c_{i-2}} - \frac{b_2b_3 \dots b_{i-1}}{c_1c_2 \dots c_{i-1}}(a_{i-1} + a_i - \lambda) \\ &= \frac{b_2b_3 \dots b_{i-1}}{c_1c_2 \dots c_{i-1}}(2b_1 - b_i - c_{i-1} - a_{i-1} - a_i + \lambda) \\ &= \frac{b_2b_3 \dots b_{i-1}}{c_1c_2 \dots c_{i-1}}(2(n - \lambda - 1) - n + c_i - n + b_{i-1} + \lambda) \\ &= \frac{b_2b_3 \dots b_{i-1}}{c_1c_2 \dots c_{i-1}}(c_i + b_{i-1} - \lambda - 2). \end{aligned}$$

Lemma 4.6 yields $2p_{i-1,i}^1 - p_{i-1,i+1}^2 - p_{i,i-2}^2 - p_{i,i-1}^2 \geq 0$ and the result follows. \blacksquare

Inequality (17) is due to TAYLOR & LEVINGSTON [16].

4.2 Induced quadrangle

Assume that there exist vertices x_1, x_2, x_3, x_4 consistent with the diagram



Now fix such a quadrangle and abbreviate

$$\left[\begin{array}{c} \text{diagram} \\ x_1 \quad x_4 \\ x_2 \quad x_3 \end{array} \right] := |\Gamma_{ijkl}(x_1, x_2, x_3, x_4)|.$$

Theorem 4.8 ([20]) *If Γ contains an induced quadrangle, then the following hold.*

$$(i) \quad (c_i - c_{i-1}) + (b_{i-1} - b_i) \geq \lambda + 2 \quad (18)$$

and equality holds if and only if

$$\left[\begin{array}{c} \text{diagram} \\ x_1 \quad x_4 \\ x_2 \quad x_3 \end{array} \right] = \left[\begin{array}{c} \text{diagram} \\ x_1 \quad x_4 \\ x_2 \quad x_3 \end{array} \right] = \left[\begin{array}{c} \text{diagram} \\ x_1 \quad x_4 \\ x_2 \quad x_3 \end{array} \right] = \left[\begin{array}{c} \text{diagram} \\ x_1 \quad x_4 \\ x_2 \quad x_3 \end{array} \right] = 0$$

for all i ($1 \leq i \leq d$).

(ii)

$$d \leq \frac{k + c_d}{\lambda + 2}$$

and equality holds if and only if equality holds in (18) for every $i = 1, 2, \dots, d$.

Proof. (i) We have

$$\begin{aligned} |\Gamma_{i,i-1,i}(x_1, x_2, x_3)| &= \left[\begin{array}{c} \text{diagram} \\ x_1 \quad x_4 \\ x_2 \quad x_3 \end{array} \right] = \left[\begin{array}{c} \text{diagram} \\ x_1 \quad x_4 \\ x_2 \quad x_3 \end{array} \right] + \left[\begin{array}{c} \text{diagram} \\ x_1 \quad x_4 \\ x_2 \quad x_3 \end{array} \right] + \left[\begin{array}{c} \text{diagram} \\ x_1 \quad x_4 \\ x_2 \quad x_3 \end{array} \right], \\ |\Gamma_{i-1,i,i-1}(x_1, x_2, x_3)| &= \left[\begin{array}{c} \text{diagram} \\ x_1 \quad x_4 \\ x_2 \quad x_3 \end{array} \right] = \left[\begin{array}{c} \text{diagram} \\ x_1 \quad x_4 \\ x_2 \quad x_3 \end{array} \right] + \left[\begin{array}{c} \text{diagram} \\ x_1 \quad x_4 \\ x_2 \quad x_3 \end{array} \right] + \left[\begin{array}{c} \text{diagram} \\ x_1 \quad x_4 \\ x_2 \quad x_3 \end{array} \right], \\ p_{i-1,i+1}^2 &= \left[\begin{array}{c} \text{diagram} \\ x_1 \quad x_4 \\ x_2 \quad x_3 \end{array} \right] = \left[\begin{array}{c} \text{diagram} \\ x_1 \quad x_4 \\ x_2 \quad x_3 \end{array} \right], \end{aligned}$$

and

$$p_{i,i-2}^2 = \left[\begin{array}{c} \text{diagram} \\ x_1 \quad x_4 \\ x_2 \quad x_3 \end{array} \right] = \left[\begin{array}{c} \text{diagram} \\ x_1 \quad x_4 \\ x_2 \quad x_3 \end{array} \right].$$

This yields

$$p_{i-1,i+1}^2 + p_{i,i-2}^2 \leq |\Gamma_{i,i-1,i}(x_1, x_2, x_3)| + |\Gamma_{i-1,i,i-1}(x_1, x_2, x_3)|. \quad (19)$$

By Lemma 4.6, we have the following equation for all $x_1 \in X$, $x_2 \in \Gamma_1(x)$ and $x_3 \in \Gamma_{21}(x_1, x_2)$:

$$|\Gamma_{i,i-1,i}(x_1, x_2, x_3)| + |\Gamma_{i-1,i,i-1}(x_1, x_2, x_3)| = 2p_{i,i-1}^1 - p_{i,i-1}^2 - p_{i,i-2}^2 - p_{i-1,i+1}^2. \quad (20)$$

From (19) and (20) we have

$$\begin{aligned}
2p_{i-1,i+1}^2 + 2p_{i,i-2}^2 &\leq 2p_{i,i-1}^1 - p_{i,i-1}^2, \\
\frac{b_2 b_3 \cdots b_{i-1}}{c_1 c_2 \cdots c_{i-1}} (2b_i + 2c_{i-1}) &\leq \frac{b_2 b_3 \cdots b_{i-1}}{c_1 c_2 \cdots c_{i-1}} (2b_1 - a_{i-1} - a_i + \lambda), \\
2b_i + 2c_{i-1} &\leq 2n - a_{i-1} - a_i - \lambda - 2, \\
b_i - b_{i-1} + c_{i-1} - c_i &\leq -\lambda - 2,
\end{aligned}$$

which yields (18). Equality holds in (19) (and with that in (18)) if and only if

$$\left[\begin{array}{c} i & & i-1 \\ & \nearrow & \searrow \\ x_1 & & x_4 \\ & \searrow & \nearrow \\ & x_2 & x_3 \end{array} \right] + \left[\begin{array}{c} i & & i \\ & \nearrow & \searrow \\ x_1 & & x_4 \\ & \searrow & \nearrow \\ & x_2 & x_3 \end{array} \right] = 0 \quad \text{and} \quad \left[\begin{array}{c} i-1 & & i-1 \\ & \nearrow & \searrow \\ x_1 & & x_4 \\ & \searrow & \nearrow \\ & x_2 & x_3 \end{array} \right] + \left[\begin{array}{c} i-1 & & i \\ & \nearrow & \searrow \\ x_1 & & x_4 \\ & \searrow & \nearrow \\ & x_2 & x_3 \end{array} \right] = 0.$$

(ii) Sum the inequalities from (18) for $i = 1, 2, \dots, d$. ■

Remark 4.9 The diameter bound of the above theorem is tight, and there are infinite families of distance-regular graphs achieving the bound. In fact, the characterization of the case of equality was done by TERWILLIGER in [21].

5 Koolen's inequalities

In [10, Lemmas 3.4, 3.5], HIRAKI re-proved Koolen's inequalities from [11, 12]. In Lemmas 5.1, 5.3 we also re-prove these inequalities using t -point counts. In fact, in this section we present how to obtain Koolen's inequalities using 5-point counts with three vertices fixed. In the follow-up paper, we will get inequalities (21) and (24) under different assumptions, i.e., we will get that an induced polygon of length $2e + 3$ yields (24) and that an induced polygon of length $2e + 2$ yields both (21) and (24).

Lemma 5.1 *If $c_e > c_{e-1}$ for some e ($2 \leq e \leq d$) then*

$$c_e \geq c_i + c_{e-i} \quad \text{for all } i = 1, 2, \dots, e-1. \quad (21)$$

Moreover, in that case $c_i + c_{e-i} = c_e$ if and only if

$$\left[\begin{array}{c} i & & w \\ & \nearrow & \searrow \\ & & e-i \\ & \searrow & \nearrow \\ & e-1 & e-i \\ & e-1 & \\ u & & v \\ & e & \end{array} \right] = 0$$

for any $u \in X$, $v \in \Gamma_e(u)$ and $w \in \Gamma_{i,e-i}(u, v)$.

Proof. Pick $u \in X$, $v \in \Gamma_e(u)$, $w \in \Gamma_{i,e-i}(u, v)$ and define $Y := C_i(w, u)$ and $Z := C_e(u, v)/C_{e-i}(w, v)$. We count the size of the set

$$\Omega := Y \times Z$$

Similarly, we can express the product $(c_e - c_{e-i})(c_e - c_{e-1})$:

$$\begin{aligned}
c_{e-i} &= \left[\begin{array}{c} i \\ \circ \\ \text{---} \\ \circ \\ u \end{array} \begin{array}{c} w \\ \circ \\ \text{---} \\ \circ \\ v \end{array} \right] = \left[\begin{array}{c} i \\ \circ \\ \text{---} \\ \circ \\ u \end{array} \begin{array}{c} w \\ \circ \\ \text{---} \\ \circ \\ v \end{array} \right], \\
c_e &= \left[\begin{array}{c} i \\ \circ \\ \text{---} \\ \circ \\ u \end{array} \begin{array}{c} w \\ \circ \\ \text{---} \\ \circ \\ v \end{array} \right] = \left[\begin{array}{c} i \\ \circ \\ \text{---} \\ \circ \\ u \end{array} \begin{array}{c} w \\ \circ \\ \text{---} \\ \circ \\ v \end{array} \right] + \left[\begin{array}{c} i \\ \circ \\ \text{---} \\ \circ \\ u \end{array} \begin{array}{c} w \\ \circ \\ \text{---} \\ \circ \\ v \end{array} \right], \\
c_e - c_{e-i} &= \left[\begin{array}{c} i \\ \circ \\ \text{---} \\ \circ \\ u \end{array} \begin{array}{c} w \\ \circ \\ \text{---} \\ \circ \\ v \end{array} \right], \\
(c_e - c_{e-i})c_e &= \left[\begin{array}{c} i \\ \circ \\ \text{---} \\ \circ \\ u \end{array} \begin{array}{c} w \\ \circ \\ \text{---} \\ \circ \\ v \end{array} \right] = \left[\begin{array}{c} i \\ \circ \\ \text{---} \\ \circ \\ u \end{array} \begin{array}{c} w \\ \circ \\ \text{---} \\ \circ \\ v \end{array} \right] + \left[\begin{array}{c} i \\ \circ \\ \text{---} \\ \circ \\ u \end{array} \begin{array}{c} w \\ \circ \\ \text{---} \\ \circ \\ v \end{array} \right], \\
(c_e - c_{e-i})c_{e-1} &= \left[\begin{array}{c} i \\ \circ \\ \text{---} \\ \circ \\ u \end{array} \begin{array}{c} w \\ \circ \\ \text{---} \\ \circ \\ v \end{array} \right] = \left[\begin{array}{c} i \\ \circ \\ \text{---} \\ \circ \\ u \end{array} \begin{array}{c} w \\ \circ \\ \text{---} \\ \circ \\ v \end{array} \right], \\
(c_e - c_{e-i})(c_e - c_{e-1}) &= \left[\begin{array}{c} i \\ \circ \\ \text{---} \\ \circ \\ u \end{array} \begin{array}{c} w \\ \circ \\ \text{---} \\ \circ \\ v \end{array} \right] + \left[\begin{array}{c} i \\ \circ \\ \text{---} \\ \circ \\ u \end{array} \begin{array}{c} w \\ \circ \\ \text{---} \\ \circ \\ v \end{array} \right] = \\
&= \left[\begin{array}{c} i \\ \circ \\ \text{---} \\ \circ \\ u \end{array} \begin{array}{c} w \\ \circ \\ \text{---} \\ \circ \\ v \end{array} \right] + \left[\begin{array}{c} i \\ \circ \\ \text{---} \\ \circ \\ u \end{array} \begin{array}{c} w \\ \circ \\ \text{---} \\ \circ \\ v \end{array} \right] + \left[\begin{array}{c} i \\ \circ \\ \text{---} \\ \circ \\ u \end{array} \begin{array}{c} w \\ \circ \\ \text{---} \\ \circ \\ v \end{array} \right] + \left[\begin{array}{c} i \\ \circ \\ \text{---} \\ \circ \\ u \end{array} \begin{array}{c} w \\ \circ \\ \text{---} \\ \circ \\ v \end{array} \right] + \\
&+ \left[\begin{array}{c} i \\ \circ \\ \text{---} \\ \circ \\ u \end{array} \begin{array}{c} w \\ \circ \\ \text{---} \\ \circ \\ v \end{array} \right] + \left[\begin{array}{c} i \\ \circ \\ \text{---} \\ \circ \\ u \end{array} \begin{array}{c} w \\ \circ \\ \text{---} \\ \circ \\ v \end{array} \right] + \left[\begin{array}{c} i \\ \circ \\ \text{---} \\ \circ \\ u \end{array} \begin{array}{c} w \\ \circ \\ \text{---} \\ \circ \\ v \end{array} \right]. \quad (23)
\end{aligned}$$

Now it is not hard to see that (22) \leq (23), i.e., that $c_i(c_e - c_{e-1}) \leq (c_e - c_{e-i})(c_e - c_{e-1})$. The result follows. \blacksquare

Corollary 5.2 *If $c_{i+q} > c_{i+q-1}$ for some $i+q$ ($2 \leq i+q \leq d$, $q \geq 1$) then*

$$c_{i+q} \geq c_i + c_q.$$

Proof. Immediate from Lemma 5.1. \blacksquare

Lemma 5.3 *If $b_e > b_{e+1}$ for some e ($2 \leq e \leq d$) then*

$$b_e \geq c_i + b_{e+i} \quad \text{for all } i = 1, 2, \dots, d - e. \quad (24)$$

Moreover, in that case $c_i + b_{e+i} = b_e$ if and only if

$$\left[\begin{array}{c} \text{Diagram: A circle with vertices } u, v, w, i. \text{ Edges } e, e+1, e+i, e+i+1. \text{ Labels } >i, \leq e+1, e+1, e+1. \end{array} \right] = 0$$

for any $u \in X$, $v \in \Gamma_e(u)$ and $w \in \Gamma_{i,e+i}(u, v)$.

Proof. Pick $u \in X$, $v \in \Gamma_e(u)$ and $w \in \Gamma_{i,e+i}(u, v)$. Let $Y := B_{e+i}(w, v) = \Gamma_{e+i+1,1}(w, v)$ and $Z := B_e(v, u) \setminus C_i(x, u)$. By counting the size of the set

$$\{(y, z) \in Y \times Z \mid \partial(y, z) \leq e + 1\}$$

in two ways, similarly to the proof of Lemma 5.1, we have

$$b_{e+i}(b_e - b_{e+1}) = \left[\begin{array}{c} \text{Diagram: Similar to the one above, but with labels } >i, \leq e+1, e+1, e+1, e+i+1. \end{array} \right]$$

and

$$(b_e - c_i)(b_e - b_{e+1}) = \left[\begin{array}{c} \text{Diagram: Similar to the one above, but with labels } >i, \leq e+1, e+1, e+1. \end{array} \right].$$

The result follows. ■

5.1 A remark about a similar technique in the literature

As far as we know, the technique and notion of t -point counts that we have in this and the follow-up paper is new. On the other hand, this technique has appeared implicitly in the literature. For the moment let $(X, \{R_i\}_{0 \leq i \leq d})$ denote a d -class P - and Q - polynomial association scheme. In [19], TERWILLIGER uses a map f from EK_4 (the set of all 2-element subsets of a 4-element set $\{1, \dots, 4\}$) to $\{1, \dots, d\}$, and defines that a 4-tuple (x_1, x_2, x_3, x_4) (of elements in X) has type f if $(x_i, x_j) \in R_{f(i,j)}$. He defines N_f as the total number of 4-tuples from X of type f and he defines N_i^* as the total number of 4-tuples from X of type i , where $i : EK_4 \rightarrow \{0, \dots, d\}$ is a constant function $i(\{s_1, s_2\}) = i$, for all $\{s_1, s_2\} \in EK_4$. Switching to our notation, we have

$$N_f = \left[\begin{array}{c} \text{Diagram: A square with vertices and edges labeled } f_{14}, f_{13}, f_{24}, f_{23}, f_{12}, f_{34}. \end{array} \right]$$

where $f_{kj} \in \{0, \dots, d\}$ are fixed, and for all $1 \leq i \leq d$,

$$N_i^* = \left[\begin{array}{c} \text{Diagram: A square with vertices and edges labeled } i. \end{array} \right].$$

In [19, Theorem 1], the author proved that for a P - and Q -polynomial association scheme Y , all the numbers N_f can be computed from the intersection numbers of Y and $N_1^*, N_2^*, \dots, N_p^*$, where $p = \min\{\lfloor \frac{d}{2} \rfloor, f_{12}, f_{13}, f_{14}, f_{23}, f_{24}, f_{34}\}$.

Next, in [10, Section 3], HIRAKI uses 5-point counts with 3 fixed vertices. In fact, the technique from (our) Section 5 is very similar to [10, Section 3]; the only difference is in notation.

Also, for example, the PhD thesis of DICKIE [7, Chapter 3] implicitly contains various 4-point counts with 3 fixed vertices (see [7, p. 25]) and 4-point counts with 2 fixed vertices (see [7, p. 27]).

It is hard to count all the references from the literature where the technique appears implicitly, and we don't know whether there are many of them, or just these few that we mention here. Let us mention one more. In [22, Equation (47)], TERWILLIGER has an equality involving two 4-point counts with 3 fixed vertices.

6 4-point counts A_i , B_i , C_i , D_i , E_i and F_i

In this section we consider the 4-point counts containing two disjoint edges and define the rational numbers

$$e_i := \frac{1}{v_\Gamma n_i c_i} \left[\begin{array}{c} i \\ \begin{array}{|c|} \hline \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \\ \hline \end{array} \\ i-1 \\ i \end{array} \right] \geq 0. \quad (25)$$

The names introduced for these 4-point counts will be used throughout.

Proposition 6.1 *With the notation of Definition 3.1 we have*

$$\begin{aligned} A_i &:= \left[\begin{array}{c} i \\ \begin{array}{|c|} \hline \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \\ \hline \end{array} \\ i-1 \\ i \end{array} \right] = v_\Gamma n_i \left(a_i^2 - c_i(b_{i-1} - b_i - e_i) - b_i(c_{i+1} - c_i - e_{i+1}) \right), \\ B_i &:= \left[\begin{array}{c} i \\ \begin{array}{|c|} \hline \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \\ \hline \end{array} \\ i-1 \\ i \end{array} \right] = \left[\begin{array}{c} i-1 \\ \begin{array}{|c|} \hline \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \\ \hline \end{array} \\ i \\ i \end{array} \right] = v_\Gamma n_i c_i \left(a_{i-1} - (c_i - c_{i-1} - e_i) \right) = v_\Gamma n_i c_i \left(a_i - (b_{i-1} - b_i - e_i) \right), \\ C_i &:= \left[\begin{array}{c} i+1 \\ \begin{array}{|c|} \hline \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \\ \hline \end{array} \\ i-1 \\ i \end{array} \right] = \left[\begin{array}{c} i \\ \begin{array}{|c|} \hline \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \\ \hline \end{array} \\ i-1 \\ i \end{array} \right] = v_\Gamma n_i c_i b_i, \\ D_i &:= \left[\begin{array}{c} i \\ \begin{array}{|c|} \hline \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \\ \hline \end{array} \\ i-1 \\ i \end{array} \right] = \left[\begin{array}{c} i-1 \\ \begin{array}{|c|} \hline \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \\ \hline \end{array} \\ i \\ i \end{array} \right] = \left[\begin{array}{c} i \\ \begin{array}{|c|} \hline \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \\ \hline \end{array} \\ i-1 \\ i \end{array} \right] = v_\Gamma n_i c_i (b_{i-1} - b_i - e_i), \\ E_i &:= \left[\begin{array}{c} i \\ \begin{array}{|c|} \hline \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \\ \hline \end{array} \\ i-1 \\ i \end{array} \right] = \left[\begin{array}{c} i-1 \\ \begin{array}{|c|} \hline \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \\ \hline \end{array} \\ i-1 \\ i \end{array} \right] = v_\Gamma n_i c_i e_i, \\ F_i &:= \left[\begin{array}{c} i \\ \begin{array}{|c|} \hline \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \\ \hline \end{array} \\ i-1 \\ i \end{array} \right] = \left[\begin{array}{c} i-1 \\ \begin{array}{|c|} \hline \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \\ \hline \end{array} \\ i-1 \\ i \end{array} \right] = v_\Gamma n_i c_i (c_i - c_{i-1} - e_i), \end{aligned}$$

for all i ($2 \leq i \leq d$).

Proof. By summing over distances (see (5) and (7)) we find

$$\begin{aligned} v_\Gamma n_{i+1} c_{i+1} c_i &= v_\Gamma n_i c_i b_i = C_i, \\ v_\Gamma n_i c_i a_i &= B_i + D_i, \\ v_\Gamma n_i c_i a_{i-1} &= F_i + B_i, \\ v_\Gamma n_i c_i b_{i-1} &= D_i + E_i + C_i, \\ v_\Gamma n_i c_i^2 &= C_{i-1} + E_i + F_i, \\ v_\Gamma n_i a_i^2 &= D_i + A_i + F_{i+1} \end{aligned}$$

(for example $v_\Gamma n_i a_i^2 = \sum_{h=0}^d \left[\begin{array}{c} h \\ \begin{array}{|c|} \hline \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \\ \hline \end{array} \\ i \end{array} \right] = \left[\begin{array}{c} i-1 \\ \begin{array}{|c|} \hline \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \\ \hline \end{array} \\ i \end{array} \right] + \left[\begin{array}{c} i \\ \begin{array}{|c|} \hline \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \\ \hline \end{array} \\ i \end{array} \right] + \left[\begin{array}{c} i+1 \\ \begin{array}{|c|} \hline \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \\ \hline \end{array} \\ i \end{array} \right]$). Now (25) implies that $E_i = v_\Gamma n_i c_i e_i$, and solving the resulting triangular linear system of equations gives the above formulas. \blacksquare

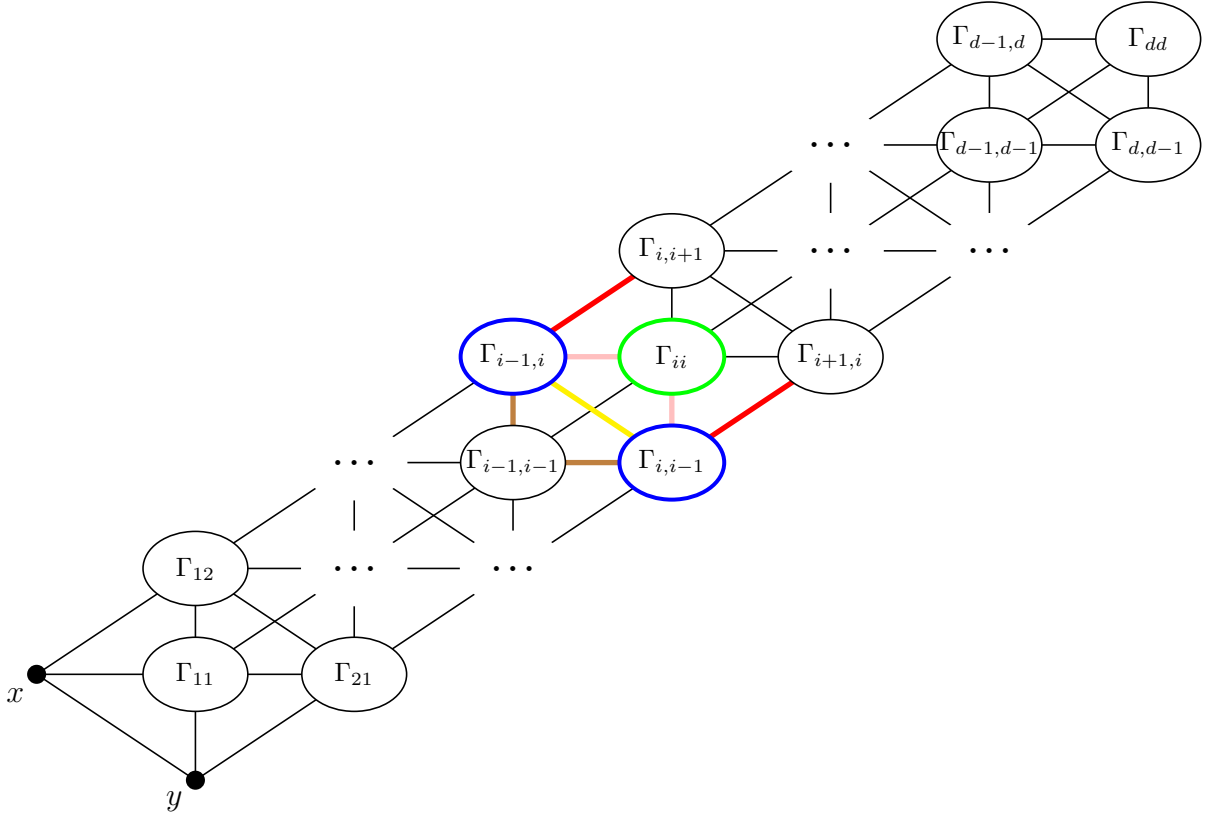


Figure 4: Intersection diagram (of rank 1) with respect to (x, y) (that is $x \in X$, $y \in \Gamma_1(x)$, $\Gamma_{ij} = \Gamma_i(x) \cap \Gamma_j(y)$). Note that: (1) $e_i > 0$ iff there exist $x \in X$ and $y \in \Gamma_1(x)$ such that there is at least one edge between $\Gamma_{i-1,i}(x, y)$ and $\Gamma_{i,i-1}(x, y)$ (a yellow edge exists); (2) $B_i > 0$ iff there exist $x \in X$ and $y \in \Gamma_1(x)$ such that there is at least one edge inside $\Gamma_{i-1,i}(x, y)$ or inside $\Gamma_{i,i-1}(x, y)$ (an edge within one of the blue sets exists); (3) $A_i > 0$ iff there exist $x \in X$ and $y \in \Gamma_1(x)$ such that there is at least one edge inside $\Gamma_{ii}(x, y)$ (an edge within the green set exists). (4) $D_i > 0$ iff there exist $x \in X$ and $y \in \Gamma_1(x)$ such that there is at least one edge between $\Gamma_{ii}(x, y)$ and $\Gamma_{i-1,i}(x, y) \cup \Gamma_{i,i-1}(x, y)$ (a pink edge exists); (5) $C_i > 0$ for every i ($1 \leq i \leq d-1$) (the red edges exist); (6) $F_i > 0$ iff there exist $x \in X$ and $y \in \Gamma_1(x)$ such that there is at least one edge between $\Gamma_{i-1,i-1}(x, y)$ and $\Gamma_{i-1,i}(x, y) \cup \Gamma_{i,i-1}(x, y)$ (a brown edge exists).

Remark 6.2 If $e_i = 0$, then the counts B_i , D_i , F_i are all constant. This makes it easier to count bigger configurations.

A Terwilliger graph is a non-complete graph Γ such that, for each pair of vertices u, v at distance two, $\Gamma_1(u) \cap \Gamma_1(v)$ is a clique of size μ (for some fixed $\mu \geq 2$). The case $e_2 = 0$ corresponds to Terwilliger graphs, which are very restricted in structure. In [21], TERWILLIGER showed that if Γ is a distance-regular graph with diameter $d \geq \frac{k + c_d}{\lambda + 2}$, then one of the following holds: (i) Γ is a Terwilliger graph; (ii) Γ is strongly-regular with smallest eigenvalue -2 ; (iii) Γ is a Hamming graph, a Doob graph, a locally Petersen graph, a Johnson graph, a half cube or the Gosset graph.

Corollary 6.3 *With reference to Proposition 6.1, let $x^+ = \max(x, 0)$. Then*

$$b_{i-1} \geq b_i, \quad c_i \geq c_{i-1}, \quad (26)$$

$$c_i(a_i - a_{i-1})^+ + b_i(a_i - a_{i+1})^+ \leq a_i^2, \quad (27)$$

$$\max \left(0, c_i - c_{i-1} - a_{i-1}, b_{i-1} - b_i - \frac{a_i^2}{c_i}, c_i - c_{i-1} - \frac{a_{i-1}^2}{b_{i-1}} \right) \leq e_i, \quad (28)$$

$$e_i \leq \min(b_{i-1} - b_i, c_i - c_{i-1}), \quad (29)$$

$$b_{i-1} = b_i \iff D_i = E_i = 0, \quad (30)$$

$$c_{i-1} = c_i \iff E_i = F_i = 0. \quad (31)$$

Proof. (26), (30), (31), and (27) follow from

$$0 \leq D_i + E_i = v_\Gamma n_i c_i (b_{i-1} - b_i),$$

$$0 \leq E_i + F_i = v_\Gamma n_i c_i (c_i - c_{i-1}),$$

$$D_i - F_i = v_\Gamma n_i c_i (a_i - a_{i-1}), \quad D_{i+1} - F_{i+1} = v_\Gamma n_i b_i (a_{i+1} - a_i),$$

$$(D_i - F_i)^+ + (F_{i+1} - D_{i+1})^+ \leq D_i + F_{i+1} + A_i = v_\Gamma n_i a_i^2.$$

(29) follows from $D_i \geq 0$, $F_i \geq 0$, and (28) from $B_i \geq 0$, $A_i + \frac{b_i}{c_{i+1}} F_{i+1} \geq 0$ and $A_{i-1} + D_{i-1} \geq 0$. \blacksquare

Inequalities (26) and (27) are due to BIGGS [2] and BROUWER & LAMBECK [4], respectively.

Corollary 6.4 (NOMURA [14]) *With reference to Proposition 6.3, the following hold.*

$$a_{i+1} \geq a_i \left(1 - \frac{a_i}{b_i} \right), \quad a_i \geq a_{i+1} \left(1 - \frac{a_{i+1}}{c_{i+1}} \right),$$

$$0 < a_i < b_i \quad \Rightarrow \quad a_{i+1} > 0$$

$$0 < a_{i+1} < c_{i+1} \quad \Rightarrow \quad a_i > 0$$

Proof. Note that (27) yields $a_i^2 \geq b_i(a_i - a_{i+1})$ and $a_{i+1}^2 \geq c_{i+1}(a_{i+1} - a_i)$. The results follow. \blacksquare

Corollary 6.5 *With reference to Proposition 6.1, the following hold.*

$$a_{i-1} \geq c_i - c_{i-1} - e_i, \quad (32)$$

$$a_i \geq b_{i-1} - b_i - e_i, \quad (33)$$

$$e_i = 0 \quad \Rightarrow \quad a_{i-1} \geq c_i - c_{i-1} \quad \text{and} \quad a_i \geq b_{i-1} - b_i, \quad (34)$$

$$c_i e_i + b_i e_{i+1} \geq b_i (c_{i+1} - c_i) + c_i (b_{i-1} - b_i) - a_i^2, \quad (35)$$

$$c_i = c_{i-1} \text{ or } b_i = b_{i-1} \quad \Rightarrow \quad e_i = 0, \quad (36)$$

$$b_i = b_{i-1} \text{ and } a_i \neq 0 \quad \Rightarrow \quad a_{i-1} > c_i - c_{i-1}, \quad (37)$$

$$c_i = c_{i-1} \text{ and } a_{i-1} \neq 0 \quad \Rightarrow \quad a_i > b_{i-1} - b_i. \quad (38)$$

Proof. (32) and (33) follow from $B_i \geq 0$. (34) follows immediately from (32) and (33). (35) follows from $A_i \geq 0$. (36) follows from (29). If $b_i = b_{i-1}$ and $a_i \neq 0$, then $e_i = 0$ and $B_i > 0$. This yields (37). Similarly, if $c_i = c_{i-1}$ and $a_{i-1} \neq 0$, then $e_i = 0$ and $B_i > 0$. This yields (38). \blacksquare

6.1 Case when $B_i = 0$

In this subsection we consider the case when $B_i = 0$.

Lemma 6.6 *With reference to Proposition 6.1, pick i ($2 \leq i \leq d-1$).*

(I) *Assume that $B_i = 0$. Then the following hold.*

- (i) *If $a_{i-1} \neq 0$ then $b_{i-1} \leq a_{i-1}$.*
- (ii) *Assume $a_{i-1} \neq 0$. Then*

$$b_{i-1} = a_{i-1} \quad \text{if and only if} \quad A_{i-1} = 0 \quad \text{and} \quad D_{i-1} = 0.$$

Moreover in this case $B_{i-1} \neq 0$.

- (iii) *$a_{i-1} \neq 0$ if and only if $F_i \neq 0$.*

(II) *Assume that $a_{i-1} \neq 0$ so that we can pick $x \in X$, $y \in \Gamma_{i-1}(x)$ and $z \in \Gamma_{i-1,1}(y, x)$. The following hold.*

$$(i) \text{ If } B_{i-1} = 0 \text{ then } c_{i-1} = \left[\begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \circ \quad \circ \\ \hline \bullet \\ \diagup \quad \diagdown \\ \circ \quad \circ \\ \hline \bullet \\ \diagup \quad \diagdown \\ \circ \quad \circ \\ \hline x \quad y \quad z \end{array} \right] = \left[\begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \circ \quad \circ \\ \hline \bullet \\ \diagup \quad \diagdown \\ \circ \quad \circ \\ \hline \bullet \\ \diagup \quad \diagdown \\ \circ \quad \circ \\ \hline x \quad y \quad z \end{array} \right].$$

$$(ii) \text{ If } B_i = 0 \text{ then } b_{i-1} = \left[\begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \circ \quad \circ \\ \hline \bullet \\ \diagup \quad \diagdown \\ \circ \quad \circ \\ \hline \bullet \\ \diagup \quad \diagdown \\ \circ \quad \circ \\ \hline x \quad y \quad z \end{array} \right] = \left[\begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \circ \quad \circ \\ \hline \bullet \\ \diagup \quad \diagdown \\ \circ \quad \circ \\ \hline \bullet \\ \diagup \quad \diagdown \\ \circ \quad \circ \\ \hline x \quad y \quad z \end{array} \right].$$

Proof. (I) By assumption $B_i = \left[\begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \hline \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \hline \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \hline \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \hline x \quad y \quad z \end{array} \right] = 0$. Assume $a_{i-1} \neq 0$. Then we can pick $x \in X$, $y \in \Gamma_{i-1}(x)$ and $z \in A_{i-1}(y, x) = \Gamma_{i-1,1}(y, x)$. We have

$$b_{i-1} = \left[\begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \circ \quad \circ \\ \hline \bullet \\ \diagup \quad \diagdown \\ \circ \quad \circ \\ \hline \bullet \\ \diagup \quad \diagdown \\ \circ \quad \circ \\ \hline x \quad y \quad z \end{array} \right] = \left[\begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \circ \quad \circ \\ \hline \bullet \\ \diagup \quad \diagdown \\ \circ \quad \circ \\ \hline \bullet \\ \diagup \quad \diagdown \\ \circ \quad \circ \\ \hline x \quad y \quad z \end{array} \right] + \underbrace{\left[\begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \circ \quad \circ \\ \hline \bullet \\ \diagup \quad \diagdown \\ \circ \quad \circ \\ \hline \bullet \\ \diagup \quad \diagdown \\ \circ \quad \circ \\ \hline x \quad y \quad z \end{array} \right]}_{=0 (B_i=0)},$$

and

$$a_{i-1} = \left[\begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \circ \quad \circ \\ \hline \bullet \\ \diagup \quad \diagdown \\ \circ \quad \circ \\ \hline \bullet \\ \diagup \quad \diagdown \\ \circ \quad \circ \\ \hline x \quad y \quad z \end{array} \right] = \left[\begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \circ \quad \circ \\ \hline \bullet \\ \diagup \quad \diagdown \\ \circ \quad \circ \\ \hline \bullet \\ \diagup \quad \diagdown \\ \circ \quad \circ \\ \hline x \quad y \quad z \end{array} \right] + \left[\begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \circ \quad \circ \\ \hline \bullet \\ \diagup \quad \diagdown \\ \circ \quad \circ \\ \hline \bullet \\ \diagup \quad \diagdown \\ \circ \quad \circ \\ \hline x \quad y \quad z \end{array} \right] + \underbrace{\left[\begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \circ \quad \circ \\ \hline \bullet \\ \diagup \quad \diagdown \\ \circ \quad \circ \\ \hline \bullet \\ \diagup \quad \diagdown \\ \circ \quad \circ \\ \hline x \quad y \quad z \end{array} \right]}_{=b_{i-1}}.$$

This yields (i). Note that $D_{i-1} = \left[\begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \hline \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \hline \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \hline \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \hline x \quad y \quad z \end{array} \right] = \left[\begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \hline \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \hline \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \hline \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \hline x \quad y \quad z \end{array} \right]$, $A_{i-1} = \left[\begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \hline \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \hline \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \hline \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \hline x \quad y \quad z \end{array} \right]$, $B_{i-1} =$

$$\left[\begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \hline \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \hline \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \hline \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \hline x \quad y \quad z \end{array} \right] \text{ and}$$

$$c_{i-1} = \left[\begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \circ \quad \circ \\ \hline \bullet \\ \diagup \quad \diagdown \\ \circ \quad \circ \\ \hline \bullet \\ \diagup \quad \diagdown \\ \circ \quad \circ \\ \hline x \quad y \quad z \end{array} \right] = \left[\begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \circ \quad \circ \\ \hline \bullet \\ \diagup \quad \diagdown \\ \circ \quad \circ \\ \hline \bullet \\ \diagup \quad \diagdown \\ \circ \quad \circ \\ \hline x \quad y \quad z \end{array} \right] + \left[\begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \circ \quad \circ \\ \hline \bullet \\ \diagup \quad \diagdown \\ \circ \quad \circ \\ \hline \bullet \\ \diagup \quad \diagdown \\ \circ \quad \circ \\ \hline x \quad y \quad z \end{array} \right].$$

This yields (ii). Claim (iii) follows from $B_i + F_i = v_{\Gamma} n_i c_i a_{i-1}$.

(II) Follows immediately from the proof of (I). ■

Lemma 6.7 *With reference to Proposition 6.1, pick i ($2 \leq i \leq d-1$) and assume that $B_i = 0$. Then the following hold.*

- (i) *If $a_i \neq 0$ then $c_i \leq a_i$.*
- (ii) *Assume $a_i \neq 0$. Then*

$$c_i = a_i \quad \text{if and only if} \quad A_i = 0 \quad \text{and} \quad F_{i+1} = 0.$$

Moreover in this case $B_{i+1} \neq 0$.

- (iii) *$a_i \neq 0$ if and only if $D_i \neq 0$.*

Proof. Similar to the proof of Lemma 6.6. ■

Corollary 6.8 *With reference to Proposition 6.1, pick i ($2 \leq i \leq d-1$) and assume that $B_i = 0$. Then the following hold.*

- (i) *If $a_{i-1} \neq 0$ and $a_i \neq 0$ then*

$$c_i > c_{i-1} \quad \text{and} \quad b_{i-1} > b_i.$$

- (ii) *$e_i = 0$ if and only if $c_i = a_{i-1} + c_{i-1}$ and $b_{i-1} = a_i + b_i$.*

Proof. Claim (i) follows immediately from (30), (31) and Lemmas 6.6(iii) and 6.7(iii). Claim (ii) follows from the formulas for B_i in Proposition 6.1. ■

Proposition 6.9 *With reference to Proposition 6.1, pick i ($2 \leq i \leq d-1$) and assume that $a_{i-1} \neq 0$. The following hold.*

- (i) *If $A_{i-1} = B_{i-1} = B_i = 0$ then*

$$a_{i-1} = c_{i-1} + b_{i-1}.$$

- (ii) *If $e_i = A_{i-1} = B_{i-1} = B_i = 0$ then*

$$d < 2i - 1.$$

Proof. (i) Similar to the proof of Lemma 6.6.

(ii) By Corollary 6.8(ii) we have $a_{i-1} = c_i - c_{i-1}$, and because of (iii) this yields $2c_{i-1} + b_{i-1} = c_i$. Thus $b_{i-1} < c_i$. The result now follows from Lemma 4.3(i). ■

7 4-point counts B_i^s , C_i^s , D_i^s , E_i^s and F_i^s

For every i ($0 \leq i \leq d$) and $s \geq 0$ define the rational numbers

$$e_i^s := \frac{1}{v_{\Gamma} n_{i+s} p_{i-1, s+1}^{i+s}} \left[\begin{array}{c} \text{Diagram 1} \\ \hline i+s-1 \\ \hline s+1 \quad i-1 \\ \hline i+s \end{array} \right] = \frac{1}{v_{\Gamma} n_{i+s} p_{i-1, s+1}^{i+s}} \left[\begin{array}{c} \text{Diagram 2} \\ \hline i+s \\ \hline s+1 \quad i \\ \hline i+s-1 \end{array} \right] \geq 0. \quad (39)$$

Note that $e_i^0 = e_i$.

Proposition 7.1 *With the notation of Definition 3.1, we have*

$$\begin{aligned}
B_i^s &:= \left[\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} \right] = v_\Gamma n_{i+s} p_{i-1, s+1}^{i+s} (a_{i+s} - (b_{i-1} - b_{i+s} - e_i^s)) = \\
&= v_\Gamma n_{i+s} p_{i-1, s+1}^{i+s} (a_{i-1} - (c_{i+s} - c_{i-1} - e_i^s)), \\
C_i^s &:= \left[\begin{array}{c} \text{Diagram 3} \\ \text{Diagram 4} \end{array} \right] = \left[\begin{array}{c} \text{Diagram 5} \\ \text{Diagram 6} \end{array} \right] = v_\Gamma n_{i+s} p_{i-1, s+1}^{i+s} b_{i+s}, \quad C_{i-1}^s = v_\Gamma n_{i+s} p_{i-1, s+1}^{i+s} c_{i-1}, \\
D_i^s &:= \left[\begin{array}{c} \text{Diagram 7} \\ \text{Diagram 8} \end{array} \right] = \left[\begin{array}{c} \text{Diagram 9} \\ \text{Diagram 10} \end{array} \right] = v_\Gamma n_{i+s} p_{i-1, s+1}^{i+s} (b_{i-1} - b_{i+s} - e_i^s), \\
E_i^s &:= \left[\begin{array}{c} \text{Diagram 11} \\ \text{Diagram 12} \end{array} \right] = \left[\begin{array}{c} \text{Diagram 13} \\ \text{Diagram 14} \end{array} \right] = v_\Gamma n_{i+s} p_{i-1, s+1}^{i+s} e_i^s, \\
F_i^s &:= \left[\begin{array}{c} \text{Diagram 15} \\ \text{Diagram 16} \end{array} \right] = \left[\begin{array}{c} \text{Diagram 17} \\ \text{Diagram 18} \end{array} \right] = v_\Gamma n_{i+s} p_{i-1, s+1}^{i+s} (c_{i+s} - c_{i-1} - e_i^s),
\end{aligned}$$

for all i ($2 \leq i \leq d$) and $s \geq 0$. Note that $B_i^0 = B_i$, $C_i^0 = C_i$, $D_i^0 = D_i$, $E_i^0 = E_i$ and $F_i^0 = F_i$.

Proof. By summing over distances (see (5) and (7)) we find

$$\begin{aligned}
B_i^s + D_i^s &= v_\Gamma n_{i+s} p_{i-1, s+1}^{i+s} a_{i+s}, \\
F_i^s + B_i^s &= v_\Gamma n_{i+s} p_{i-1, s+1}^{i+s} a_{i-1}, \\
C_i^s &= v_\Gamma n_{i+s} p_{i-1, s+1}^{i+s} b_{i+s}, \\
D_i^s + E_i^s &= v_\Gamma n_{i+s} p_{i-1, s+1}^{i+s} (b_{i-1} - b_{i+s}), \\
E_i^s + F_i^s &= v_\Gamma n_{i+s} p_{i-1, s+1}^{i+s} (c_{i+s} - c_{i-1}).
\end{aligned}$$

Now (39) implies that $E_i^s = v_\Gamma n_{i+s} p_{i-1, s+1}^{i+s} e_i^s$, and solving the resulting triangular linear system of equations gives the above formulas. \blacksquare

Corollary 7.2 *With reference to Proposition 7.1, if $i + s \leq d$ then*

$$\max \left\{ 0, c_{i+s} - c_{i-1} - a_{i-1}, b_{i-1} - b_{i+s} - a_{i+s} \right\} \leq e_i^s \leq \min \{ b_{i-1} - b_{i+s}, c_{i+s} - c_{i-1} \}.$$

Proof. Immediate from the nonnegativity of 4-point counts from Proposition 7.1. \blacksquare

Corollary 7.3 *Let i and s denote positive integers with $2 \leq i \leq d-1$ and $i + s \leq d-1$. Then the following hold.*

- (i) *If $b_{i-1} = b_{i+s}$ and $a_{i+s} \neq 0$ then $a_{i-1} > c_{i+s} - c_{i-1}$.*
- (ii) *If $c_{i-1} = c_{i+s}$ and $a_{i-1} \neq 0$ then $a_{i+s} > b_{i-1} - b_{i+s}$.*

Proof. With reference to Proposition 7.1, if $b_{i-1} = b_{i+s}$ or $c_{i-1} = c_{i+s}$ then $e_i^s = 0$. Now if $(b_{i-1} = b_{i+s} \text{ and } a_{i+s} \neq 0)$ or $(c_{i-1} = c_{i+s} \text{ and } a_{i-1} \neq 0)$ then $B_i^s > 0$ and the results follow. \blacksquare

Corollary 7.4 *With reference to Proposition 7.1, if $i + s \leq d$ then the following hold.*

- (i) $c_{i+s} > c_{i-1}$ if and only if $F_i^s \neq 0$ or $E_i^s \neq 0$.
- (ii) $b_{i-1} > b_{i+s}$ if and only if $E_i^s \neq 0$ or $D_i^s \neq 0$.
- (iii) $a_{i+s} > 0$ if and only if $B_i^s \neq 0$ or $D_i^s \neq 0$.
- (iv) $a_{i-1} > 0$ if and only if $F_i^s \neq 0$ or $B_i^s \neq 0$.

Proof. Immediate from the proof of Proposition 7.1. ■

Corollary 7.5 *With reference to Proposition 7.1, the following hold.*

- (i) $c_i = c_{2i-1}$ if and only if $F_{i+1}^{i-2} = 0$ and $E_{i+1}^{i-2} = 0$.
- (ii) $b_i = b_{2i-1}$ if and only if $D_{i+1}^{i-2} = 0$ and $E_{i+1}^{i-2} = 0$.
- (iii) If $c_{i-1} = c_{2i-1}$ and $b_{i-1} > b_i = b_{2i-1}$ then $F_i^{i-1} = 0$, $E_i^{i-1} = 0$, $D_i^{i-1} \neq 0$.

Proof. Immediate from Corollary 7.4. ■

So, for example, if we want to show that a given graph Γ with the property $c_{i-1} = c_{2i-1}$ and $b_{i-1} > b_i = b_{2i-1}$ does not exist, it is enough to prove that $D_i^{i-1} = 0$, contradicting Corollary 7.5(iii).

In (40) we have a generalization of half of the Brouwer–Lambeck inequality (27).

Proposition 7.6 *Let i and s denote positive integers with $2 \leq i \leq d - 1$ and $i + s \leq d$. Then the following holds:*

$$c_i(a_{i+s} - a_{i-1}) \leq a_{i+s}(a_i + \dots + a_{i+s} - a_1 - \dots - a_s). \quad (40)$$

Proof. With reference to Proposition 7.1, note that

$$v_{\Gamma} n_{i+s} p_{i-1, s+1}^{i+s} (a_{i+s} - a_{i-1}) = D_i^s - F_i^s \leq D_i^s \leq \left[\begin{array}{c} i+s \\ \frac{s+1}{i+s} \quad \frac{i-1}{i+s} \end{array} \right] = v_{\Gamma} n_{i+s} p_{i, s+1}^{i+s} a_{i+s}.$$

Now, from Lemma 4.4, on one side we have

$$p_{i-1, s+1}^{i+s} = \frac{c_i c_{i+1} \cdots c_{i+s}}{c_1 c_2 \cdots c_{s+1}} = \frac{c_i}{c_{s+1}} p_{is}^{i+s}$$

and on the another side we have

$$p_{i, s+1}^{i+s} = p_{is}^{i+s} \frac{a_i + \dots + a_{i+s} - a_1 - \dots - a_s}{c_{s+1}}.$$

The result follows. ■

7.1 Case when $e_i > 0$ for some i ($1 \leq i \leq d$)

Note that (28) involves e_i , hence it is not a feasibility condition for the intersection array. However, when the lower bound in (28) is positive, we may apply the following results. We use the new counts

$$K_{jk} := \left[\begin{array}{c} \text{Diagram 1} \end{array} \right] \quad \text{and} \quad H_{jk} := \left[\begin{array}{c} \text{Diagram 2} \end{array} \right]_2. \quad (41)$$

Theorem 7.7 *With reference to Proposition 6.1, we have*

$$e_i > 0, \quad c_{s+1} > \min(c_i - c_{i-1}, b_{i-1} - b_i) \implies c_{i+s} > c_i, \quad (42)$$

$$e_i > 0, \quad i > 1 \implies c_{2i-1} > c_i, \quad (43)$$


$$e_i > 0 \implies b_1 \geq b_i + c_{i-1}, \quad (44)$$

$$e_i > 0 \implies b_{i-1} - b_i + c_i - c_{i-1} \geq \lambda + 2. \quad (45)$$

Proof. If $e_i > 0$ then by (29),

$$e := \min(c_i - c_{i-1}, b_{i-1} - b_i) \geq e_i > 0,$$

and with that $c_i > c_{i-1}$ and $b_{i-1} > b_i$. Also, $e_i > 0$ implies that we may choose $xyvw$ consistent

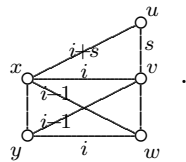
with  (since $E_i > 0$). Note that

$$\left[\begin{array}{c} \text{Diagram 3} \end{array} \right] \leq c_i - c_{i-1} = \left[\begin{array}{c} \text{Diagram 4} \end{array} \right] \quad (46)$$

and

$$\left[\begin{array}{c} \text{Diagram 5} \end{array} \right] \leq b_{i-1} - b_i = \left[\begin{array}{c} \text{Diagram 6} \end{array} \right]. \quad (47)$$

Now consider a vertex u consistent with



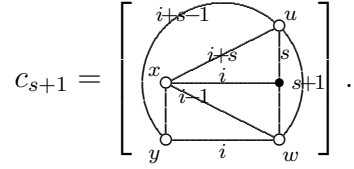
We have

$$p_{i+s,s}^i = \left[\begin{array}{c} \text{Diagram 7} \end{array} \right] = \left[\begin{array}{c} \text{Diagram 8} \end{array} \right]_{s+1}.$$

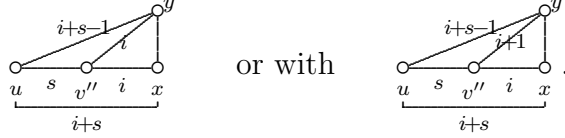
All $p_{i+s,s}^i$ choices for u yield $\partial(u, y) = i + s - 1$ and $\partial(w, u) = s + 1$. Consider a vertex $v' \in \Gamma_{1s}(w, u)$ with $\partial(v', y) = i - 1$. For such a choice of v' , the triple xwv' implies $\partial(x, v') \in \{i - 2, i - 1, i\}$. On the other hand, the triple $xv'u$ yields $\partial(x, v') \geq i$. Now, because of (46) and (47), the number of $v' \in \Gamma_{1,s,i-1}(w, u, y)$ is at most e , i.e.,

$$\left[\begin{array}{c} \text{Diagram 9} \end{array} \right] = \left[\begin{array}{c} \text{Diagram 10} \end{array} \right] \leq e.$$

Also, note that



If $c_{s+1} > e$, there is some $v'' \in \Gamma_{1s}(w, u)$ with $\partial(v'', y) \geq i$. In other words, there is a vertex v'' such that $wv''xy$ is consistent with



That is, we have

$$F_{i+1}^{s-1} > 0 \quad \text{or} \quad E_{i+1}^{s-1} > 0.$$

This yields $c_{i+s} > c_i$ (see Corollary 7.4(i)). Thus (42) holds, and (43) follows by taking $s = i - 1 > 0$.

To derive the remaining statements we note that (41) gives

$$H_{i,i+1} = E_i b_i,$$

$$H_{i-2,i-1} = E_i c_{i-1},$$

$$H_{i-1,i-1} + H_{i-1,i} + H_{i,i-1} + H_{ii} = E_i (b_1 - b_i - c_{i-1}), \quad (48)$$

$$K_{i-1,i-1} + H_{i-1,i-1} + K_{i,i-1} + H_{i,i-1} = E_i (c_i - c_{i-1} - 1), \quad (49)$$

$$K_{i,i-1} + H_{i,i-1} + K_{ii} + H_{ii} = E_i (b_{i-1} - b_i - 1), \quad (50)$$

$$K_{i-1,i-1} + K_{i-1,i} + K_{i,i-1} + K_{ii} = E_i \lambda. \quad (51)$$

By swapping the edges in the diagram in (41), we observe that

$$K_{i-1,i} = K_{i,i-1}. \quad (52)$$

Now (44) follows from the nonnegativity of (48), and (45) follows from (49)+(50)≥(51) which holds in view of (52). ■

It would be interesting to know which graphs have the property $e_i = 0$ for all i ($1 \leq i \leq d$).

Research problem 7.8 *Classify all distance-regular graphs for which $e_i = 0$ for all i ($1 \leq i \leq d$).*

Research problem 7.9 *Pick $i \geq 2$. Using t -point counts, prove (or disprove) that the following holds:*

$$e_i = 0 \quad \text{and} \quad c_i > c_{i-1} \quad \implies \quad d < 2i - 1 \quad \text{or} \quad c_{2i-1} > c_i.$$

8 Note about the case $a_i > a_j$

Pick i ($2 \leq i \leq d$). In this section we show that

$$a_i > a_{i-1} \implies b_{i-1} > b_i$$

and

$$a_i < a_{i-1} \implies c_i > c_{i-1}.$$

Moreover, we have

$$a_j > a_i \implies b_i > b_j$$

and

$$a_j < a_i \implies c_j > c_i$$

for any i, j ($1 \leq i < j \leq d$). To our knowledge, these two claims are new!

Lemma 8.1 *Let Γ denote a distance-regular graph. Then the following hold.*

(i) $a_i > a_{i-1} \implies b_{i-1} > b_i.$

(ii) $a_i < a_{i-1} \implies c_i > c_{i-1}.$

Proof. We prove claim (i). The proof of (ii) is similar.

From Proposition 6.1 we have $D_i - F_i = v_{\Gamma} n_i c_i (a_i - a_{i-1})$, and since $a_i > a_{i-1}$, we have $D_i > F_i \geq 0$. This yields that $v_{\Gamma} n_i c_i (b_{i-1} - b_i - e_i) = D_i > 0$ and the result follows. \blacksquare

Theorem 8.2 *Let Γ denote a distance-regular graph, and pick i, j ($1 \leq i < j \leq d$). Then the following hold.*

(i) $a_j > a_i \implies b_i > b_j.$

(ii) $a_j < a_i \implies c_j > c_i.$

Proof. Pick s such that $i+s = j$. From Proposition 7.1 we have $D_{i+1}^{s-1} - F_{i+1}^{s-1} = v_{\Gamma} n_{i+s} p_{i,s}^{i+s} (a_{i+s} - a_i) = v_{\Gamma} n_j p_{i,j-i}^j (a_j - a_i)$. If $a_j > a_i$ then $D_{i+1}^{s-1} > F_{i+1}^{s-1}$, which yields $b_i - b_j > e_{i+1}^{s-1} \geq 0$ and the result follows. The proof for the case $a_j < a_i$ is similar. \blacksquare

Research problem 8.3 *Using t -point counts, explain under which restrictions on intersection numbers (and indices i, j) the following hold:*

$$a_j > a_i \implies b_{i+1} > b_j,$$

$$a_j < a_i \implies c_j > c_{i+1}.$$

9 Hiraki's first inequality

In [10, Lemma 3.1] HIRAKI proved that if $1 \leq q \leq d-1$, $c_{q+1} > c_q$ and $a_q \leq c_{q+1} - c_q$ then $c_q + c_i \leq c_{q+1}$ for all $2 \leq i \leq d-q$. In this section, we re-prove Hiraki's inequality using t -point counts and we investigate different sub-cases. We will prove Hiraki's second inequality in the follow-up paper.

Lemma 9.1 *Let $\Gamma = (X, R)$ be a distance-regular graph of diameter d , and let q be an integer with $1 \leq q \leq d-1$. Suppose $c_q < c_{q+1}$. The following hold.*

(i) *If $a_q \leq c_{q+1} - c_q$ then*

$$c_{q+i} \geq c_i + c_q \quad \text{for all } i \text{ (} 2 \leq i \leq d-q \text{)}.$$

(ii) *If $a_q > c_{q+1} - c_q$ then*

$$c_{q+i} > \frac{c_{q+1} - c_q}{a_q} c_i + c_q \quad \text{for all } i \text{ (} 2 \leq i \leq d-q \text{)}.$$

Moreover, if $a_q = c_{q+1} - c_q$ then $c_{q+i} = c_i + c_q$ if and only if

$$\begin{aligned} \left[\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} \right] &= 0, & \left[\begin{array}{c} \text{Diagram 3} \\ \text{Diagram 4} \end{array} \right] &= 0, \\ \left[\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} \right] &= 0 & \text{and} & \left[\begin{array}{c} \text{Diagram 3} \\ \text{Diagram 4} \end{array} \right] &= 0. \end{aligned}$$

Proof. Pick $u \in X$, $v \in \Gamma_i(u)$ and $w \in \Gamma_{q+i,q}(u, v)$. We consider the sets $Y := C_i(u, v)$, $Z := C_{q+i}(u, w) \setminus C_q(v, w)$, $Z_A := \Gamma_{q+i-1, q, 1}(u, v, w)$ and $Z_B := \Gamma_{q+i-1, q+1, 1}(u, v, w)$. Note that $Z = Z_A \cup Z_B$ and $|Z| = |Z_A| + |Z_B|$. We count the size of the set

$$\Omega = \{(y, z) \in Y \times Z \mid \partial(y, z) = q\} = \left\{ \begin{array}{c} \text{Diagram 5} \end{array} \right\}$$

in two ways. From

$$\begin{aligned} c_i &= |Y| = |C_i(u, v)| = \left[\begin{array}{c} \text{Diagram 6} \end{array} \right], \\ c_i c_{q+1} &= \left[\begin{array}{c} \text{Diagram 7} \end{array} \right] \quad \text{and} \quad c_i c_q = \left[\begin{array}{c} \text{Diagram 8} \end{array} \right] \end{aligned}$$

we have

$$c_i(c_{q+1} - c_q) = \left[\begin{array}{c} \text{Diagram 9} \end{array} \right] + \left[\begin{array}{c} \text{Diagram 10} \end{array} \right].$$

Similarly, from

$$c_{q+i} = |C_{q+i}(u, w)| = \left[\begin{array}{c} \text{Diagram 11} \end{array} \right], \quad c_q = |C_q(v, w)| = \left[\begin{array}{c} \text{Diagram 12} \end{array} \right]$$

and

$$\begin{aligned} |Z| = c_{q+i} - c_q = |C_{q+i}(u, w) \setminus C_q(v, w)| &= \underbrace{\left[\begin{array}{c} \text{Diagram 13} \end{array} \right]}_{=|Z_A|} + \underbrace{\left[\begin{array}{c} \text{Diagram 14} \end{array} \right]}_{=|Z_B|}, \end{aligned}$$

we have

$$\begin{aligned}
|Z_A|a_q + |Z_B|(c_{q+1} - c_q) &= \left[\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} \right] + \left[\begin{array}{c} \text{Diagram 3} \\ \text{Diagram 4} \end{array} \right] = \\
&= \left[\begin{array}{c} \text{Diagram 5} \\ \text{Diagram 6} \\ \text{Diagram 7} \end{array} \right] + \left[\begin{array}{c} \text{Diagram 8} \\ \text{Diagram 9} \\ \text{Diagram 10} \end{array} \right] + \left[\begin{array}{c} \text{Diagram 11} \\ \text{Diagram 12} \\ \text{Diagram 13} \end{array} \right] + \\
&\left[\begin{array}{c} \text{Diagram 14} \\ \text{Diagram 15} \\ \text{Diagram 16} \end{array} \right].
\end{aligned}$$

Now, it is not hard to see that

$$c_i(c_{q+1} - c_q) = |\Omega| \leq |Z_A|a_q + |Z_B|(c_{q+1} - c_q). \quad (53)$$

Note that $c_i(c_{q+1} - c_q) = |Z_A|a_q + |Z_B|(c_{q+1} - c_q)$ if and only if

$$\begin{aligned}
\left[\begin{array}{c} \text{Diagram 17} \\ \text{Diagram 18} \end{array} \right] &= 0, & \left[\begin{array}{c} \text{Diagram 19} \\ \text{Diagram 20} \end{array} \right] &= 0, \\
\left[\begin{array}{c} \text{Diagram 21} \\ \text{Diagram 22} \end{array} \right] &= 0 & \text{and} & \left[\begin{array}{c} \text{Diagram 23} \\ \text{Diagram 24} \end{array} \right] = 0.
\end{aligned}$$

(i) If $a_q \leq c_{q+1} - c_q$ then, by (53), we have $c_i(c_{q+1} - c_q) \leq (|Z_A| + |Z_B|)(c_{q+1} - c_q) = (c_{q+i} - c_q)(c_{q+1} - c_q)$, and the result follows.

(ii) If $a_q > c_{q+1} - c_q$ then, by (53), we have $c_i(c_{q+1} - c_q) < (|Z_A| + |Z_B|)a_q = (c_{q+i} - c_q)a_q$, and the result follows. \blacksquare

Corollary 9.2 *With reference to Lemma 9.1, suppose $c_{q+1} > c_q$. For $2 \leq i \leq d - q$ let E_{q+1}^{i-1} and F_{q+1}^{i-1} be as in Proposition 7.1. Fix i ($2 \leq i \leq d - q$). Then*

$$c_i(c_{q+1} - c_q) \leq \frac{1}{v_\Gamma n_{i+q} p_{i,q}^{i+q}} (F_{q+1}^{i-1} a_q + E_{q+1}^{i-1} (c_{q+1} - c_q)). \quad (54)$$

Proof. Recall that

$$E_{q+1}^{i-1} = \left[\begin{array}{c} \text{Diagram 25} \\ \text{Diagram 26} \end{array} \right] = \left[\begin{array}{c} \text{Diagram 27} \\ \text{Diagram 28} \end{array} \right]$$

and

$$F_{q+1}^{i-1} = \left[\begin{array}{c} \text{Diagram 29} \\ \text{Diagram 30} \end{array} \right] = \left[\begin{array}{c} \text{Diagram 31} \\ \text{Diagram 32} \end{array} \right].$$

Similarly as in the proof of Lemma 9.1, it is not hard to show that

$$v_{\Gamma} n_{i+q} p_{i,q}^{i+q} c_i (c_{q+1} - c_q) = \left[\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} \right] + \left[\begin{array}{c} \text{Diagram 3} \\ \text{Diagram 4} \end{array} \right],$$

and

$$F_{q+1}^{i-1} a_q + E_{q+1}^{i-1} (c_{q+1} - c_q) = \left[\begin{array}{c} \text{Diagram 5} \\ \text{Diagram 6} \end{array} \right] + \left[\begin{array}{c} \text{Diagram 7} \\ \text{Diagram 8} \end{array} \right].$$

The result follows. ■

Corollary 9.3 *With reference to Lemma 9.2, suppose $c_{q+1} > c_q$. For $2 \leq j \leq d - q$ let e_{q+1}^{j-1} be as in (39). Fix i ($2 \leq i \leq d - q$). The following hold.*

- (i) *If $c_{i+q} - c_q > e_{q+1}^{i-1}$ then $F_{q+1}^{i-1} > 0$.*
- (ii) *If $c_{i+q} - c_q = e_{q+1}^{i-1}$ then $F_{q+1}^{i-1} = 0$.*
- (iii) *If $c_{i+q} - c_q > e_{q+1}^{i-1}$ and $a_q < c_{q+1} - c_q$ then $c_{i+q} > c_i + c_q$.*
- (iv) *If $c_{i+q} - c_q = e_{q+1}^{i-1}$ then $c_{i+q} \geq c_i + c_q$.*
- (v) *If $e_{q+1}^{i-1} = 0$ then $a_q \neq 0$ and $c_{q+i} \geq \frac{c_{q+1} - c_q}{a_q} c_i + c_q$.*

Proof. Recall

$$e_{q+1}^{j-1} = \frac{1}{v_{\Gamma} n_{j+q} p_{j,q}^{j+q}} \left[\begin{array}{c} \text{Diagram 9} \\ \text{Diagram 10} \end{array} \right] = \frac{1}{v_{\Gamma} n_{j+q} p_{j,q}^{j+q}} \left[\begin{array}{c} \text{Diagram 11} \\ \text{Diagram 12} \end{array} \right] \geq 0. \quad (55)$$

Claims (i) and (ii) follow immediately from Proposition 7.1. If $c_{i+q} - c_q > e_{q+1}^{i-1}$ then F_{q+1}^{i-1} is nonzero. Note that

$$\frac{1}{v_{\Gamma} n_{i+q} p_{i,q}^{i+q}} (F_{q+1}^{i-1} + E_{q+1}^{i-1}) = c_{i+q} - c_q.$$

Together with (54) and Lemma 9.1(i), this yields (iii). If $c_{i+q} - c_q = e_{q+1}^{i-1}$ then $F_{q+1}^{i-1} = 0$ so (54) becomes

$$\begin{aligned} c_i (c_{q+1} - c_q) &\leq \frac{1}{v_{\Gamma} n_{i+q} p_{i,q}^{i+q}} E_{q+1}^{i-1} (c_{q+1} - c_q) = \\ &= \frac{1}{v_{\Gamma} n_{i+q} p_{i,q}^{i+q}} (F_{q+1}^{i-1} + E_{q+1}^{i-1}) (c_{q+1} - c_q) = (c_{q+i} - c_q) (c_{q+1} - c_q) \end{aligned}$$

which implies (iv). Finally, if $e_{q+1}^{i-1} = 0$ then E_{q+1}^{i-1} is zero. Together with (54), this yields (v). ■

10 New diameter bound in the case when $c_{q+1} > c_q$ and $a_q \leq c_{q+1} - c_q$

In this section we show that if Γ is a distance-regular graph such that $c_q < c_{q+1}$ and $a_q \leq c_{q+1} - c_q$ (where $q \geq 2$), then

$$d \leq (k + 1 - c_{q+1})q + 1 \leq (k - 1)q + 1.$$

This bound is tight. Let's recall some well-known diameter bounds. The following result for the bipartite case is due to HIRAKI [10] ($\lfloor n \rfloor$ denotes the maximal integer i such that $i \leq n$).

Theorem 10.1 ([10, Theorem 1.2]) *Let Γ be a bipartite distance-regular graph of diameter d , valency $k \geq 3$ and $c_{q+1} > c_q$ where $q \geq 2$. Suppose Γ is not the doubled Odd graph. Then*

$$d \leq \left\lfloor \frac{k+2}{2} \right\rfloor q + 1.$$

The next result, also for the bipartite case, is due to KOOLEN [11].

Theorem 10.2 ([11, Theorem 6]) *Let Γ be a bipartite distance-regular graph of diameter d , valency $k \geq 3$ and girth $2q \geq 6$. If Γ is not the doubled Odd graph, then*

$$d \leq (q - 1)(k - 1) - \left\lfloor \frac{k - 3}{2} \right\rfloor.$$

If the girth of Γ is $2q \geq 6$, then $c_q > c_{q-1}$ ($q \geq 3$), but the converse is not true. The following result of TERWILLIGER [17, 18] for bipartite distance-regular graphs is well-known.

Theorem 10.3 ([17, 18]) *Let Γ be a bipartite distance-regular graph of diameter $d \geq 3$ and valency $k \geq 3$ with $c_{q+1} > c_q$. Then*

$$d \leq (k - 1)q + 1. \tag{56}$$

The bound (56) is tight. The hypercubes and the doubled Odd graphs satisfy $d = (k - 1)q + 1$ with $q = 1, 2$, respectively.

In Theorem 10.6 we derive a slightly stronger diameter bound (from (56)) which, in certain cases, also applies to non-bipartite distance-regular graphs. The bounds (56) and (57) appear to be equivalent only when $c_{q+1} = 2$, which is the minimal value for which the conditions can be met.

The best known general bound that can be related with the condition $c_q < c_{q+1}$ (without any restriction on a_q) is the bound given by BANG, HIRAKI and KOOLEN in [1].

Theorem 10.4 ([1, Corollary 1.4]) *Let Γ be a distance-regular graph of diameter $d \geq 2$, valency $k \geq 3$, and assume that $c_{q+1} > c_q = 1$. Then*

$$d \leq \frac{1}{2}k^\alpha q + 1$$

where $\alpha := \min\{x > 0 \mid 4^{\frac{1}{x}} - 2^{\frac{1}{x}} \leq 1\}$ (note that $1.44 < \alpha < 1.441$).

To prove our claim, we need Lemma 10.5.

Lemma 10.5 *Let Γ be a distance-regular graph of diameter $d \geq 2$, let t be a positive integer and let q be an integer with $2 \leq q \leq d - 1$. If $c_q < c_{q+1}$, $a_q \leq c_{q+1} - c_q$ and $tq + 1 \leq d$ then*

$$c_{tq+1} \geq t + 1.$$

Proof. Note that $c_{q+1} \geq 2$. By assumption $a_q \leq c_{q+1} - c_q$, and from Lemma 9.1(i) we have $c_{i+q} \geq c_i + c_q$ for all i ($2 \leq i \leq d - q$).

We prove the inequality by induction on t . For the induction basis, note that for $t = 1$ we have $c_{tq+1} = c_{q+1} \geq 2 = t + 1$, while for $t = 2$ we have $c_{tq+1} = c_{2q+1} \geq c_{q+1} + c_q \geq c_{q+1} + 1 \geq 3 = t + 1$. For the induction step, assume that $c_{sq+1} \geq s + 1$ for every $2 \leq s \leq t - 1$, and prove that inequality also holds for t . Indeed, $c_{tq+1} = c_{(t-1)q+1+q} \geq c_{(t-1)q+1} + c_q \geq c_{(t-1)q+1} + 1 \geq t + 1$ and the result follows. ■

Note some similarity between Corollary 10.5 and [10, Lemma 4.2(2)]. Our proof of Theorem 10.6 goes along the same lines as the proof of [10, Theorem 1.2] (i.e., for the bipartite case).

Theorem 10.6 *Let Γ be a distance-regular graph of diameter d with valency $k \geq 3$, and let q be an integer with $2 \leq q \leq d - 1$. If $c_q < c_{q+1}$ and $a_q \leq c_{q+1} - c_q$ then*

$$d \leq (k + 1 - c_{q+1})q + 1. \quad (57)$$

Proof. Let $t := k - c_{q+1}$. To derive a contradiction, suppose $(t + 1)q + 2 \leq d$. Then $tq + 1 \leq d - q - 1$. Thus, because of Lemma 4.3(i) and (26), we have $b_{tq+1} \geq b_{d-q-1} \geq c_{q+1}$, and $c_{tq+1} \geq t + 1$ by Lemma 10.5. It follows that

$$t + 1 + c_{q+1} \leq c_{tq+1} + b_{tq+1} \leq k = t + c_{q+1}.$$

This is a contradiction. The theorem is proved. ■

Remark 10.7 Some of the graphs of diameter $d \geq 5$ that are not bipartite and that satisfy the conditions of Theorem 10.6 (for the smallest possible index q) are

- the Biggs–Smith graph,
 - the intersection array is $\{3, 2, 2, 2, 1, 1, 1; 1, 1, 1, 1, 1, 1, 3\}$,
 - the bound (57) is tight,
 - $d = 7$, $k = 3$, $q = 6$,
 - $c_6 < c_7 = 3$, $1 = a_6 < c_7 - c_6$, $(k + 1 - c_{q+1})q + 1 = 7$,
- the Odd graph on 13 points,
 - the intersection array is $\{7, 6, 6, 5, 5, 4; 1, 1, 2, 2, 3, 3\}$,
 - $d = 6$, $k = 7$, $q = 2$,
 - $c_2 < c_3 = 2$, $a_2 < c_3 - c_2$, $(k + 1 - c_{q+1})q + 1 = 13$,
- the Odd graph on 11 points,
 - the intersection array is $\{6, 5, 5, 4, 4; 1, 1, 2, 2, 3\}$,
 - $d = 5$, $k = 6$, $q = 2$,
 - $c_2 < c_3 = 2$, $a_2 < c_3 - c_2$, $(k + 1 - c_{q+1})q + 1 = 11$,
- the generalized dodecagon of order $(2, 1)$,
 - the intersection array is $\{4, 2, 2, 2, 2, 2; 1, 1, 1, 1, 1, 2\}$,
 - $d = 6$, $k = 4$, $q = 5$,
 - $c_5 < c_6 = 2$, $a_5 = c_6 - c_5$, $(k + 1 - c_{q+1})q + 1 = 16$
- and the dodecahedron,
 - the intersection array is $\{3, 2, 1, 1, 1; 1, 1, 1, 2, 3\}$,
 - $d = 5$, $k = 3$, $q = 3$,
 - $a_3 = c_4 - c_3$, $(k + 1 - c_{q+1})q + 1 = 7$.

Besides the Biggs–Smith graph, the bound (57) is tight for the doubled Odd graphs; for this family (which is both antipodal and bipartite) we have $d = (k - 1)q + 1$ with $q = 2$. One of the graphs of diameter $d \geq 5$ which does not satisfy the conditions of Theorem 10.6 (for the smallest possible index q) is the Ivanov-Ivanov-Faradjev graph ($d = 8, k = 7, q = 3, c_3 < c_4 = 2, 2 = a_3 > c_3 - c_2$). On the other hand, for $q = 4$ we have $c_4 < c_5 = 4$ and $1 = a_4 < c_5 - c_4$.

10.1 Case when $c_{i+q} - c_q > e_{q+1}^{i-1}$ for all $i \in \{q + 1, 2q + 1, \dots, (m - 1)q + 1\}$

In this subsection we show that if Γ is a distance-regular graph such that $c_q < c_{q+1}, a_q < c_{q+1} - c_q$ ($q \geq 2$) and $c_{i+q} - c_q > e_{q+1}^{i-1}$ for all $i \in \{q + 1, 2q + 1, \dots, (m - 1)q + 1\}$ (where m is the largest positive integer such that $mq + 1 \leq d$), then

$$d \leq \left\lfloor \frac{k + 4 - c_{q+1}}{2} \right\rfloor q + 1 \leq \left\lfloor \frac{k + 2}{2} \right\rfloor q + 1.$$

Lemma 10.8 *Let Γ be a distance-regular graph of diameter d with valency $k \geq 3$, and let q be an integer with $2 \leq q \leq d - 1$. If $c_q < c_{q+1}, a_q < c_{q+1} - c_q$ and $c_{i+q} - c_q > e_{q+1}^{i-1}$ for all $i \in \{q + 1, 2q + 1, \dots, (m - 1)q + 1\}$, where m is the largest positive integer such that $mq + 1 \leq d$, then*

$$c_{tq+1} \geq 2t \quad (1 \leq t \leq m).$$

Proof. Note that $c_{q+1} \geq 2$. By assumption $a_q < c_{q+1} - c_q$ and $c_{i+q} - c_q > e_{q+1}^{i-1}$ for all $i \in \{q + 1, 2q + 1, \dots, (m - 1)q + 1\}$, so Corollary 9.3(iii) yields $c_{i+q} > c_i + c_q$ for all $i \in \{q + 1, 2q + 1, \dots, (m - 1)q + 1\}$. This yields $c_{i+q} \geq c_i + c_q + 1$ for all $i \in \{q + 1, 2q + 1, \dots, (m - 1)q + 1\}$.

We prove the inequality by induction on t . For the induction basis, note that for $t = 1$ we have $c_{tq+1} = c_{q+1} \geq 2 = 2t$, while for $t = 2$ we have $c_{2q+1} = c_{2q+1} \geq c_{q+1} + c_q + 1 \geq c_{q+1} + 2 \geq 4 = 2t$. For the induction step, assume that $c_{sq+1} \geq 2s$ for every $2 \leq s \leq t - 1$, and prove that inequality also holds for t . Indeed, $c_{tq+1} = c_{(t-1)q+1+q} \geq c_{(t-1)q+1} + c_q + 1 \geq c_{(t-1)q+1} + 2 \geq 2(t - 1) + 2$ and the result follows. \blacksquare

Theorem 10.9 *With reference to Lemma 10.8, assume that $c_q < c_{q+1}$ and $c_{i+q} - c_q > e_{q+1}^{i-1}$ for all $i \in \{q + 1, 2q + 1, \dots, (m - 1)q + 1\}$, where $mq + 1 \leq d$. If $a_q < c_{q+1} - c_q$ then*

$$d \leq \left\lfloor \frac{k + 4 - c_{q+1}}{2} \right\rfloor q + 1. \quad (58)$$

Proof. Let $t := \left\lfloor \frac{k+4-c_{q+1}}{2} \right\rfloor - 1$. This implies $k \leq 2t - 1 + c_{q+1}$. To derive a contradiction, suppose $(t + 1)q + 2 \leq d$. Then $tq + 1 \leq d - q - 1$. Thus, because of Lemma 4.3(i) and (26) we have $b_{tq+1} \geq b_{d-q-1} \geq c_{q+1}$, and $c_{tq+1} \geq 2t$ from Lemma 10.8. It follows that

$$2t + c_{q+1} \leq c_{tq+1} + b_{tq+1} \leq k \leq 2t - 1 + c_{q+1}.$$

This is a contradiction. The theorem is proved. \blacksquare

Research problem 10.10 *Let q be the smallest integer such that $c_{q+1} > c_q$ ($2 \leq q \leq d - 1$), and assume $a_q < c_{q+1} - c_q$. Using t -point counts, explain under which restrictions on intersection numbers (and index i) the following holds:*

$$c_{i+q} - c_q > e_{q+1}^{i-1} \quad \text{for all } i \in \{q + 1, 2q + 1, \dots, (m - 1)q + 1\},$$

where m is the largest positive integer such that $mq + 1 \leq d$.

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