$t\frac{1}{2}$-Designs*

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SUMMARY

Every $(t + 1)$-design $\mathscr{B}$ satisfies

(+) If T is a set of t points, and B a block of $\mathscr{B}$ then the number $\alpha(T, B)$ of flags (x, A) with $x \notin T$, $x \in B$, $T \cup \{x\} \subseteq A$ depends only on $|T \cap B|$.

A t-design with property (+) is called a $t\frac{1}{2}$-design. The most interesting general classes of t-designs are $t\frac{1}{2}$-designs: Hadamard 3-designs are $3\frac{1}{2}$-designs, symmetric 2-designs are $2\frac{1}{2}$-designs, and dual 2-designs, transversal designs, and partial geometries are $1\frac{1}{2}$-designs; in fact, $1\frac{1}{2}$-designs share most properties of partial geometries.

$1\frac{1}{2}$-designs are studied in detail, and their connection with strongly regular graphs is investigated.

It is shown that $t\frac{1}{2}$-designs behave like t-designs with respect to derivation, residuals, and complementation.

Various characterizations of partial geometries, generalized quadrangles, symmetric 2-designs, and Hadamard 3-designs are given in terms of $t\frac{1}{2}$-designs.

The paper ends with a proof that $t\frac{1}{2}$-designs with $t \geqslant 4$ are already $(t + 1)$-designs.

1. PRELIMINARIES

1.1. An *incidence structure* [4] is a triple $(\mathscr{P}, \mathscr{B}, \mathscr{F})$ with $\mathscr{F} \subseteq \mathscr{P} \times \mathscr{B}$. Elements of $\mathscr{P}$, $\mathscr{B}$, and $\mathscr{F}$ are called *points*, *blocks*, and *flags*, respectively. We write pIb or $p \in b$ or $b \ni p$ if $(p, b) \in \mathscr{F}$.

The *dual* structure of $(\mathscr{P}, \mathscr{B}, \mathscr{F})$ is $(\mathscr{B}, \mathscr{P}, \mathscr{F}^d)$ with $\mathscr{F}^d = \{(b, p) | (p, b) \in \mathscr{F}\}$.

* This research was done while the author was at Westfield College, London.

1.2. A *block design* over P is a collection $\mathscr{B}$ of subsets of P (allowing repeated subsets). We call elements of P *points*, elements of $\mathscr{B}$ *blocks*, and pairs $(a, B) \in P \times \mathscr{B}$ with $a \in B$ *flags*.

To every incidence structure $(\mathscr{P}, \mathscr{B}, \mathscr{F})$ there is an associated block design over $\mathscr{P}$, namely $\mathscr{B}' = \{\{B_b \mid b \in \mathscr{B}\}\}$ with $B_b = \{p \in \mathscr{P} \mid p\mathrm{I}b\}$. (Here double brackets $\{\{\ \}\}$ mean that we count a block as often as it occurs. We also use the convention that $\#\mathscr{B} = |\mathscr{B}|$ denotes the number of elements in a collection $\mathscr{B}$, counting repeated blocks according to their multiplicity.) We consider incidence structures and their associated block designs as equivalent, and switch sometimes (without stating this) from one concept to the other; e.g., we speak of an incidence structure $\mathscr{B}$ over $\mathscr{P}$ instead of $(\mathscr{P}, \mathscr{B}, \mathscr{F})$.

1.3. A *tactical configuration* [4] is an incidence structure $\mathscr{B}$ satisfying: For each point $p \in \mathscr{P}$ there are exactly r blocks $b \in \mathscr{B}$ with $p\mathrm{I}b$, and, dually, for each block $b \in \mathscr{B}$ there are exactly k points $p \in \mathscr{P}$ with $p\mathrm{I}b$. We call $v = |\mathscr{P}|$, k, $b = |\mathscr{B}|$, and r the *parameters* of $\mathscr{B}$. The parameters satisfy $vr = bk$. The dual of $\mathscr{B}$ is a tactical configuration with parameters $v' = b$, $k' = r$, $b' = v$, $r' = k$.

1.4. A t-(v, k, λ)-design (or simply a *t-design*, $t \geqslant 0$) [7] is a block design $\mathscr{B}$ over a v-set P satisfying: every block contains exactly k points, and, for every t-set T of points there are exactly λ blocks containing T. A t-(v, k, λ)-design $\mathscr{B}$ over P is called *trivial* if, for some integer n, $\mathscr{B}$ consists of exactly n times every k-subset of P.

A 1-(v, k, λ)-design with b blocks is (interpreted as an incidence structure) nothing else than a tactical configuration with parameters $(v, k; b, r = \lambda)$.

1.5. We list some well-known facts about t-designs [7]. Let $\mathscr{B}$ be a t-(v, k, λ)-design over P.

1. If $\mathscr{B}$ is nontrivial then $t + 1 \leqslant k \leqslant v - 1 - t$, $\lambda \geqslant 1$.

2. $\mathscr{B}$ is an s-(v, k, λ_s)-design with $\lambda_s = \binom{k-s}{t-s}^{-1}\binom{v-s}{t-s}\lambda$, for every $s \leqslant t$. We use the letters $b = \lambda_0$, $r = \lambda_1$, $\lambda = \lambda_t$.

3. Suppose R is an r-subset, and S is an s-subset of P. If $R \cap S = \varnothing$ and $r + s \leqslant t$ then the number of blocks $B \in \mathscr{B}$ with $R \subseteq B$, $S \cap B = \varnothing$ is $\lambda_{rs} = \binom{v-t}{k-t}^{-1}\binom{v-r-s}{k-r}\lambda$.

4. If $\mathscr{B}$ is nontrivial then the *complementary* design $\mathscr{B}^c = \{\{P\backslash B \mid B \in \mathscr{B}\}\}$ is a t-$(v, v-k, \lambda^c)$-design over P with $\lambda^c = \binom{k}{t}^{-1}\binom{v-k}{t}\lambda$.

5. If $1 \leqslant t \leqslant k$, and $a \in P$ then the *residual* design (at point a) $\mathscr{B}^a = \{\{B \in \mathscr{B} \mid a \notin B\}\}$ is a $(t-1)-(v-1, k, \lambda^r)$-design over $P\backslash\{a\}$ with $\lambda^r = (v-k)\lambda/(k-t+1)$.

6. If $1 \leqslant t \leqslant k$, and $a \in P$ then the *derived* design (at point a) $\mathscr{B}_a = \{\{B\backslash\{a\} \mid a \in B \in \mathscr{B}\}\}$ is a $(t-1)-(v-1, k-1, \lambda)$-design over $P\backslash\{a\}$.

7. If every derived design of $\mathscr{B}$ is a t-design then $\mathscr{B}$ is a $(t+1)$-design.

8. If every residual design of $\mathscr{B}$ is a t-design then $\mathscr{B}$ is a $(t+1)$-design.

1.6. A *symmetric 2-design* (projective design in [4]) is a 2-(v, k, λ)-design with the property that any two distinct blocks have exactly μ common points (then $\mu = \lambda$).

An *affine 2-design* [4] is a 2-design $\mathscr{B}$ over P such that any two blocks have either 0 or μ common points (disjoint points are then called parallel), and if $B \in \mathscr{B}$, $a \in P\backslash B$ then there is exactly one $L \in \mathscr{B}$ parallel to B which contains a.

A *Hadamard 2-design* is a 2-$(4n-1, 2n-1, n-1)$-design, or its complement, a 2-$(4n-1, 2n, n)$-design.

A *Hadamard 3-design* is a 3-$(4n, 2n, n-1)$-design.

1.7. (see [4]).

1. (Fisher inequality) Every nontrivial 2-(v, k, λ)-design satisfies $b \geqslant v$, $r \geqslant k$, with equality iff it is a symmetric 2-design.

2. A symmetric 2-design satisfies $\lambda(v-1) = k(k-1)$, and, with $n = k - \lambda$: $4n - 1 \leqslant v \leqslant n^2 + n + 1$. Equality holds on the left hand side iff the design is a Hadamard 2-design, and on the right hand side iff $\lambda = 1$.

3. Every Hadamard 3-design is a self-complementary affine 2-design with 2 blocks in every parallel class (namely, a block and its complement). Conversely, an affine 2-design with two blocks in a parallel class is a Hadamard 3-design.

1.8. Let $(\mathscr{P}, \mathscr{B}, \mathscr{F})$ be an incidence structure, and $a, b \in \mathscr{P}$, $A, B \in \mathscr{B}$. We define

$\delta_{ab} = 1$ if $a = b$, $= 0$ otherwise;

$\delta_{AB} = 1$ if $A = B$, $= 0$ otherwise;

$i_{aB} = 1$ if $a\mathrm{I}B$, $= 0$ otherwise;

$\lambda_{ab}(\lambda_{abc}, \ldots) =$ number of blocks incident with $a, b(c, \ldots)$;

$\mu_{AB}(\mu_{ABC}, \ldots) =$ number of points incident with $A, B(C, \ldots)$.

We denote an all-one vector by j, an all-one matrix (of any size) by J, the identity matrix (of any size) by I, and the matrix $(i_{aB})_{a \in \mathscr{P}, B \in \mathscr{B}}$ by A. A is called the *incidence matrix* of $(\mathscr{P}, \mathscr{B}, \mathscr{F})$.

The incidence matrix of the dual structure is A^{T}. $(\mathscr{P}, \mathscr{B}, \mathscr{F})$ is a 1-design iff $AJ = rJ$, $JA = kJ$, a 2-design iff $AA^{\mathrm{T}} = (r - \lambda)I + \lambda J$, and a symmetric 2-design iff $AA^{\mathrm{T}} = A^{\mathrm{T}}A = nI + \lambda J$.

1.9. A *transversal design* [6], $T[k, \lambda; u]$ is a triple $(P, \mathscr{G}, \mathscr{B})$ consisting of a point set P, a partition $\mathscr{G}$ of P in k classes of size u each, and a collection

$\mathscr{B}$ of subsets of P (blocks) such that every block contains exactly one point from every class, and if a, b are points from different classes then there are exactly λ blocks containing a and b.

Every transversal design is a 1-design with parameters $v = ku$, $b = \lambda u^2$, $r = \lambda u$, k.

The dual of a transversal design with $\lambda = 1$ is known as a *net* [1].

1.10. A *strongly regular graph* [1] is a (simple, undirected) graph G such that every vertex is adjacent with the same number of other vertices, and the number of vertices adjacent with two distinct vertices a, b depends only on whether a, b are adjacent or not.

The *adjacency matrix* of a graph is the matrix $M = (m_{ab})$ with $m_{ab} = 1$ or 0 depending on whether a and b are adjacent or not. M is a symmetric (0, 1)-matrix with zero diagonal. The *eigenvalues* of a graph are the eigenvalues of its adjacency matrix.

A graph is strongly regular iff its adjacency matrix M satisfies equations of the form $MJ = pJ$, $M^2 = qM + rJ + sI$.

1.11. An incidence structure is called *geometric* (or, a *partial plane*) if it satisfies one of the two equivalent conditions: Two distinct points have at most one common incident block, and two distinct blocks have at most one common incident point. The blocks of a geometric incidence structure are also called *lines*.

A *partial geometry* $PG(r, k, \alpha)$ [1] is a geometric tactical configuration with parameters $(v, k;, b, r)$ such that for any nonincident point-line-pair (a, B) there are exactly $\alpha \geqslant 1$ lines containing a and intersecting B. In the literature we find often the parameters $s := k - 1$, $t := r - 1$ (and sometimes t instead of α).

A partial geometry with $\alpha = 1$ is called a *generalized quadrangle*.

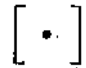
1.12. In [1], the following results are proved:

1. A $PG(r, k, \alpha)$ has $v = k + k(k-1)(r-1)/\alpha = (s+1)(st+\alpha)/\alpha$ points, and $b = r + r(r-1)(k-1)/\alpha = (t+1)(st+\alpha)/\alpha$ blocks.
2. The dual of a $PG(r, k, \alpha)$ is a partial geometry $PG(k, r, \alpha)$.
3. A $PG(r, k, \alpha)$ is a 2-design (with $\lambda = 1$) iff $\alpha = k = s + 1$, the dual of a 2-design iff $\alpha = r = t + 1$, a transversal design (with $\lambda = 1$) iff $\alpha = k - 1 = s$, a net iff $\alpha = r - 1 = t$.
4. For a $PG(r, k, \alpha)$, the graph G with vertex set P, and edges $\overline{ab}$ iff a and b are incident with the same line, is strongly regular, and some eigenvalue of G has multiplicity $\sigma = k(k-1)r(r-1)/\alpha(k + r - \alpha - 1) = st(s+1)(t+1)/\alpha(s + t + 1 - \alpha)$. In particular, σ is an integer.

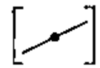
1.13. We occasionally use diagrams to illustrate proofs or definitions. In diagrams we represent fixed points by small circles, fixed blocks by dashed lines or arcs, and variable points or blocks by full circles, resp. full lines or arcs. Square brackets mean "number of configurations of the type in the bracket".

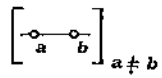
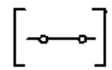
The incidences shown in a diagram are always required but there may be more incidences than indicated. If we want to exclude this we do it (for the fixed elements) in the text, and (for the variable elements) by giving the restrictions as subscripts.

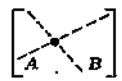
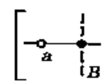
To make the reader familiar with such diagrams we interpret some of them below, for the case of 2-(v, k, λ)-designs.

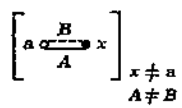
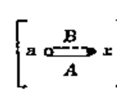
 $= v$,  $= b$,

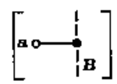
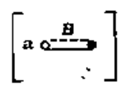
 $= r$,  $= k$,

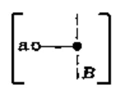
 $= vr = bk$,

 $= \lambda$,  $= \lambda_{ab} = \lambda + (r - \lambda)\,\delta_{ab}$,

 $= \mu_{AB}$,  $= \lambda k$ (if $a \notin B$),

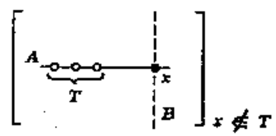
 $= (\lambda - 1)(k - 1)$,  $= \lambda(k - 1) + r$,

 $=$  for $a \in B$, whence in general,

 $= \lambda k + (r - \lambda)\, i_{aB}$.

2. $t\frac{1}{2}$-Designs

2.1. If T and B are any sets of points of a block design $\mathscr{B}$ then we call a flag (x, A) compatible with (T, B) if A contains T, and x lies in B but not in T. We denote by $\alpha(T, B)$ the number of flags compatible with (T, B), i.e.,

$\alpha(T, B) =$ 

2.2. Lemma. *Let $\mathscr{B}$ be a $(t + 1) - (v, k, \lambda_{t+1})$-design over P. If B is any w-subset, and T is a t-subset of P with $|B \cap T| = i$ then $\alpha(T, B) = (w - i)\lambda_{t+1}$.*

Proof. There are exactly $w - i$ points $x \in B$ which are not in T, and for every such x there are exactly λ_{t+1} blocks A containing the $(t + 1)$-set $T \cup \{x\}$. Hence there are $(w - i)\lambda_{t+1}$ flags (x, A) compatible with (T, B).

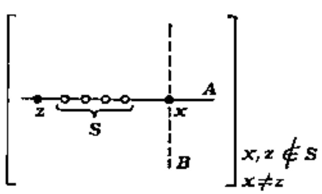
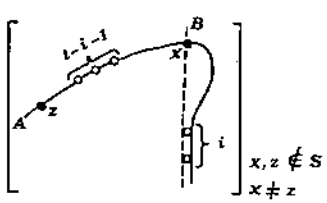
In particular, $\alpha(T, B)$ depends only on $|B \cap T|$. We shall look at t-designs which have this property at least for blocks B.

2.3. A *$t\frac{1}{2}$-design* is a t-design $\mathscr{B}$ over P for which there are integers $\alpha_0, \ldots, \alpha_t$ such that $\alpha(T, B) = \alpha_i$ for every block $B \in \mathscr{B}$ and every t-subset T of P with $|T \cap B| = i$ $(i = 0, \ldots, t)$.

We call a $t\frac{1}{2}$-design *nontrivial* if it is nontrivial as a t-design. A $t\frac{1}{2}$-design which is a t-(v, k, λ)-design is called a $t\frac{1}{2}$-(v, k, λ)-design.

Of course, every $t\frac{1}{2}$-design is a t-design, and, by 1.5.2 and 2.2, every $(t + 1) - (v, k, \lambda_{t+1})$-design is a $t\frac{1}{2}$-(v, k, λ)-design with $\lambda = ((v-t)/(k-t))\lambda_{t+1}$, $\alpha_i = (k - i)\lambda_{t+1} = ((k - i)(k - t)/(v - t))\lambda$.

2.4. The parameters $\alpha_0, \ldots, \alpha_t, v, k, \lambda$ of a $t\frac{1}{2}$-design are not independent. To show this we fix a $(t - 1)$-set S and a block B with $|S \cap B| = i$ $(0 \leqslant i \leqslant t - 1)$, and count in two ways the number N_i of triples $(z, x, A) \in P \times P \times \mathscr{B}$ with $x, z \notin S$, $x \neq z$, $x \in B$, $S \cup \{x, z\} \subseteq A$; i.e.,

$N_i =$  $=$ 

For every (x, A) compatible with this figure the number of $z \in A$ with $z \notin S \cup \{x\}$ is $k - t$, and the number of compatible (x, A) is $\alpha(S, B) = (k - i)\lambda$ since $\mathscr{B}$ is a t-design. Hence $N_i = (k - i)(k - t)\lambda$.

On the other hand, for fixed $z \notin S$, the number of suitable flags (x, A) is α_i if $z \notin B$, and α_{i+1} if $z \in B$. Now there are $k - i$ points $z \notin S$ which are on B, and $(v - k) - (t - i - 1)$ points $z \notin S$ which are not on B. Hence $N_i = (v - k - t + i + 1)\alpha_i + (k - i)\alpha_{i+1}$. By comparison,

$$\alpha_{i+1} = (k - t)\lambda - \frac{v - k - t + i + 1}{k - i}\alpha_i \qquad (i = 0, \ldots, t - 1). \tag{1}$$

To turn this into an explicit formula we introduce the integers $\epsilon_i = (v - t)\alpha_i - (k - i)(k - t)\lambda$ so that

$$\alpha_i = \frac{1}{v - t}(\epsilon_i + (k - i)(k - t)\lambda). \tag{2}$$

By 2.3, $\epsilon_i = 0$ for a $t\frac{1}{2}$-design which is actually a $(t + 1)$-design, so ϵ_i measures in a sense the deviation of a $t\frac{1}{2}$-design from a $(t + 1)$-design (cf. Lemma 4.4).

Inserting (2) in (1) gives $\epsilon_i(v - k - t + i + 1) = -(k - 1)\epsilon_{i+1}$, and a simple induction argument gives Eq. (3) of:

2.5. Lemma. *Let $\mathscr{B}$ be a $t\frac{1}{2}$-(v, k, λ)-design with parameters $\alpha_0, \ldots, \alpha_t$. Then*

$$\alpha_{i+1} = (k - t)\lambda - \frac{v - k - t + i + 1}{k - i}\alpha_i \qquad (i = 0, \ldots, t - 1), \tag{1}$$

$$\alpha_i = \frac{1}{v - t}(\epsilon_i + (k - i)(k - t)\lambda) \qquad (i = 0, \ldots, t), \tag{2}$$

with integers ϵ_i satisfying

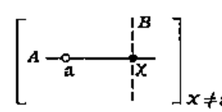
$$\epsilon_i = (-1)^{t-i}\binom{v - k}{t - i}^{-1}\binom{k - i}{k - t}\epsilon_t \qquad (i = 0, \ldots, t). \tag{3}$$

In particular, a $(t + 1)$-design viewed as a $t\frac{1}{2}$-design has $\epsilon_i = 0$ for all i.

Now we give some examples of $t\frac{1}{2}$-designs which are not $(t + 1)$-designs.

2.6. Example. The dual $\mathscr{B}$ of a 2-(v, k, λ)-design (with b blocks and r blocks through any point) is a $1\frac{1}{2}$-design with parameters $v' = b$, $k' = r$, $b' = v$, $r' = k$, $\alpha_0 = k\lambda$, $\alpha_1 = k(\lambda - 1) + r - \lambda$.

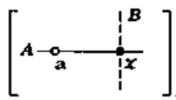
Proof. Of course, $\mathscr{B}$ is a 1-design with the stated parameters. Given a point a and a block B we count

$\alpha(a, B) =$ 

The number of blocks A containing a is $r' = k$. If $a \notin B$ then $A \neq B$, and A and B have λ common points none of which can be a. Hence $\alpha_0 = k\lambda$. If $a \in B$ then one of the k blocks is $A = B$ with $k' - 1 = r - 1$ points $x \neq a$ in $A \cap B$, and the remaining $k - 1$ blocks $A \ni a$ have $\lambda - 1$ points $x \neq a$ in $A \cap B$. Hence $\alpha_1 = r - 1 + (\lambda - 1)(k - 1) = k(\lambda - 1) + r - \lambda$.

2.7. Example. If $(P, \mathscr{G}, \mathscr{B})$ is a transversal design $T[k, \lambda; u]$ then $\mathscr{B}$ is a $1\frac{1}{2}$-design over P with parameters $v' = uk$, $k' = k$, $b' = u^2\lambda$, $r' = u\lambda$, $\alpha_0 = (k-1)\lambda$, $\alpha_1 = (k-1)\lambda$.

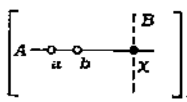
Proof. Again, $\mathscr{B}$ is a 1-design with the stated parameters. Given a point a and a block B we count

$\alpha(a, B) =$  $_{x \neq a}$.

Suppose $C \in \mathscr{G}$ is the class in which a lies. Then $|B \cap C| = 1$, say $z \in B \cap C$. Through a and any of the $k - 1$ points $x \neq z$ on B there are exactly λ blocks A giving $\lambda(k-1)$ admissible flags (x, A). $x = z$ is only possible if $z \neq a$, and then there is no block through a and z so that there is no additional admissible flag. Hence $\alpha(a, B) = (k-1)\lambda$, independently of a, B.

2.8. Example. A symmetric 2-(v, k, λ)-design is a $2\frac{1}{2}$-design with parameters $v, k, b = v, r = k, \lambda, \alpha_0 = \lambda^2, \alpha_1 = \lambda(\lambda - 1), \alpha_2 = \lambda^2 - 3\lambda + k$.

Proof. Given two distinct points a, b, and a block B we count

$\alpha(a, b, B) =$  $_{x \neq a, b}$.

There are λ blocks A through a and b.

If $a, b \in B$ then $\lambda - 1$ of them are distinct from B and intersect B in λ points, $\lambda - 2$ of which are distinct from a and b; the remaining block through a and b is B, and then there are $k - 2$ choices for x. Hence $\alpha_3 = (\lambda - 1)(\lambda - 2) + k - 2 = \lambda^2 - 3\lambda + k$.

If a, b are not both in B then all λ blocks through a and b are distinct from B and intersect B in λ points. Hence we have for x λ choices each if $a, b \notin B$, and $\lambda - 1$ if one of a, b is in B. Hence $\alpha_0 = \lambda^2$, $\alpha_1 = \lambda(\lambda - 1)$.

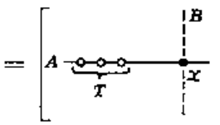
From this, we obtain the well-known

2.9. Corollary. *A nontrivial symmetric 2-design can never be a 3-design.*

Proof. A symmetric 2-(v, k, λ)-design $\mathscr{B}$ is a $2\frac{1}{2}$-design with $\epsilon_0 = (v-2)\alpha_0 - k(k-2)\lambda = \lambda(k - \lambda)$. A 3-design is a $2\frac{1}{2}$-design with $\epsilon_0 = 0$. Hence, if $\mathscr{B}$ is a 3-design then $\lambda = 0$ or $\lambda = k$. In the second case we have $v = k$ so that in both cases $\mathscr{B}$ is trivial.

2.10. Example. A Hadamard 3-design is a $3\frac{1}{2}$-design with parameters $v = 4n$, $k = 2n$, $b = 8n - 2$, $r = 4n - 1$, $\lambda_2 = 2n - 1$, $\lambda = n - 1$, $\alpha_0 = n(n-2)$, $\alpha_1 = (n-1)^2$, $\alpha_2 = (n-1)(n-2)$, $\alpha_3 = n^2 - 3n + 3$.

Proof. Given a set T of 3 distinct points, and a block B. Again we count

$\alpha(T, B) =$  $_{x \notin T}$.

There are just $\lambda = n - 1$ blocks A containing T, and by 1.7.3, we have for any block A:

$|A \cap B| = 2n$ iff $A = B$, $= 0$ iff $A = P - B$, and $= n$, otherwise. The case $A = B$ occurs iff $T \subseteq B$, and $A = P - B$ iff $T \cap B = \varnothing$. Given A there are just $|A \cap B| - i$ possibilities for x, where $i = |T \cap B|$. Hence

if $i = 0$ then $\alpha(T, B) = (n-2)(n-0) + 1(0-0) = n(n-2)$,

if $i = 1$ then $\alpha(T, B) = (n-1)(n-1) = (n-1)^2$,

if $i = 2$ then $\alpha(T, B) = (n-1)(n-2)$, and

if $i = 3$ then $\alpha(T, B) = (n-2)(n-3) + 1(2n-3) = n^2 - 3n + 3$.

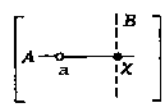
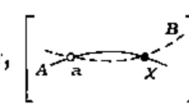
2.11. We mention the following trivial construction. If $\mathscr{B}$ is a $t\frac{1}{2}$-design with parameters v, k, λ, α_i $(i = 0, \ldots, t)$ then the collection $s \times \mathscr{B}$ containing each block of $\mathscr{B}$ s times (called a *multiple* of $\mathscr{B}$) is a $t\frac{1}{2}$-design with parameters $v, k, \lambda s, s\alpha_0, \ldots, s\alpha_t$.

Note that (unlike for t-designs) the disjoint union of two different $t\frac{1}{2}$-designs over the same point set P need not be a $t\frac{1}{2}$-design!

3. $1\frac{1}{2}$-Designs

3.1. There is an alternative, self-dual definition of a $1\frac{1}{2}$-design:

A $1\frac{1}{2}$-*design* is a tactical configuration $(\mathscr{P}, \mathscr{B}, \mathscr{F})$ satisfying $s(a, B) = \alpha$ if $a \notin B$, $= \beta$ if $a \in B$, where $s(a, B)$ is the number of flags (x, A) with $x \in B$, $a \in A$, and $x \neq a$, $A \neq B$. That is, we require

 $_{a \notin B} = \alpha$,  $_{x \neq a,\ A \neq B} = \beta$,

for all choices of a, B compatible with the figures. (Note that if B is a repeated block with multiplicity e then $A \neq B$ means that, *as a point set*, A may be $(e-1)$ times chosen as B!)

One observes immediately that $\alpha(\{a\}, B) = s(a, B) + (k-1)\, i_{aB}$ so that the two definitions of $1\frac{1}{2}$-designs are equivalent, and

$$\alpha_0 = \alpha, \qquad \alpha_1 = \beta + k - 1. \tag{1}$$

3.2. For reasons which will later become clear we introduce the integer

$$n = r + k + \beta - \alpha - 1 = r - \alpha_0 + \alpha_1 . \tag{2}$$

A $1\frac{1}{2}$-design which has as a tactical configuration the parameters (v, k, b, r) is said to have parameters (r, k, α, β, n), where α, β, n are as above. If $\alpha > 0$, $3 \leqslant k \leqslant v - 3$, $3 \leqslant r \leqslant b - 3$ then we call the $1\frac{1}{2}$-design *proper* (the improper $1\frac{1}{2}$-designs are easily classified).

The dual of a $1\frac{1}{2}$-design is again a $1\frac{1}{2}$-design; α, β, n are preserved, v, b and k, r are interchanged by dualization. Also, the complement of a $1\frac{1}{2}$-design is a $1\frac{1}{2}$-design with $n' = n$ (This is a simple consequence of Theorem 3.11 below).

3.3. If we count in two ways the number of triples $(x, y, A) \in \mathscr{P} \times \mathscr{P} \times \mathscr{B}$ satisfying $x, y \in A$, $x \in B$, $x \neq y$, $A \neq B$ (for a fixed block B) we obtain

$$(v-k)\alpha + k\beta = k(k-1)(r-1). \tag{3}$$

This also follows from Lemma 2.5. Using $vr = bk$, $v > k > 0$, $\alpha > 0$, $\beta \geqslant 0$ we obtain for a proper $1\frac{1}{2}$-design from Eqs. (1)–(3):

$$v = \frac{k(kr-n)}{\alpha}, \tag{4i}$$

$$b = \frac{r(kr-n)}{\alpha}, \tag{4ii}$$

$$\beta = \alpha + n + 1 - r - k, \tag{4iii}$$

$$k + r \leqslant n + \alpha + 1 \leqslant kr. \tag{5}$$

3.4. We have the following examples of $1\frac{1}{2}$-designs (for A and B see Section 2):

A. 2-(v, k, λ)-designs with parameters v, k, $b = \lambda v(v-1)/(k-1)$, $r = \lambda(v-1)/(k-1)$, $\alpha = \lambda k$, $\beta = (\lambda - 1)(k-1)$, $n = r - \lambda = \lambda(v-k)/(k-1)$ (we chose the letter n for Expression (2) since it is the standard expression for $r - \lambda$ in a 2-design).

B. Transversal designs $T[k, \lambda; u]$ with parameters $v = ku$, k, $b = \lambda u^2$, $r = \lambda u$, $\alpha = \lambda(k-1)$, $\beta = (\lambda-1)(k-1)$, $n = \lambda u$.

C. The duals of 2-designs and transversal designs.

D. The symmetric complete bipartite graph $K_{n,n}$ on $2n$ vertices with parameters $v = 2n$, $k = 2$, $b = n^2$, $r = n$, $\alpha = 1$, $\beta = 0$, n.

E. The dual of $K_{n,n}$, a quadratic grid of side n, with parameters $v = n^2$, $k = n$, $b = 2n$, $r = 2$, $\alpha = 1$, $\beta = 0$, n. ($K_{n,n}$ and its dual are examples of improper $1\frac{1}{2}$-designs).

F. Partial geometries. In fact we have:

3.5. Lemma. (a) *An incidence structure is a partial plane iff $s(a, B) = 0$ for every flag (a, B).*

(b) *A partial plane $\mathscr{B}$ is a $1\frac{1}{2}$-design iff it is a partial geometry.*

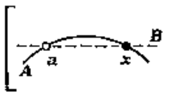
(c) *For a $1\frac{1}{2}$-design, the following statements are equivalent:*

(1) $\beta = 0$,

(2) *$\mathscr{B}$ is a partial plane,*

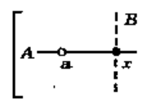
(3) *$\mathscr{B}$ is a partial geometry.*

Proof. (a) There are two distinct points a, x and two distinct blocks A, B with $a, x \in A, B$ iff

$s(a, B) =$ []$_{x \neq a,\ A \neq B}$ ≥ 1

for some flag (a, B).

(b) If $\mathscr{B}$ is a partial geometry then for $a \in B$, $s(a, B) = 0$ since $\mathscr{B}$ is a partial plane. For $a \notin B$,

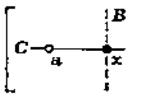
$s(a, B) =$ [] .

There are exactly α lines $A \ni a$ meeting B, since $\mathscr{B}$ is a partial plane, each such line determines a unique $x \in A \cap B$. Hence $s(a, B) = \alpha$. Conversely, if $\mathscr{B}$ is a $1\frac{1}{2}$-design and $a \notin B$ then as above, $s(a, B)$ is the number of lines $A \ni a$ meeting B. Hence this is a constant α.

Part (c) follows at once from (a) and (b).

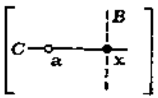
3.6. Lemma. *A proper $1\frac{1}{2}$-design $\mathscr{B}$ is a generalized quadrangle iff $\alpha = 1$ (then automatically $\beta = 0$).*

Proof. (a) Suppose first $\mathscr{B}$ has no repeated blocks. If $A \neq B$ then there is a point $a \in A$, $a \notin B$, and

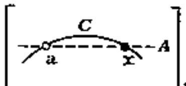
$$1 = \alpha(a, B) = \Big[$$  $$\Big].$$

The possibility $C = A$ gives a solution for every $x \in A \cap B$, hence $|A \cap B| \leqslant 1$. Therefore $\mathscr{B}$ is a partial plane, and by Lemma 3.5 a partial geometry. By definition, β is a generalized quadrangle ($\alpha = 1$).

(b) Suppose now A is a repeated block of $\mathscr{B}$. Let B be any block intersecting A in some point x. If A contains a point $a \notin B$ then

$$1 = \alpha(a, B) = \Big[$$  $$\Big] \geqq 2$$

since we can take for C at least twice the repeated block A, a contradiction. Hence B contains all the points of A. Since there are r blocks through a point $x \in A$, A has multiplicity r. But if $a \in A$ then

$$\alpha_1 = \alpha(a, A) = \Big[$$  $$\Big]_{x \neq a} = r(k-1)$$

whence by (2), (4iii), and (4i) $n = rk - 1$, $v = k$, and $\mathscr{B}$ is improper.

(c) The converse is true by definition and Lemma 3.5.

There are some more conditions on the parameters of a proper $1\frac{1}{2}$-design.

3.7. LEMMA. *The parameters of a proper $1\frac{1}{2}$-design $\mathscr{B}$ satisfy*

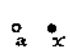
$$\alpha \geqslant k(r-n), \tag{6a}$$

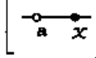
with equality iff $\mathscr{B}$ is a 2-design (in this case $\lambda = r - n$). Dually,

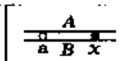
$$\alpha \geqslant r(k-n), \tag{6b}$$

with equality iff $\mathscr{B}$ is a dual 2-design (with $\lambda^{\text{dual}} = k - n$).

Proof. Fix $a \in P$. We define for $i = 0, 1, 2$, $s_i = \sum_{x \neq a} \lambda_{ax}^i$. Then

$$s_0 = \sum_{x \neq a} 1 = \Big[$$  $$\Big]_{x \neq a} = v - 1,$$

$$s_1 = \sum_{x \neq a} \lambda_{ax} = \Big[$$  $$\Big]_{x \neq a} = r(k-1),$$

$$s_2 - s_1 = \sum_{x \neq a} \lambda_{ax}(\lambda_{ax} - 1) = \Big[$$  $$\Big]_{\substack{x \neq a \\ A \neq B}} = \sum_{B \ni a} s(a, B) = r\beta.$$

Now

$$0 \leqslant \sum_{x \neq a} \left(\lambda_{ax} - \frac{s_1}{s_0}\right)^2 = s_2 - 2\frac{s_1}{s_0} s_1 + \frac{s_1}{s_0{}^2} s_0$$
$$= \frac{s_2 s_0 - s_1{}^2}{s_0} = \frac{r(v-k)}{k(v-1)} (\alpha - k(r-n)),$$

where we simplified using (4i) and (4iii). This implies (6a). Moreover if $\alpha = k(r-n)$ then the sum of squares is zero whence $\lambda_{ax} = s_1/s_0$ for every $x \neq a$. Since a was arbitrary, $\mathscr{B}$ must be a 2-design. On the other hand, every 2-design has $\lambda = r - n$ and satisfies (6a) with equality.

3.8. COROLLARY. (1) (Fisher inequality [4]) *A nontrivial 2-(v, k, λ)-design satisfies $b \geqslant v$, with equality iff $\mathscr{B}$ is a dual 2-(v, k, λ)-design (and hence a symmetric 2-design).*

(2) (Hanani [6]) *A transversal design $TD[k, \lambda; u]$ with $u > 1$ satisfies $k \leqslant (\lambda u^2 - 1)/(u - 1)$, with equality iff it is a dual 2-design (and hence a dual affine design).*

Proof. Apply (6b) of Lemma 3.7 to $\mathscr{B}$ considered as a $1\frac{1}{2}$-design and simplify with 3.4.

3.9. We give now characterizations of transversal designs and affine designs.

A *point parallelism* of a block design $\mathscr{B}$ over P is a partition $\mathscr{G}$ of P into *classes* with the property: $A \in \mathscr{G}$, $B \in \mathscr{B}$ implies $|A \cap B| = 1$; i.e., every block contains exactly one point from every class.

Of course $\mathscr{G}$ contains exactly k classes.

LEMMA. *Let $\mathscr{B}$ be a proper $1\frac{1}{2}$-design over P with parameters $(v, k, b, r; \alpha, \beta, n)$, and $\mathscr{G}$ be a point parallelism of $\mathscr{B}$. Then every class contains $u = v/k$ points. Moreover,*

$$n \geqslant r, \tag{7}$$

with equality iff $(P, \mathscr{G}, \mathscr{B})$ is a transversal design $T[k, \lambda; u]$ with $\lambda = rk/v$.

Proof. Let A be a fixed class of $\mathscr{G}$, and $|A| = u$. If we count in two ways the number of flags $(x, B) \in A \times \mathscr{B}$ we obtain $b = ru$, whence $u = b/r = v/k$. For fixed $a \in A$, we have, in the notation of Lemma 3.7,

$$\sum_{x \notin A} 1 = v - u, \qquad \sum_{x \notin A} \lambda_{ax} = s_1, \qquad \sum_{x \notin A} \lambda_{ax}^2 = s_2. \tag{8}$$

Using (4i) and (4iii) we obtain

$$0 \leqslant \sum_{x \notin A} (\lambda_{ax} - rk/v)^2 = r(n - r)(1 - k/v). \tag{9}$$

Hence $n \geqslant r$. If $n = r$ then (9) implies $\lambda_{ax} = rk/v$ for all $x \notin A$. Since $A \in \mathscr{G}$, $a \in A$ were arbitrary, $(P, \mathscr{G}, \mathscr{B})$ is a transversal design. Conversely, a transversal design satisfies (7) with equality, by 3.4.

3.10. *Remarks.* (1) A partial geometry with $n = r$ (i.e., $\alpha = k - 1$) has the point parallelism induced by the equivalence relation $\equiv$ on the points defined by $x \equiv y$ iff $\lambda_{xy} = 0$; hence a $1\frac{1}{2}$-design with $n = r$ and $\beta = 0$ is always a transversal design.

(2) The dual concept of a point parallelism is a (block) parallelism. A *parallelism* of a block design $\mathscr{B}$ over P is an equivalence relation $\parallel$ of the blocks satisfying the parallel axiom: For every $x \in P$ and every $B \in \mathscr{B}$ there is exactly one block $A \parallel B$ with $x \in A$.

Applying Lemma 3.9 to the dual of a 2-design with parallelism we obtain the well-known

COROLLARY [4]. *The parameters of a proper 2-(v, k, λ)-design with parallelism satisfy*

$$r \geqslant k + \lambda \ (\textit{or, equivalently, } b \geqslant v + r - 1), \tag{10}$$

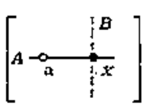
with equality iff the design is an affine design.

3.11. THEOREM. *An incidence structure $\mathscr{B}$ is a $1\frac{1}{2}$-design with parameters $(v, k, b, r; \alpha, \beta, n)$ iff its incidence matrix A satisfies the equations*

$$AJ = rJ, \ JA = kJ, \ AA^{\mathrm{T}}A = nA + \alpha J. \tag{11}$$

The remaining parameters are then defined by Eq. (4i)–(4iii).

Proof. A satisfies $AJ = rJ$ and $JA = kJ$ iff $\mathscr{B}$ is a 1-design with parameters v, k, b, r. The (a, B)-entry of $AA^{\mathrm{T}}A$ is

$$\sum_A \sum_x i_{aA} i_{xA} i_{xB} =$$  $$= s(a, B) + (r + k - 1)\, i_{aB},$$

as one calculates easily. On the other hand, the (a, B)-entry of $nA + \alpha J$ is $ni_{aB} + \alpha$. Hence $AA^{\mathrm{T}}A = nA + \alpha J$ is equivalent to $s(a, B) + (r + k - 1)i_{aB} = ni_{aB} + \alpha$ for all a, B, or $s(a, B) = \alpha$ if $a \notin B$, $= \alpha + n + 1 - r - k = \beta$ if $a \in B$, i.e., equivalent to $\mathscr{B}$ being a $1\frac{1}{2}$-design.

Another nontrivial parameter condition for $1\frac{1}{2}$-designs (the *divisibility condition*) comes from

3.12. LEMMA. *If A is the incidence matrix of a proper $1\frac{1}{2}$-design then $N = AA^{\mathrm{T}}$ satisfies*

$$NJ = krJ, \ N^2 = nN + \alpha rJ. \tag{12}$$

N has the eigenvalues kr, n, 0 with multiplicities 1, σ, and $v - 1 - \sigma$ respectively, where

$$\sigma = r(v - k)/n. \tag{13}$$

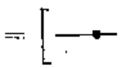
In particular,

$$n > 0, \tag{14}$$

$$r(v - k) \equiv 0 \bmod n. \tag{15}$$

Proof. We obtain from (11) $NJ = AA^{\mathrm{T}}J = krJ$, $N^2 = AA^{\mathrm{T}}AA^{\mathrm{T}} = nAA^{\mathrm{T}} + JA^{\mathrm{T}} = nN + \alpha rJ$. This gives (12). Define $f(x) = x(x - n)/\alpha r$. Then $f(N) = J$. From $NJ = krJ$ follows $Nj = krj$, whence kr is an eigenvalue of N corresponding to the simple eigenvalue $f(kr) = v$ of J. Hence kr has multiplicity 1. The only other eigenvalue of J is 0, with multiplicity $v - 1$, whence any eigenvalue $\rho \neq kr$ of N satisfies $f(\rho) = 0$. Hence $\rho = n$ or $\rho = 0$. If $\rho = n$ has multiplicity σ then $\rho = 0$ has multiplicity $v - 1 - \sigma$.

Now the trace of N, $\mathrm{tr}(N)$, is the sum of the eigenvalues which is $1 \cdot kr + \sigma \cdot n + (v - 1 - \sigma) \cdot 0 = kr + \sigma n$. On the other hand,

$$\mathrm{tr}(N) = \mathrm{tr}(AA^{\mathrm{T}}) = \sum_a \sum_B i_{aB}^2 =$$  $$= vr.$$

Therefore $kr + \sigma n = vr$, or $\sigma n = (v - k)r$. Since $\mathscr{B}$ is proper we have $\sigma n > 0$, and since σ is nonnegative, $n > 0$ and $\sigma = (v - k)r/n$. This implies (13)–(15).

3.13. Lemma 3.12 can be used for an alternative proof of 3.7. The eigenvalue 0 of N has multiplicity $0 \leqslant v - 1 - \sigma = v(\alpha - k(r - n))/kn$, whence $\alpha \geqslant k(r - n)$ with equality iff N is nonsingular. In this case we obtain from (12): $AA^{\mathrm{T}} = N = N^{-1}N^2 = N^{-1}(nN + \alpha rJ) = nI + N^{-1}$. $(\alpha r/kr)\, NJ = nI + (\alpha/k)J$. By 1.8, $\mathscr{B}$ must be a 2-design.

3.14. Suppose now that $\mathscr{B}$ is a $1\frac{1}{2}$-design over P which is not a 2-design, and satisfies $\lambda_{ab} \in \{\lambda_1, \lambda_2\}$ for all $a \neq b$, and $\lambda_1 > \lambda_2$. (Examples are transversal designs and partial geometries; see Remarks 2, 3, below.)

Fix $a \in P$, and suppose there are m points $b \neq a$ with $\lambda_{ab} = \lambda_1$, and $v - 1 - m$ points $b \neq a$ with $\lambda_{ab} = \lambda_2$. Then $0 < m < v - 1$ since $\mathscr{B}$ is not a 2-design. Using the notation of 3.7, we have

$$r(k-1) = s_1 = m_1 + (v - 1 - m)\,\lambda_2, \tag{16a}$$

$$r\beta = s_2 - s_1 = m\lambda_1(\lambda_1 - 1) + (v - 1 - m)\,\lambda_2(\lambda_2 - 1). \tag{16b}$$

Hence, independently of a,

$$m = \frac{(k-1)\,r - (v-1)\,\lambda_2}{\lambda_1 - \lambda_2}. \tag{17}$$

Substituting $\lambda_2 = \lambda_1 - d$ in (16a) we obtain

$$\lambda_1 = \frac{r(k-1) + (v-1-m)\,d}{v-1}, \qquad \lambda_2 = \frac{r(k-1) - md}{v-1}. \tag{18}$$

Substituting this into (16b) gives, using (4)

$$d^2 = \frac{r(v-k)(\alpha - k(r-n))}{km(v-1-m)}, \qquad d > 0. \tag{19}$$

Note that by (6a) the expression for d^2 is always positive. The fact that λ_1, λ_2 must be integers, and that (19) must be an integer square, restricts the possibilities for m.

3.15. With the supposition of 3.14 define the *point graph* G of $\mathscr{B}$ in the following way: Vertices are the points of $\mathscr{B}$, and two vertices are adjacent iff they are contained in exactly λ_1 blocks; i.e., $\overline{ab} \in G$ iff $\lambda_{ab} = \lambda_1$.

Denote by M the adjacency matrix of G. Then

$$N = AA^{\mathrm{T}} = dM + \lambda_2 J + (r - \lambda_2)I. \tag{20}$$

Substituting this in (12) gives, using (17)–(19),

$$MJ = mJ, \tag{21a}$$

$$M^2 = \frac{2\lambda_2 - 2r + n}{d} M + \frac{\lambda_2^2 - \lambda_1(rk + md + r - n) + \alpha r}{d^2} J + \frac{(r-\lambda_2)(n - r + \lambda_2)}{d^2} I. \tag{21b}$$

By 1.10, G is a strongly regular graph.

From (12) and (20) we get $M = f(N)$ with $f(x) = (1/d)(x - (r - \lambda_2) - (\lambda_2/\alpha r)x(x-n))$. Hence M has the eigenvalues

$$f(kr) = m, f(n) = (1/d)(n - r + \lambda_2), f(0) = (1/d)(\lambda_2 - r) \tag{22}$$

with multiplicities $1, \sigma, v - \sigma$, respectively.

3.16. *Remarks.* (1) If $\mathscr{B}$ satisfies $\mu_{AB} \in \{\mu_1, \mu_2\}$ for all $A \neq B$, and $\mu_1 > \mu_2$ then the *block graph* of $\mathscr{B}$ is the graph whose vertices are the blocks of $\mathscr{B}$, and two vertices $A \neq B$ are adjacent iff $\mu_{AB} = \mu_1$. Arguments dual to the above show that the block graph is also strongly regular.

(2) A transversal design has $\lambda_1 = \lambda$, $\lambda_2 = 0$, $m = u(k-1)$. The point graph of a transversal design is the symmetric complete multipartite graph corresponding to the partition $\mathscr{G}$ of P. (This graph is a rather trivial strongly regular graph.)

(3) A partial geometry has $\lambda_1 = 1$, $\lambda_2 = 0$, $m = r(k-1)$. Hence:

COROLLARY [1]. *The point graph (and, dually, the line graph) of a partial geometry is a strongly regular graph.*

Note that the above discussion implies 1.12.4.

(4) A 2-design with $\mu_{AB} \in \{\mu_1, \mu_2\}$ for all $A \neq B$ is called a *quasi-symmetric design.* For example, all 2-designs with $\lambda = 1$ are quasi-symmetric. Since a 2-design is a $1\frac{1}{2}$-design, we have:

COROLLARY [5]. *The block graph of a quasi-symmetric design (in particular of a 2-design with $\lambda = 1$) is a strongly regular graph.*

4. DERIVED AND RESIDUAL DESIGNS; CHARACTERIZATIONS AND NONEXISTENCE

4.1. THEOREM. *Let $\mathscr{B}$ be a nontrivial $t\frac{1}{2}$-(v, k, λ)-design over P, $t \geq 1$. Then*

(i) *Every derived design is a $(t - \frac{1}{2}) - (v-1, k-1, \lambda)$-design with*

$$\alpha_i^{\text{der}} = \alpha_{i+1}, \qquad \epsilon_i^{\text{der}} = \epsilon_{i+1}.$$

(ii) *Every residual design is a $(t - \frac{1}{2}) - (v-1, k, ((v-k)/k - t + 1))\lambda)$-design with*

$$\alpha_i^{\text{res}} = \lambda(k - i) - \alpha_i, \ \epsilon_i^{\text{res}} = -\epsilon_i.$$

Proof. (i) Consider the derived design $\mathscr{B}_a$ with respect to $a \in P$. Given a $(t-1)$-set $T \subseteq P - \{a\}$, and a block $B' = B - \{a\} \in \mathscr{B}_a$ with $|T \cap B'| = i$

we have $|(T \cup \{a\}) \cap B| = i + 1$, whence $\alpha^{\text{der}}(T, B') = \alpha(T \cup \{a\}, B) = \alpha_{i+1}$. Hence the derived design is a $(t - \frac{1}{2})$-design, and the ϵ_i^{der} can be computed from Lemma 2.5.

(ii) Consider the residual design $\mathscr{B}^a$ with respect to $a \in P$. Given a $(t-1)$-set $T \subseteq P - \{a\}$, and a block $B \in \mathscr{B}^a$ with $|T \cap B| = i$ we count in two ways the number of pairs $(x, A) \in P \times \mathscr{B}$ with $T \cup \{x\} \subseteq A$, $a \notin A$, $x \in B$, $x \notin T \cup \{a\}$ and obtain $\alpha^{\text{res}}(T, B) = \lambda_i(k - i) - \alpha_i$, independent of T and B. Hence $\mathscr{B}^a$ is a $(t - \frac{1}{2})$-design, and the ϵ_i^{res} can be computed from Lemma 2.5.

Remark. It can be shown that the complement of a $t\frac{1}{2}$-design is again a $t\frac{1}{2}$-design. But the proof is much more complicated and it is, in view of the complete characterization of $t\frac{1}{2}$-designs with $t > 1$, omitted. The case $t = 1$ was considered in Section 3.

4.2. LEMMA. *Let $\mathscr{B}$ be a nontrivial t-design over P, $t \geqslant 2$.*

(a) *If every derived design of $\mathscr{B}$ is a $(t - \frac{1}{2})$-design then $\alpha(T, B)$ depends only on $|T \cap B|$, provided that $B \in \mathscr{B}$ and $T \cap B \neq \varnothing$.*

(b) *If every residual design of $\mathscr{B}$ is a $(t - \frac{1}{2})$-design then $\alpha(T, B)$ depends only on $|T \cap B|$, provided that $B \in \mathscr{B}$ and $T \not\subseteq B$.*

(c) *If every derived design, and every residual design of $\mathscr{B}$ is a $(t - \frac{1}{2})$-design then $\mathscr{B}$ is a $t\frac{1}{2}$-design.*

Proof. (a) We use for the moment $\alpha^a(T, B)$ for $\alpha(T, B)$ defined over the derived design with respect to $a \in P$. Let $a, b \in P$ be two distinct points, B be a block containing a, b, and T be a t-subset of B containing a, b. Then $\alpha_{t-1}^a = \alpha^a(T - \{a\}, B - \{a\}) = \alpha(T, B) = \alpha^b(T - \{b\}, B - \{b\}) = \alpha_{t-1}^b$. By Lemma 2.5 this implies that $\alpha_i^a = \alpha_i^b$ for all $i \leqslant t - 1$. Hence $\alpha_i^a =: \alpha_i^{\text{der}}$, independent of a. Now let B be any block, and T be any t-subset of P with $|T \cap B| = i > 0$. Take some $a \in T \cap B$. Then $\alpha(T, B) = \alpha^a(T - \{a\}, B - \{a\}) = \alpha_{i-1}^a = \alpha_{i-1}^{\text{der}}$, independent of T, B.

(b) Now we use $\alpha^a(T, B)$ for $\alpha(T, B)$ defined over the residual design with respect to $a \in P$. Let $a, b \in P$ be two distinct points, choose a block B not containing a, b, and a t-subset T, disjoint with B, containing a and b. This is possible since $\mathscr{B}$ is nontrivial, hence $v > k + t$. Then $\alpha_0^a = \alpha^a(T - \{a\}, B) = \alpha(T - \{a\}, B) - \alpha(T, B) = k\lambda - \alpha(T, B)$, by 2.2, and similarly $\alpha_0^b = k\lambda - \alpha(T, B)$. Hence $\alpha_0^a = \alpha_0^b$, and by Lemma 2.5, $\alpha_i^a = \alpha_i^b$ for all $i \leqslant t - 1$. Hence $\alpha_i^a =: \alpha_i^{\text{res}}$ is independent of a.

Now let B be any block, and T be any t-subset of P with $|T \cap B| = i < t$. Choose $a \in T - B$. Then $\alpha(T, B) = \alpha(T - \{a\}, B) = (k - i)\lambda - \alpha_i^a = (k - i)\lambda - \alpha_i^{\text{res}}$, independent of T, B.

4.3. *Remarks.* (1) A consequence of the proof of Lemma 4.2 is that if a t-design contains two or more points $a \in P$ such that the derived (residual) design with respect to a is a $(t - \frac{1}{2})$-design then all these derived (residual) designs have the same parameters.

(2) A t-design such that every derived design is a $(t - \frac{1}{2})$-design need not be a $t\frac{1}{2}$-design. Examples are affine 2-designs. The only affine 2-designs which are $2\frac{1}{2}$-designs are those with exactly two blocks in a parallel class (and these are in fact Hadarmard $3\frac{1}{2}$-designs); but every derived design of any affine 2-design is a dual 2-design and hence a $1\frac{1}{2}$-design.

4.4. LEMMA. *The parameters of a nontrivial $t\frac{1}{2}$-design $\mathscr{B}$ with $t \geqslant 1$ satisfy $\epsilon_t \geqslant 0$, with equality iff $\mathscr{B}$ is a $(t + 1)$-design.*

Proof. (a) Every $(t + 1)$-design is a $t\frac{1}{2}$-design with $\epsilon_t = 0$; this was already mentioned in 2.4.

(b) We use induction to prove the converse. For $t = 1$ we have $\epsilon_1 = (v - 1)\alpha_1 - (k - 1)^2 r = ((v - k)/k)(\alpha - k(r - n))$, using (1) and (4) of Section 3. By Lemma 3.7, $\epsilon_1 \geqslant 0$ with equality iff $\mathscr{B}$ is a 2-design.

For arbitrary t, we know that every derived design is a $(t - \frac{1}{2})$-design with $0 \leqslant \epsilon_{t-1}^{\text{der}} = \epsilon_t$. We have equality iff every derived design is a t-design. But this implies (by 1.5.7) that $\mathscr{B}$ is a $(t + 1)$-design.

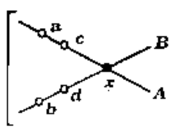
4.5. COROLLARY. *If at least one derived design or residual design of a nontrivial $t\frac{1}{2}$-design $\mathscr{B}$, $t \geqslant 1$, is a t-design then $\mathscr{B}$ is a $(t + 1)$-design.*

Proof. The hypothesis implies $\epsilon_i = 0$ for all $i \neq 0$ (resp. $i \neq t$), by Theorem 4.1. By Lemma 2.5 we have then $\epsilon_t = 0$ so that 4.4 applies.

We call a $t\frac{1}{2}$-design *genuine* if it is not a $(t + 1)$-design. Every genuine $t\frac{1}{2}$-design is nontrivial.

4.6. LEMMA. *For a genuine $2\frac{1}{2}$-design, $\alpha_0 = k(k - 1)/(v - 1)$, $\alpha_1 = \alpha_0 - \lambda$, $\alpha_2 = \alpha_0 - 3\lambda + r$.*

Proof. Let $\mathscr{B}$ be a nontrivial $2\frac{1}{2}$-design over P. For any four distinct points $a, b, c, d \in P$ define

$s(a, b, c, d) = \Big[$  $\Big]_{x \neq a, b, c, d}$, the number of triples $(x, A, B) \in P \times \mathscr{B} \times \mathscr{B}$ satisfying $a, c, x \in A$, $b, d, x \in B$, $x \neq a, b, c, d$. We have

$$s(a, b, c, d) = \sum_{A \supseteq a,c} \alpha(\{b, d\}, A) - \#\{(x, A, B) \mid x \in \{a, c\} \subseteq A, \{b, d, x\} \subseteq B\}$$

$$= \alpha_2\lambda_{abcd} + \alpha_1(\lambda_{abc} + \lambda_{acd} - 2\lambda_{abcd}) + \alpha_0(\lambda - \lambda_{abc} - \lambda_{acd} + \lambda_{abcd}) - \lambda(\lambda_{abd} + \lambda_{bcd}).$$

Therefore, $s(a, b, c, d) - s(b, a, d, c) = (\alpha_1 + \lambda - \alpha_0)(\lambda_{abc} + \lambda_{acd} - \lambda_{abd} - \lambda_{bcd})$. But $s(a, b, c, d) = s(b, a, d, c)$ so that $\lambda_{abc} + \lambda_{acd} = \lambda_{abd} + \lambda_{bcd}$ if $\alpha_1 \neq \alpha_0 - \lambda$. Interchanging c and d gives $\lambda_{abd} + \lambda_{acd} = \lambda_{abc} + \lambda_{bcd}$; and subtracting this from the previous equation gives (after division by (2) $\lambda_{abc} = \lambda_{abd}$ for any four distinct points $a, b, c, d \in P$. But this implies that $\mathscr{B}$ is a 3-design. (For, if a, b, c, d, e, f are six distinct points then $\lambda_{abc} = \lambda_{abf} = \lambda_{aef} = \lambda_{def}$.) Hence, if $\mathscr{B}$ is genuine, we must have $\alpha_1 = \alpha_0 - \lambda$. Using the relations of Lemma 2.5.(1), we find $\alpha_0 = k(k-1)/(v-1)$, and then $\alpha_2 = \alpha_0 - 3\lambda + r$.

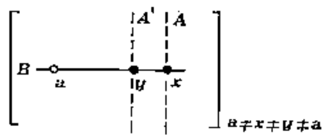
4.7. Theorem. *Every genuine $2\frac{1}{2}$-design is a multiple of a symmetric 2-design.*

Proof. Let $\mathscr{B}$ be a genuine $2\frac{1}{2}$-design over P. If $a, b \in P$, $a \neq b$, $B \in \mathscr{B}$, then

$$\alpha(\{a, b\}, B) = \alpha_0 - \lambda(i_{aB} + i_{bB}) + (r - \lambda)\, i_{aB} i_{bB}. \tag{1}$$

This can be verified easily by distinguishing the cases $a \in B \ni b$, $a \in B \not\ni b$, $a \notin B \ni b$, and $a \notin B \not\ni b$, using the previous lemma.

We define for all triples $(a, A, A') \in P \times \mathscr{B} \times \mathscr{B}$ the number

$t(a, A, A') =$  ,

the number of triples $(x, y, B) \in P \times P \times \mathscr{B}$ with $x \in A, B$, $y \in A', B$, $a \neq x \neq y \neq a$.

Using (1) we obtain $t(a, A, A') = \sum_{y \in A', y \neq a} \alpha(\{a, y\}, A) = \alpha_0(k - i_{aA'}) - \lambda_{i_{aA}}(k - i_{aA'}) - \lambda(\mu_{AA'} - i_{aA} i_{aA'}) + (r - \lambda)\, i_{aA}(\mu_{AA'} - i_{aA} i_{aA'})$. Therefore $0 = t(a, A, A') - t(a, A', A) = -\alpha_0(i_{aA'} - i_{aA}) - \lambda k(i_{aA} - i_{aA'}) + (r - \lambda)\,\mu_{AA'}(i_{aA} - i_{aA'}) = (i_{aA} - i_{aA'})((r - \lambda)\,\mu_{AA'} + \alpha_0 - \lambda k)$.

Hence either $\mu_{AA'} = (\lambda k - \alpha_0)/(r - \lambda) = k(k-1)/(v-1) =: \mu$, or $i_{aA} = i_{aA'}$ for all $a \in P$, i.e., A and A' consist of the same points, i.e., $\mu_{AA'} = k$. If A is a block of multiplicity d then $rk = \sum_{A' \in \mathscr{B}} \mu_{AA'} = dk + (b - d)\mu$, whence $d = (rk - b\mu)/(k - \mu) = r/k$. Therefore every block has the same multiplicity $d = r/k$. Hence $\mathscr{B} = r/k \times \mathscr{B}^*$ with a $2\frac{1}{2}$-design $\mathscr{B}^*$ which has $r = k$, $b = v$. By Corollary 3.8, $\mathscr{B}^*$ is a symmetric 2-design.

Corollary. *A $2\frac{1}{2}$-design is either a 3-design or a multiple of a symmetric 2-design. The derived and residual designs of a $2\frac{1}{2}$-design are either 2-designs or the multiples of dual 2-designs.*

4.8. Theorem. *Every genuine $3\frac{1}{2}$-design is a multiple of a Hadamard 3-design.*

Proof. Let $\mathscr{B}$ be a genuine $3\frac{1}{2}$-(v, k, λ)-design over P. By Theorem 4.1 and Corollary 4.5, every derived design is a genuine $2\frac{1}{2}$-$(v-1, k-1, \lambda)$-design with $\alpha_0^{\text{der}} = \alpha_1$, and every residual design of $\mathscr{B}$ is a genuine $2\frac{1}{2}$-$(v-1, k, ((v-k)/(k-2))\lambda)$-design with $\alpha_1^{\text{res}} = \lambda(k-1) - \alpha_1$. Lemma 4.6 gives $\alpha_0^{\text{der}} = ((k-1)(k-2)/(v-2))\lambda$, $\alpha_1^{\text{res}} = (k(k-1)/(v-2) - 1)((v-k)/(k-2))\lambda$. Hence $\lambda(k-1) = \alpha_0^{\text{der}} + \alpha_1^{\text{res}} = \lambda(k-1) - \lambda(v-k)(v-2k)/(v-2)(k-2)$ (after some simplification). This implies $v = 2k$ since $\mathscr{B}$ is nontrivial. If B is any block of $\mathscr{B}$, then the residual with respect to a point $x \notin B$ is a genuine $2\frac{1}{2}$-$(2k-1, k, k\lambda/(k-2))$-design. By the proof of Theorem 4.7, B has multiplicity $2\lambda/(k-2)$. Therefore, $\mathscr{B} = 2\lambda/(k-2) \times \mathscr{B}'$ with a $3\frac{1}{2}$-design $\mathscr{B}'$. The parameters of $\mathscr{B}'$ are $(v, k, (k-2)/2) = (4n, 2n, n-1)$, where $n = k/2$, whence $\mathscr{B}'$ is a Hadamard 3-design.

Corollary. *A $3\frac{1}{2}$-design is either a 4-design or a multiple of a Hadamard 3-design.*

4.9. Theorem. *Every $t\frac{1}{2}$-design with $t \geqslant 4$ is a $(t+1)$-design.*

Proof. Let $\mathscr{B}$ be a $t\frac{1}{2}$-(v, k, λ)-design. If $t = 4$ then the derived designs are $3\frac{1}{2}$-designs with $v' = v - 1$, $k' = k - 1$, and the residual designs are $3\frac{1}{2}$-designs with $v'' = v - 1$, $k'' = k$. At most one of these can be a multiple of a Hadamard 3-$(4n, 2n, n-1)$-design; therefore the other must be a 4-design. By Corollary 4.5, $\mathscr{B}$ is a 5-design. If $t > 4$ then a derived design is a $(t - \frac{1}{2})$-design with $t - 1 \geqslant 4$, hence (by induction) a t-design. By Corollary 4.5, $\mathscr{B}$ is a $(t+1)$-design.

5. Remarks and Problems

5.1. Most of 1.1–1.7, and 3.8.1, 3.10.3, can be found in [4, 7]. For transversal designs see, e.g., [6]; there is also a proof of 3.8.2. For strongly regular graphs and partial geometries we refer to [1]. This paper contains also 3.16.3. Result 3.16.4 is in [5]. After the preparation of the paper I learned that [2, 3] consider $1\frac{1}{2}$-designs under the name *partial geometric* *designs*. I haven't seen [3]; [2] contains among others the results of 3.5, 3.11, and 3.12. The other results seem to be new.

5.2. The class of $1\frac{1}{2}$-designs seems to have many interesting properties, and the existence and structure of $1\frac{1}{2}$-designs pose a lot of problems. We mention a few.

1. Find more examples of $1\frac{1}{2}$-designs.
2. Make a systematic study of $1\frac{1}{2}$-designs with small k.
3. Give other necessary conditions for $1\frac{1}{2}$-designs, or prove the non-existence of particular cases.
4. Look at the automorphism group of $1\frac{1}{2}$-designs. Use them to construct examples or to characterize particular $1\frac{1}{2}$-designs.
5. Find conditions under which the disjoint union of $1\frac{1}{2}$-designs over the same point set is again a $1\frac{1}{2}$-design (cf. 2.11).
6. Is every $1\frac{1}{2}$-design with $n = r$ a transversal design? For partial geometries the answer is yes.
7. Give a combinatorial interpretation of σ. (No nonalgebraic proof that σ is an integer is known. A combinatorial proof of this would have consequences also for strongly regular graphs.)
8. Consider subdesigns of $1\frac{1}{2}$-designs.

5.3. Since $t\frac{1}{2}$-designs with $t \geqslant 2$ are completely classified in Section 5, one could argue that the defining axioms are too restrictive. There are two ways in which the axioms could be weakened:

(1) If in the definition 2.3 of a $t\frac{1}{2}$-design the requirement that $\mathscr{B}$ is a t-design is dropped, can we get then more examples?

(2) In view of Lemma 4.2 we could consider the class of all 2-designs with the property that every derived design is a $1\frac{1}{2}$-design. This class contains many 2-designs which are not $2\frac{1}{2}$-designs, e.g., all affine 2-designs (cf. Remark 4.3.2).

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