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Cliques and Claws in Edge-transitive Strongly Regular Graphs

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1. Strongly Regular Graphs

A *strongly regular graph* is a graph Γ (finite, undirected, without loops and multiple edges) with v vertices (points) such that every vertex is adjacent with exactly k vertices, and the number of vertices adjacent to two distinct vertices x, y is λ or μ, depending on wether x and y are adjacent or not. We assume here that Γ is connected, and that Γ is neither complete nor a conference graph (i.e. the parameters are not $v=4\mu+1$, $k=2\mu$, $\lambda=\mu-1$, μ). Then the parameters of Γ can be expressed in terms of three integral parameters m, n, and μ as follows (here k, $n-m$, and $-m$ are the eigenvalues of the $(0,1)$-adjacency matrix of Γ):

1.1. Lemma (Neumaier [6]).

$$v=\mu^{-1}(\mu+(m-1)(n-m))(\mu+m(n+1-m)),$$
$$k=\mu+m(n-m), \qquad \lambda=\mu+n-2m, \mu,$$
$$2\leqq m\leqq n, \qquad 1\leqq \mu.$$

A *clique* in Γ is a complete subgraph. A *grand clique* is a maximal clique C of size $|C|>\frac{1}{2}n+\mu+1-m$ (this differs from the definition in Bose [1]; but assuming his inequalities, all his grand cliques are grand cliques in the present sense). A clique C is called *regular* if every point not in C is adjacent with the same number $e>0$ of points of C.

1.2. Lemma (Neumaier [7]). i) *A clique C is regular if and only if it contains exactly $|C|=1+m^{-1}k$ points; in this case, $e=m^{-1}\mu$.*

ii) *A non-regular clique contains less than $1+m^{-1}k$ points.*

A graph Γ is called *vertex-transitive*, resp. *edge-transitive* if any two vertices, resp. (unordered) edges can be mapped onto each other by an automorphism of Γ. Note that all rank 3 groups of even order give rise to edge-transitive strongly regular graphs (cf. Hubaut [5]).

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1.3. Lemma. *A connected edge-transitive strongly regular graph Γ is also vertex-transitive.*

Proof. By a result of Folkman [4], Γ is either vertex-transitive, or bipartite. But the only bipartite strongly regular graphs are the symmetric complete bipartite graphs which are vertex-transitive.

1.4. Lemma. (Neumaier [7]). *A vertex- and edge-transitive graph containing a regular clique is strongly regular.*

Since a connected bipartite graph contains a regular clique only if it is complete bipartite, the only edge-transitive graphs containing a regular clique which are not strongly regular are the nonsymmetric complete bipartite graphs, by Folkman [4], and 1.4.

2. Special $1\frac{1}{2}$-designs and Partial Geometries

A *special* $1\frac{1}{2}$*-design* consists of a set of v points and a collection of b blocks (sets with K points each) such that every point is in R blocks, two distinct points are in 0 or Λ blocks ($0<\Lambda<R$), and every point not in a given block B is adjacent to exactly e points of B. Here two points are called *adjacent* if they are distinct and contained in a common block. This defines a graph, the *point graph* of the design. A *partial geometry* is a special $1\frac{1}{2}$-design with $\Lambda=1$. A graph is called *geometric* if it is the point graph of a partial geometry.

2.1. Lemma (Neumaier [7]). *The point graph Γ of a special $1\frac{1}{2}$-design (a partial geometry* (Bose [1])) *is strongly regular, and each block of the design is a regular clique of Γ.*

2.2. Lemma (Bridges and Shrikhande [2]). *A two-class partially balanced design* (Bose [1]) *with $\lambda_1=0$ which has more points than blocks is a special $1\frac{1}{2}$-design.*

2.3. Theorem (Bose [1]). *A strongly regular graph with $m|\mu\leq m^2$ and*

$$n>\tfrac{1}{2}m(m-1)(\mu+1)+m-1 \tag{1}$$

is geometric.

It is possible to remove the condition $m|\mu\leq m^2$ from Theorem 2.3, and to specify the possible graphs. A *Latin square graph* is the point graph of a partial geometry with $e=R-1$, and a *Steiner graph* is the point graph of a partial geometry with $e=R$. (In other words, Latin square graphs and Steiner graphs are the line graphs of transversal designs and 2-designs with $\lambda=1$, cf. [6]). Note that a Latin square graph has $\mu=m(m-1)$, and a Steiner graph has $\mu=m^2$.

2.4. Theorem (Neumaier [6]). *A strongly regular graph whose parameters satisfy* (1) *is a Latin square graph or a Steiner graph.*

3. Properties of Cliques

In this, and the following section, let Γ be a strongly regular graph with parameters given by 1.1.

3.1. Lemma. *Let $s \geqq 1$ be an integer. Then every edge contains at most s maximal cliques of size larger than*

$$\gamma_s = 2 + \frac{\lambda}{s+1} + \frac{s}{2}(\mu - 2). \tag{2}$$

Proof. Assume that the edge ab contains $s+1$ maximal cliques $C_0, \ldots, C_s$ of size $> \gamma_s$ each. Define $D_i = C_i - \{a, b\}$. All points of D_i are adjacent to a and b whence $|\cup D_i| \leqq \lambda$. Since the C_i are maximal, there is (for each pair $i \neq j$) a pair of nonadjacent points in $C_i \cup C_j$ which all points of $C_i \cap C_j$ are joined. Hence $|D_i \cap D_j| = |C_i \cap C_j| - 2 \leqq \mu - 2$.

Now

$$\sum_i |D_i| \leqq |\cup D_i| + \sum_{i<j} |D_i \cap D_j|, \tag{3}$$

since every point which is in p sets D_i is counted p times in the left, and $1 + p(p-1)/2 \geqq p$ times in the right formula. But the left hand side of (3) is $> (s+1)(\gamma_s - 2)$ by assumption, and the right hand side is $\leqq \lambda + \frac{1}{2}s(s+1)(\mu - 2) = (s+1)(\gamma_s - 2)$ by the above counting arguments. This is a contradiction.

3.2. Corollary. *Every edge is in at most one grand clique.*

Proof. Apply Lemma 3.1 with $s=1$ and use $\lambda = \mu + n - 2m$ (from Lemma 1.1).

Directly (similar to 3.1), or from Neumaier [7], we get:

3.3. Lemma. *If the size K of a regular clique satisfies $K > \mu + 1 - m$ then every edge is in at most one regular clique.*

3.4. Theorem. *Suppose that Γ is edge-transitive and contains a clique of size $\geqq K$. If there is an integer $s \geqq 1$ such that*

$$K > \gamma_s, \qquad K(K-1) > ks \tag{4}$$

then Γ is geometric.

Proof. Since (4) remains valid when K is replaced by a greater integer, we may assume the existence of a maximal clique C of size K. Consider the orbit $\mathscr{B}$ of C under the automorphism group of Γ. If we call the cliques in $\mathscr{B}$ blocks, then by vertex-transitivity (Lemma 1.3), every point is in the same number R of blocks, and, by construction, every block contains K points. Also two points are in 0 blocks if they are nonadjacent, and, by edge-transitivity, in a constant number Λ of blocks if they are adjacent. Hence, since Γ is strongly regular, we have a partially balanced design with two classes and $\lambda_1 = 0$, $\lambda_2 = \Lambda$. By 3.1, $\Lambda \leqq s$ since $K > \gamma_s$. Counting flags (incident point-block pairs) through a given point gives $R(K-1) = k\Lambda \leqq ks < K(K-1)$ by (4), whence $R < K$. Counting all flags gives $vR = bK$, where b is the number of blocks in $\mathscr{B}$. Hence $v > b$, and by 2.2, we have a special $1\frac{1}{2}$-design. In particular, the blocks are regular cliques. Now if $s \geqq 2$ then $K > \gamma_s \geqq 2 + \frac{1}{2}s(\mu - 2) \geqq \mu > \mu + 1 - m$, so 3.3 implies $\Lambda \leqq 1$. If $s = 1$ then $\Lambda \leqq s = 1$. Hence $\Lambda = 1$ and we have a partial geometry.

3.5. Theorem. i) *If Γ contains a grand clique then $n > 2\mu(m-1)/m$.*

ii) *If Γ is geometric, and $n > 2\mu(m-1)/m$ then the grand cliques are just the blocks of the corresponding partial geometry.*

iii) *If Γ is edge-transitive and contains a grand clique then either Γ is geometric, or $\mu < m$.*

Proof. i) If Γ contains a grand clique C then $\frac{1}{2}n+\mu+1-m<|C|\leqq m^{-1}k+1 = n+1-m+m^{-1}\mu$ (by 1.2 and 1.1) which implies $n>2\mu(m-1)/m$.

ii) The block size is $m^{-1}k+1>\frac{1}{2}n+\mu+1-m$ (see (i)) whence the blocks are grand cliques. There are no other grand cliques since an edge is in some block but in at most one grand clique.

iii) Assume $\mu \geqq m$. Then by i), $n \geqq 2m-2$. Now a grand clique has size $K>\frac{1}{2}n+\mu+1-m=\gamma_1$, and by 1.1, $K(K-1)-k=(K-1)^2+(K-1)-k>\frac{1}{4}n^2+(\frac{1}{2}n+\mu+1-m)-(\mu+m(n-m))>\frac{1}{4}(n+1-2m)^2\geqq 0$. Hence Theorem 3.4 applies.

4. Cliques Constructed from Claws

A *d-claw* is a pair (a, S) consisting of a point a, and a set S of d points adjacent to a which are mutually nonadjacent. The next two lemmas are straightforward extensions of results by Bose [1] and Bumiller [3].

4.1. Lemma. *Let d be the maximal integer such that there is a d-claw. Then Γ contains a clique of size $\geqq 2+\lambda-(d-1)(\mu-1)$.*

Proof. Let (a, S) be a d-claw with maximal d. Choose $b \in S$. For every $x \in T = S - \{b\}$ there are $\leqq \mu-1$ points adjacent to a, b, and x, whence there are at least $\lambda-(d-1)(\mu-1)$ points adjacent to a, b but not to any element of T. Call this set of vertices C_0. If $p, q \in C_0$ then p and q are adjacent since otherwise $(a, T\cup\{p, q\})$ would be a $(d+1)$-claw. Therefore, $C=C_0\cup\{a, b\}$ is a clique with the required size.

4.2. Lemma. *Let $s \leqq m$ be an integer with*

$$n>(2m-3)\mu+m+\frac{(s-2)((s-3)(\mu-1)+2m-2)}{2(m+1-s)}. \tag{5}$$

Then $d \leqq 2m-s$ for every d-claw.

Proof. Suppose (a, S) is a d-claw. Denote by T the set of all $x \notin S$ which are adjacent with a. For $x \in T$, define a_x as the numer of points of S adjacent to x. Then an easy counting argument shows that

$$\Sigma 1 = k-d,$$
$$\Sigma a_x = d\lambda,$$
$$\Sigma a_x(a_x-1) \leqq d(d-1)(\mu-1),$$

where the sum extends over all $x \in T$. Hence $0 \leqq \Sigma(a_x-1)(a_x-2) \leqq d(d-1)(\mu-1) -2d\lambda+2(k-d)$. If we insert $d=2m+1-s$, use Lemma 1.1 to simplify, and solve for n, we obtain the negation of (5). Hence (5) implies that there is no $(2m+1-s)$-claw. This proves the lemma.

4.3. Corollary. *If $n>(2m-3)\mu+m$ then $d\leqq 2m-2$ for every d-claw.*

Essentially by combining Theorem 3.4 (for $s=1$) with 4.1 and 4.2 $\left(\text{with } s=\left[\frac{2m+5}{3}\right]\right)$, Bumiller [3] obtains the following extension of Theorem 2.3:

4.4. Theorem. *If Γ is a rank 3 graph with parameters 1.1 satisfying*

$$n\geqq(4m-5)(2\mu+1)/3+3/(m-2)+m+3 \tag{6}$$

then Γ is geometric.

We prove a similar result which extends Theorem 2.4.

4.5. Theorem. *Suppose Γ is an edge-transitive strongly regular graph with parameters 1.1. Let s be the smallest integer with $4m\leqq(s+1)^2$. If*

$$\mu\geqq 2+\tfrac{1}{4}(2m-1)s, \tag{7}$$

$$n\geqq(2m-1)\mu, \tag{8}$$

then Γ is a Latin square graph or a Steiner graph.

Proof. By definition of s, $s^2<4m$ whence

$$s^2+3\leqq 4m\leqq(s+1)^2 \tag{9}$$

since $s^2\equiv 0$ or $1 \bmod 4$. By Lemma 4.2, $d\leqq 2m-s$ for every d-claw. For otherwise the right hand side of (5) would be $\geqq n\geqq(2m-1)\mu$. Hence we would have $\frac{(s-2)((s-3)(\mu-1)+2m-2)}{2(m+1-s)}\geqq 2(\mu-1)-(m-2)$, whence $(s-2)(s-3)$ $(\mu-1)$ $+(s-2)$ $(2m-2)\geqq 4(m+1-s)$ $(\mu-1)-2(m+1-s)$ $(m-2)$ and $2m^2-6m+2s$ $\geqq(4m-s^2+s-2)$ $(\mu-1)\geqq(4m-s^2+s-2)$ $(1+\frac{1}{4}(2m-1)s)$ by (9) and (7). Multiplying this with 8 and writing $4m=x+s^2$ so that $x\geqq 3$ we obtain $x^2+(2s^2-12)x+s^4-12s^2+16s\geqq(x+s-2)$ $(sx+s^3-2s+8)$, or $(s-1)x^2+(s^3-s^2-4s+20)x-2s^3+10s^2-4s-16\leqq 0$. This is monotone in x, and positive for $x=2$, a contradiction.

Now Lemma 4.1 implies the existence of a clique of size $\geqq 2+\lambda-(d_{\max}-1)$ $(\mu-1)\geqq 2+\lambda-(2m-s-1)$ $(\mu-1)=K$. By Theorem 3.4, Γ is geometric once we know that $K>\gamma_s$ and $K(K-1)>ks$. To show this we remark first that (8) implies

$$\lambda=\mu+n-2m\geqq 2m(\mu-1), \tag{10}$$

$$K=2+\lambda-(2m-s-1)(\mu-1)\geqq 2+(s+1)(\mu-1). \tag{11}$$

Assume that $K\leqq\gamma_s$, i.e. $2+\lambda-(2m-s-1)(\mu-1)\leqq 2+\frac{\lambda}{s+1}+\frac{s}{2}(\mu-2)$. Subtract 2, multiply by $2(s+1)$, sort for λ, and use (10) to get $4ms(\mu-1)\leqq 2s\lambda\leqq 2(s+1)$ $(2m-s-1)(\mu-1)+s(s+1)(\mu-2)$. This implies $s(s+1)+(\mu-1)(s^2+3s+2$

$-4m)\leqq 0$, contradicting (9). Next assume that $K(K-1)\leqq ks$. Then by (11), (7) and 1.1, $((s+1)(\mu-1)+2)((s+1)(\mu-1)+1-ms)\leqq K(K-1-ms)\leqq(k-mK)s$ $=((2m^2-2m+1-sm)(\mu-1)+m^2+3m+1)s$. Hence by (7) and (9), $4m+sm(2m-1)=(1+\frac{1}{4}(2m-1)s)\cdot 4m \leqq (\mu-1)(4m(\mu-1)-ms(2m-1)) \leqq (\mu-1)((s+1)^2(\mu-1)-ms(2m-1))\leqq m^2s-ms+s-2-(2s+3)(\mu-1)\leqq m^2s-ms$, which is impossible.

Hence Γ is geometric. Since $n>2(m-1)(\mu+1-m)$, and $n>m(m-1)$ if $\mu=m$, the proof of Theorem 4.7 of Neumaier [6] applies and shows that the corresponding partial geometry has $e=R-1$ or $e=R$, whence Γ is a Latin square graph or a Steiner graph.

4.6. Corollary. *If Γ is edge-transitive and $n\geqq(2m-1)\mu$ then*

$$\mu=m^2, \quad \mu=m(m-1) \quad \text{or } \mu<2+\tfrac{1}{4}(2m-1)s.$$

Remark. (8) is always better than (6), and better than (1) if and only if $m\geqq 5$. (6) is better than (1) if $m\geqq 6$.

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