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# REGULAR CLIQUES IN GRAPHS AND SPECIAL $1\frac{1}{2}$ - DESIGNS

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Many strongly regular graphs constructed from designs contain cliques with the property that every point not in the clique is adjacent to the same number of points of the clique. In the first section we give some examples and investigate various properties of such regular cliques. In particular, parameter relations and inequalities are discussed. Section two defines special $1\frac{1}{2}$-designs as a certain class of designs with connection numbers 0 and $\Lambda$, generalizing partial geometries. Notable examples of special $1\frac{1}{2}$-designs are transversal designs and classical polar spaces. It is shown that the point graph of a special $1\frac{1}{2}$-design is strongly regular, and the blocks are regular cliques in the point graph.

As applications we reprove a result of Higman on partial geometries with isomorphic point graphs, and improve an inequality of Cameron and Drake concerning the parameters of partial $\Lambda$ - geometries.

## 1. REGULAR CLIQUES

In this paper, all graphs are finite, undirected, without loops or multiple edges. A *strongly regular graph* (SRG) is a graph with $v$ *vertices (or points)* such that

(R1) every vertex is adjacent to $k$ other vertices;

(R2) the number of vertices adjacent to two adjacent vertices is always $\lambda$;

(R3) the number of vertices adjacent to two nonadjacent vertices is always $\mu$.

(i.e. a complete subgraph) is called *regular* if every point not in $C$ is adjacent to the same number $e > 0$ of points in $C$; we call $e$ the *nexus* of $C$. Of course, $C$ is a maximal clique unless $e$ equals the size of $C$.

If $\Gamma$ is a graph with regular clique $C$, and we remove some edges disjoint from $C$, then in the new graph, $C$ is still regular. In a complete graph, all cliques are regular. In the complement of the hexagon, there are regular cliques of sizes 2 and 3 (with nexus 1). In contrast to this, we have

**THEOREM 1.1:** *Let $\Gamma$ be an edge-regular graph which is not complete. If $\Gamma$ has a regular clique, then*

(i) *all regular cliques have the same size $K$ and nexus $e$;*

(ii) *every clique has size at most $K$;*

(iii) *the regular cliques are exactly the cliques of size $K$.*

**PROOF:** Let $C$ be a regular clique of smallest size $K$, with nexus $e$. If we count in two ways the number of all edges $xy$ with $x \notin C$, $y \in C$, and the number of all triangles $xyz$ with $x \notin C$, $y,z \in C$, we find

$$K(k+1-K) = (v-K)e, \tag{1}$$

$$K(K-1)(\lambda+2-K) = (v-K)e(e-1). \tag{2}$$

Now let $C$ be any clique of size $K$. For $x \notin C$, denote by $e_x$ the number of vertices in $C$ adjacent to $x$. Counting the same numbers as before, we find

$$K(k+1-K) = \Sigma e_x,$$

$$K(K-1)(\lambda+2-K) = \Sigma e_x(e_x-1),$$

where the sum extends over all $v-k$ vertices $x \notin C$. Hence by (1) and (2), $\Sigma(e_x-e)^2 = 0$; so $e_x = e$ for all $x \notin C$. Hence $C$ is regular of nexus $e$.

If there is a clique of size $> K$ then any subclique $C$ of size $K$ is regular. But since $C$ is not maximal, $e = K$, and since $\Gamma$ is assumed to be regular, $\Gamma$ is complete. □

*Note:* For similar results see Bose [1], Delsarte [6], Neumaier [9], Wilbrink and Brouwer [15].

*Problem:* Investigate the possibilities when $\Gamma$ is regular but not edge-regular.

**COROLLARY 1.2:** *Let $\Gamma$ be an edge-regular graph containing a regular clique of size $K$ and nexus $e$.*

(i) *There is a number $m > 1$ such that $(e-1, K-1)m$ is integral and*

$$v = K + \frac{(m-1)K(K-1)}{e}, \quad k = m(K-1), \lambda = K-2+(e-1)(m-1). \tag{3}$$

(ii) *If $\Gamma$ is strongly regular, then*

$$\mu = em. \tag{4}$$

*Moreover, $m$ and*

$$f = m(m-1)K(K-1)/e(K+m-1-e), \tag{5}$$

*are integers.*

(iii) *Among the strongly regular graphs which are neither complete nor complete multipartite, only finitely many contain a regular clique of given size $K$.*

**PROOF:** (i) is a consequence of (1) and (2) if we choose $m = k/(K-1)$. Note that $(e-1, K-1)m$ is the greatest common diviser of $K$ and $\lambda - K + e + 1$. To prove (ii) we count in two ways the number of edges $xy$ with $x$ adjacent and $y$ nonadjacent to a given point, and obtain $(v-1-k)\mu = k(k-1-\lambda)$. If we solve this for $\mu$, and simplify with (i), we obtain $\mu = em$. Since $m$ can be written as $m = K-1-e+\mu-\lambda$, it is an integer. From standard formulas for the eigenvalues and multiplicities of a SRG (see [8], [10], or [13]), we see that the eigenvalues of the adjacency matrix are $k = m(K-1)$, $K-1-e$, and $-m$, with respective (integral) multiplicities $1, f$, and $v-1-f$, where $f$ is given by (5). In particular, $K = m-1-e$ divides $m(m-1)K(K-1)$, and hence

$$K + m - 1 - e \mid K(K-1)(K-e)(K-e-1) . \qquad (6)$$

If $K > e+1$, then we have only finitely many possibilities for $e$, and, by (6), for $m$. Hence $v$ takes only finitely many values, and (iii) holds in this case. If $K \le e+1$, then the following theorem completes the proof.

**THEOREM 1.3:** *Let $\Gamma$ be an edge-regular graph containing a regular clique $C$ of size $K$ and nexus $e$.*

(i) $e = K$ *if and only if* $\Gamma$ *is complete.*

(ii) $e = K-1$ *if and only if* $\Gamma$ *is complete multipartite with* $K$ *classes of the same size.*

**PROOF:** (i) If $e = k$ then every point in $C$ is adjacent to all other points, whence the valency is $v-1$. Hence $\Gamma$ is complete.

(ii) If $e = K-1$, then each point not in $C$ is nonadjacent to exactly one point of $C$. Hence no edge of $C$ is in an induced subgraph  whence by edge-regularity, there are no such induced subgraphs. For $a \in C$, denote by $A_a$ the class consisting of $a$ and all points which are not adjacent to $a$. Then the $A_a$ partition the points of $\Gamma$. If $x \in A_a$, $y \in A_b$, $b \neq a$ then $xy$ is an edge since otherwise $xay$ is the forbidden subgraph. Now each point $a \in C$ is unjoined with just the points of $A_a - \{a\}$, whence by regularity, all $A_a$ have same size $s$, and every point $x \in \Gamma$ is unjoined to exactly $s-1$ points. But the only candidates are the $s-1$ points $\neq x$ in the class of $x$. Hence no $A_a$ contains an edge, and $\Gamma$ is complete multipartite. Conversely, in a complete multipartite graph with $K$ classes, the maximal cliques are just the sets containing one point from every class, and these are regular with size $K$ and nexus $e = K-1$. □



*Problem:* (cf. Corollary 2.4) Is every edge-regular graph with a regular clique strongly regular?

**LEMMA 1.4:** *Let $C$ be a regular clique of size $K$ with nexus $e$ in an edge-regular graph with parameters (3). Then the points of $C$, together with the blocks* $B_a = \{x \in C \mid X \text{ adjacent to } a\}$ *for $a \notin C$, form a $2-(K, e, (e-1)(m-1))$-design.* □

This straightforward result is essentially due to Wilbrink and Brouwer [15]. Usually, the design will have repeated blocks.

**LEMMA 1.5:** *Let $\Gamma$ be an edge-regular graph which is not complete and which has regular cliques of size $K$ and nexus $e$. Then*

(i) *two distinct regular cliques have at most $e$ common points;*

(ii) *if two regular cliques have $d \ge 2$ common points,* $K \le (m-1)(e-1) + d$;

(iii) *if* $K-1 > m(e-1)$, *every edge is in at most one regular clique.*

**PROOF:** (i) is obvious. If $xy$ is an edge contained in two regular cliques which intersect in $d$ points then $x$ and $y$ are adjacent to at least $2K-d-2 \le \lambda$ points. This implies (ii), and (iii) if we observe that $d \le e$. □

**LEMMA 1.6:** *In an edge-regular graph which is not complete and which has regular cliques of size $K$ and nexus $e$, suppose that any two distinct regular cliques have at most $d$ common points, and two points are in at most $\Lambda$ regular cliques. Then*

$$e - 1 \ge \frac{(\Lambda-1)(K-2)(K-d)}{(m-1)(K-2+(\Lambda-1)(d-2))} \qquad (7)$$

*and*

$$\Lambda \le \frac{(K-d)}{(K-2)^2 - \lambda(d-2)} \quad \text{if } d-2 < \frac{(K-2)^2}{\lambda} . \qquad (8)$$

PROOF: Choose an edge $xy$, and denote by $a_z$ the number of distinct regular cliques containing $x, y, z$. If we count in two ways the number $N_0$ of points $z$ adjacent to $x$ and $y$, the number $N_1$ of pairs $(z,A)$ consisting of a point $z \neq x,y$, and a regular clique $A$ containing $x,y,z$, and the number $N_2$ of triples $(z,A,B)$ consisting of a point $z \neq x,y$, and two distinct regular cliques $A, B$ containing $x,y,z$, we find

$$\begin{aligned} N_0 &= \Sigma 1 = \lambda = K-2 + (m-1)(e-1), \\ N_1 &= \Sigma a_z = \Lambda(K-2), \\ N_2 &= \Sigma a_z(a_z - 1) \le \Lambda(\Lambda - 1)(d-2), \end{aligned} \tag{9}$$

where the summation is over all points $z$ adjacent to $x$ and $y$. Now the average of the $a_z$ is $a = N_1 / N_0$, and we have $0 \le \Sigma (a_z - a)^2 = N_2 + (1-2a) N_1 + a^2 N_0 = N_1 + N_2 - N_1^2/N_0$. Hence $N_1^2 \le N_0(N_1 + N_2)$, and if we use (9) and solve for $e-1$ and $\Lambda$, we obtain (7) and (8). □

The proof also yields the following.

COROLLARY 1.7: *Equality in (7) or (8) holds if and only if any two distinct regular cliques intersect in* 0, 1, *or* d *points, and every triangle is in the same number of regular cliques.* □

*Remarks:* (1) By Lemma 1.5, we may always take $d = e$.

(2) The proof shows that the same result holds if we require that the regular cliques belong to a distinguished set of cliques.

(3) In some cases the condition $i(i+1)N_0 - 2iN_1 + N_2 = \Sigma (a_z - i)(a_z - 1 - i) \ge 0$ for all integers $i$ will exclude some possibilities which are not excluded by (7) or (8).

EXAMPLE 1.8: The two SRGs with $K = 4$, $m = 3$, $e = 2$ are obtained from the two groups $X$ of order 4 as follows: the points are the triples $(a,b,c) \in X^3$ with $a+b+c = 0$, adjacent if they have the same entry in some place. Regular cliques are the 12 sets consisting of the points with fixed entry in some place (type 1), and the sets $\{(a,b,c,), (a, b+i, c+i), (a+i, b, c+i),$



Hence in the cyclic case, all edges are in a unique regular clique of type 1, and in at most one regular clique of type 2. In the noncyclic case, all edges are in a unique clique of each type. This agrees with Lemma 1.6, which states that $\Lambda \le 2$.

EXAMPLE 1.9: A SRG with parameters $K$, $m = K-1$, $e = K-2 \ge 1$ is obtained from a K-set $X$ as follows: the points are the ordered pairs $(a,b) \in X^2$, adjacent if and only if they have distinct entries in both places. Regular cliques are the sets $\{(a, \pi a) \mid a \in X\}$, where $\pi$ is a permutation of $X$. Hence two regular cliques may have up to $K-2$ common points, and every edge is in $\Lambda = (K-2)!$ regular cliques. Lemma 1.6 gives no bound for $\Lambda$ unless $K = 3$ or $K = 4$, where the bound is exact.

EXAMPLE 1.10: A SRG with parameters $K$, $m = 2K-3$, $e = K-2 \ge 1$ is obtained from a 2K-set $X$ as follows: the points are the 2-subsets of $X$, adjacent if and only if they are disjoint. Regular cliques are the partitions of $X$ into 2-subsets. Hence two regular cliques may contain up to $K-2$ common points, and every edge is in $\Lambda = 1 \cdot 3 \cdot 5 \ldots (2K-5)$ regular cliques. Lemma 1.6 gives no bound for $\Lambda$ unless $K = 3$ or $K = 4$, where the bound is exact.

## 2. SPECIAL 1½ - DESIGNS

A *special 1½-design* consists of a set of $v$ points, a collection of $b$ blocks, and an incidence relation $\epsilon$ between points and blocks such that

(S1) blocks contain $K$ points, points are in $R$ blocks,

(S2) two distinct points are in 0 or $\Lambda$ blocks, $0 < \Lambda < R$, and both possibilities occur.

(S3) If $B$ is a block then every point outside the block is adjacent to exactly $e$ points on $B$.

Here two points are called *adjacent* if they are distinct and contained in some block. This defines a graph, the *point graph* of the design

**THEOREM 2.1 :** *The point graph* $\Gamma$ *of a special* $1\frac{1}{2}$*-design* $\mathcal{B}$ *is strongly regular. Every block of* $\mathcal{B}$ *is a regular clique in* $\Gamma$.

**PROOF :** For a point $a$, the number of pairs $(A,x)$ with $a, x \in A$ is $k\Lambda = R(K-1)$, where $k$ is the number of points adjacent to $a$. Hence $k$ is independent of $a$. For an edge $ab$, the number of pairs $(A,x)$ with $a, x \in A$, $x$ adjacent to $b$, and $b \notin A$, is $(k-1-\lambda)\Lambda = (R-\Lambda)(K-e)$, where $\lambda$ is the number of points adjacent to $a$ and $b$. Hence $\lambda$ is independent of $ab$. For a nonedge $ab$, the number of pairs $(A,x)$ with $a, x \in A$ and $b$ adjacent with $x$ is

$$\mu\Lambda = Re, \qquad (10)$$

where $\mu$ is the number of points adjacent to $a$ and $b$. Hence $\mu$ is independent of $ab$. Therefore $\Gamma$ is strongly regular. By (S3), the blocks are regular cliques in $\Gamma$. □

**COROLLARY 2.2 :** *There is an integer* $m \geq 2$ *such that the point graph has parameters given by Corollary 1.2, and*

$$v = K + (m-1)K(K-1)/e, \quad b = \Lambda m + \Lambda m(m-1)(K-1)/e, \quad R = \Lambda m.$$

**PROOF :** Apply Corollary 1.1. By (10), $R = \Lambda m$, and since the number of pairs $(x,A)$ with $x \in A$ is $vR = bK$, the formula for $b$ follows. □

**THEOREM 2.3 :** *Let* $\Gamma$ *be a graph with a point- and edge-transitive automorphism group* $G$, *and let* $C$ *be a regular clique in* $\Gamma$. *If we define the images of* $C$ *under* $G$ *as blocks we obtain a special* $1\frac{1}{2}$*-design with point graph* $\Gamma$.

**PROOF :** Point-transitivity implies (S1), edge-transitivity implies (S2), and by construction, all blocks are regular cliques whence (S3) holds. □

**COROLLARY 2.4 :** *A point- and edge-transitive graph which contains a regular clique is strongly regular.* □

A *transversal design* $T[K,\Lambda;m]$ consists of a set of points, a partition of the points into $K>1$ classes of size $m$ each, and a collection of blocks (point sets) such that every block contains exactly one point from every class, and any two points from different classes are in exactly $\Lambda$ blocks. Neumaier [11] shows that every transversal design $T[K,\Lambda;m]$ is a special $1\frac{1}{2}$-design with $R = \Lambda m$, $e = K-1$.

**THEOREM 2.5 :** *For a special* $1\frac{1}{2}$*-design* $\mathcal{B}$, $e \leq K-1$, *with equality if and only if* $\mathcal{B}$ *is a transversal design.*

**PROOF :** Since there are nonadjacent points, $e \leq K-1$. If $e = K-1$ then the point graph is complete multipartite, and the regular cliques are just the sets which contain one point from every class. By Theorem 2.1, $\mathcal{B}$ is a transversal design. □

**THEOREM 2.6 :** *Points and maximal subspaces of a classical, finite polar space form a special* $1\frac{1}{2}$*-design.*

**PROOF :** For the axioms and basic properties of a polar space see e.g. Tits [14]. If the block set of $\mathcal{B}$ is the set of maximalsubspaces of a polar space of rank $s$ over $GF(q)$ then the blocks are $(s-1)$-dimensional projective spaces whence $K = (q^s-1)/(q-1)$. The points of a block which are adjacent to a given point outside form an $(s-2)$-dimensional subspace of the block whence $e = (q^{s-1}-1)/(q-1)$. Every point is in the same number $R$ of blocks since the automorphism group is transitivie on points. Every pair of points which is in a block is in a unique line, and since the automorphism group is transitive on lines, in the same number $\Lambda$ of blocks. Hence $\mathcal{B}$ is a special $1\frac{1}{2}$-design. If $\mathcal{B}$ is not a polar space over $GF(q)$ then the polar space is a generalized quadrangle, i.e. a special $1\frac{1}{2}$-design with $\Lambda = 1$, $e = 1$. □

In some cases [3], [14], the blocks can be split into two sets $\mathcal{B}_1$, $\mathcal{B}_2$ such that every (s-2)-dimensional subspace is in exactly one block of $\mathcal{B}_1$, and one block of $\mathcal{B}_2$. In this case a similar proof shows that the *half polar spaces* $\mathcal{B}_1$ and $\mathcal{B}_2$ also form special $1\frac{1}{2}$-designs.

By a result of Tits [14], all polar spaces of rank $> 2$ can be embedded into projective spaces. Thas and De Clerck [13] classified all partial geometries consisting of some lines, and all points on these lines, of a projective space. It would be interesting to classify similarly the special $1\frac{1}{2}$-designs consisting of some subspaces, and all points on these subspaces, of a projective space, or a polar space.

Special $1\frac{1}{2}$-designs with $\Lambda = 1$ are just the partial geometries defined by Bose [1], and those with $\Lambda = 1$, $e = 1$ are the generalized quadrangles. If we take $\Lambda$ identical copies of the blocks of a partial geometry $\mathcal{B}$ we obtain a special $1\frac{1}{2}$-design, called a *multiple* of $\mathcal{B}$. From Theorem 2.1 and Lemma 1.5, we now get immediately

**THEOREM 2.7 :**

(i) *A special $1\frac{1}{2}$-design $\mathcal{B}$ with $K-1 > m(e-1)$ is a multiple of a partial geometry.*

(ii) $e = 1$ *if and only if $\mathcal{B}$ is a multiple of a generalized quadrangle.* □

**COROLLARY 2.8 :** (Higman [8]) *If two partial geometries have the same point graph, and satisfy $K-1 > R(e-1)$, then they are isomorphic.*

**PROOF :** The point graph of a partial geometry has $m = R$ whence the union of the blocks of two partial geometries with the same point graph is a special $1\frac{1}{2}$-design with $\Lambda' = 2$, $K' = K$, $m' = R$, $e' = e$ (count repeated blocks twice!). By Theorem 2.7 (i), every block is repeated twice whence the partial geometries are isomorphic. □



*Remark :* Corollary 2.8 is in some sense best possible. For, the points of the graph of Example 1.7 (noncyclic case) together with the regular cliques of type 1, respectively type 2, as blocks define two distinct partial geometries with $K = 4$, $R = 3$, $e = 2$, hence $K-1 = R(e-1)$, with the same point graph.

We note another consequence of Theorem 2.1 .

**COROLLARY 2.9 :** *Every special $1\frac{1}{2}$-design is a 2-class partially balanced design with $\lambda_1 = 0$.* □

In general, the converse is not true, but Bridges and Shrikhande [2] proved a result (their Corollary 2.4) which easily implies

**THEOREM 2.10 :** *A 2-class partially balanced design with $\lambda_1 = 0$ which has more points than blocks is a special $1\frac{1}{2}$-design.* □

This result again implies Theorem 2.6 .

## 3. PARTIAL $\Lambda$-GEOMETRIES AND $1\frac{1}{2}$-DESIGNS

A $1\frac{1}{2}$-*design* consists of a set of points, a collection of blocks, and an incidence relation $\epsilon$ between points and blocks such that blocks have $K$ points, points are in $R$ blocks, and, for a given point $a$ and block $B$, the number of flags (i.e. incident point-line pairs) $(x,A)$ with $a \neq x \in B$, $a \in A \neq B$ is $\alpha$ if $a \notin B$, and $\beta$ if $a \in B$. Neumaier [11] introduces also the parameter

$$N = R + K + \beta - \alpha - 1 . \qquad (11)$$

**LEMMA 3.1 :**

(i) *A special $1\frac{1}{2}$-design with parameters $\Lambda$, $K$, $m$, $e$ is a $1\frac{1}{2}$-design with parameters*

$$K,\ R = \Lambda m,\ \alpha = \Lambda e,\ \beta = (\Lambda-1)(K-1),\ N = \Lambda(K+m-1-e). \qquad (12)$$

(ii) *Every $1\frac{1}{2}$-design with property (S2) is special.*

**PROOF:** (i) is straightforward. (ii) Let $a$ be a point not on $B$, and suppose that there are $e$ points on $B$ adjacent with $a$. Through $a$ and each such point there are exactly $\Lambda$ blocks, whence $\alpha = \Lambda e$. Hence $e$ is independent of $a$ and $B$, i.e. (S3) holds. □

**LEMMA 3.2:** *The parameters of a special* $1\frac{1}{2}$*-design satisfy*

$$\Lambda - 1 \geq \frac{(e-m)(m-1)}{m(K+m-1-e)}, \tag{13}$$

*with equality if and only if any two blocks intersect in the same number of points, which is* $e/m$.

**PROOF:** By Lemma 3.7 of Neumaier [11], $\alpha \geq R(K-N)$, with equality if and only if any two blocks intersect in the same number of points, which is $K-N = \alpha/R$. Now use the parameters (12). □

This lower bound is sharp, since there are many transversal designs which satisfy (13) with equality, namely the duals of affine 2-designs. An upper bound for $\Lambda$ can often be obtained from Lemma 1.6, if there are no repeated blocks.

**COROLLARY 3.3:** *For a special* $1\frac{1}{2}$*-design,* $e > m$ *implies* $\Lambda > 1$ *and* $K-1 \leq m(e-1)$.

**PROOF:** $\Lambda > 1$ by Lemma 3.2. If $K-1 > m(e-1)$, then by Theorem 2.7(i) we have a multiple of a partial geometry with $e > m$, contradicting 3.2 (with $\Lambda = 1$). □

The dual of a $1\frac{1}{2}$-design is obtained by interchanging the roles of points and blocks, and reversing incidence. It is again a $1\frac{1}{2}$-design with $K$, $R$ interchanged, and some $\alpha, \beta, N$. In particular, the dual of a special $1\frac{1}{2}$-design is special if and only if the axiom dual to (S2),

(S2*) two distinct blocks intersect in $0$ or $\Lambda^*$ points $(0 < \Lambda^* < K)$, and both possibilities occur,

holds. In this case the dual parameters $\Lambda^*$ and $e^*$ can be calculated from (12) and its dual: By comparing the two expressions for $\alpha$ and $\beta$ we obtain

$$(R-1)(\Lambda^*-1) = (K-1)(\Lambda-1), \quad \Lambda^* e^* = \Lambda e. \tag{14}$$

Cameron and Drake [3] define a *partial* $\Lambda$*-geometry* as a design with (S1), (S2), (S2*) and (S3), with $\Lambda^* = \Lambda$. For $\Lambda = 1$, this is just a partial geometry, and for $\Lambda > 1$, (14) implies $e^* = e$, $R = K$. Cameron and Drake [3] show that half polar spaces of rank 4 over $GF(q)$ are partial $\Lambda$-geometries with $\Lambda = q+1$, $K = R = q^3+q^2+q+1$, $e = q^2+q+1$. They also have examples of partial $\Lambda$-geometries with $\Lambda = 2$, $e = 3$, $K \in \{8, 24\}$. Drake [7] calls the number $i = K-e$ the *index* of the partial $\Lambda$-geometry, and constructs infinitely many partial $\Lambda$-geometries with $\Lambda > 1$ and index 1. On the other hand, we have

**LEMMA 3.4:** *For given* $i > 1$, *there are only finitely many partial* $\Lambda$*-geometries with* $\Lambda > 1$ *and index* $i$.

**PROOF:** From Theorems 1.1 and 2.1, we can see that, in the notation of Neumaier [10], the point graph has parameters $m$, $n = K+m-1-e = i+m-1$, $\mu = em$, whence its complement has parameters $\overline{m} = i$, $\overline{n} = i+m-1$, $\overline{\mu} = i(i-1)(m-1)/e > 0$. If $\overline{\mu} \in \{i(i-1), i^2\}$ then $e \leq m-1$, $(\Lambda-1)m = K-m = e+i-m \leq i-1$. Since $\Lambda > 1$, there are only finitely many possibilities for $e$ and $m$, and hence for $m, n$ and $\mu$. Therefore, we have only finitely many such graphs. If $\mu \notin \{i(i-1), i^2\}$, then, by Theorem 5.1 of Neumaier [10], we also have only finitely many such graphs. Now (S2*), and Theorem 2.1 imply that the blocks are distinct regular cliques, whence there are only finitely many patial $\Lambda$-geometries with $\Lambda > 1$ and index $i$. □

It can be shown that for $\Lambda > 1$ there is no partial $\Lambda$-geometry with index 2, and the only possibilities with index 3 have parameters

$$v = 50, \quad \Lambda = 5, \quad K = 15, \quad e = 12, \quad \text{or}$$
$$v = 85, \quad \Lambda = 3, \quad K = 15, \quad e = 12.$$

On the other hand, De Clerck [5] gives a complete classification of the (infinitely many) partial $\Lambda$-geometries with $\Lambda = 1$ and index 2.

**LEMMA 3.5:** *For a partial $\Lambda$-geometry with $\Lambda > 1$ we have*

$$K = R = \Lambda m, \tag{15}$$

$$e - 1 \geq A(m) = \frac{(\Lambda - 1)(m\Lambda - 2)}{m + \Lambda - 3}. \tag{16}$$

**PROOF:** (15) has already been proved. By (S2*) and Lemma 1.6, inequality (7) holds for $K = \Lambda m$, $d = \Lambda^* = \Lambda$, which implies (16).□

**COROLLARY 3.6:** *For a partial $\Lambda$-geometry with $\Lambda > 1$ and index $i > 1$, we have*

$$i \leq m(m-2), \tag{17}$$

$$e \geq 3\Lambda - 3. \tag{18}$$

**PROOF:** By (16), $i = \Lambda m - e \leq \Lambda m - 1 - A(m) = (m-1)^2 - (m-1)(m-2)^2/(\Lambda + m - 3)$. For $m = 2$, this implies $i \leq 1$, contrary to our assumptions. For $m \geq 3$, this implies $i < (m-1)^2$ whence $i \leq m(m-2)$. Since $A(m)$ increases as $m$ increases, $e-1 \geq A(m) \geq A(3) = 3\Lambda - 5 + 2/\Lambda > 3\Lambda - 5$. Hence $e \geq 3\Lambda - 3$.□

*Remark:* Some instances with $e = 3\Lambda - 3$ can be excluded by using Remark (3) after Corollary 1.7: The substitution $x = 1$ contradicts $\Lambda = 3, e = 6$, and $\Lambda = 4, e = 9$, and the substitution $x = 2$ contradicts $\Lambda = 7, e = 18$. With this method, Cameron and Drake [3] proved the result $e \geq 2\Lambda + 1$ if $\Lambda > 2$, which is better than (18) only for $\Lambda = 3$ and $\Lambda = 4$.

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