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Distance Matrices and n-dimensional Designs

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We consider two classes of n-dimensional designs containing all projective and polar spaces, and investigate their relations to distance matrices, Delsarte matrices and association schemes.

1. M_n-DESIGNS

All sets considered are finite. A *design* consists of a set P of *points* and a collection $\mathcal{B}$ of subsets of P, called *blocks*. A *quasilattice* is a design, together with a set of subsets of P, called *subspaces*, such that the empty set, the sets consisting of a single point, the intersection of subspaces and the intersection of a subspace with a block are subspaces. In particular, this implies that the set of all subspaces contained in a given block, together with this block, is a lattice. *A variety* is either a subspace, a block or the set P. Note that if any two blocks intersect in a subspace then the varieties form an atomic lattice. A quasilattice is called *short* if every subspace is the intersection of two blocks.

A quasilattice is called *regular* of *dimension* n if the following axioms (L), (B), (K) and (R) are satisfied:

(L) If x is a variety then all maximal chains $\varnothing = x_0 < x_1 < \cdots < x_i = x$ of varieties have the same length i (we call such a variety an *i-variety*, and we write X_i for the set of all i-varieties).

(B) The n-varieties are the blocks.

(K) For $i \leqslant n$, every i-variety contains exactly K_i points; $0 = K_0 < K_1 < \cdots < K_n$.

(R) For $i \leqslant n$, every i-variety is in exactly $R_i > 0$ blocks. Note that R_0 is the total number of blocks.

An *M_m-design* is a regular quasilattice of dimension n satisfying

(M_n) If x is an i-variety, z a block containing x, and $p \in z$ a point not in x then there is an $(i+1)$-variety $y \leqslant z$ containing x and p.

Note that by the intersection property, there is (for $i \leqslant n-2$) at most one $(i+1)$-variety y containing x and p; so this y is in all blocks z containing x.

It is easy to see that a regular quasilattice of dimension n is an M_n-design if and only if the lattice of subspaces of every block is a matroid (see e.g. Welsh [8] for a definition).

EXAMPLES

M1. Every 1-design is an M_2-design; in fact the two concepts are the same.

M2. The set of all proper subsets of an $(n+1)$-set P is an M_n-design with $K_i = i$.

M3. The set of all partial transversals of a partition of a v-set P into $n < v$ classes of v/n points each is an M_n-design with $K_i = i$; blocks are the complete transversals.

M4. The set of all proper subspaces of a projective or affine space of dimension n over a finite field $GF(q)$ is an M_n-design with $K_i = (q^i - 1)/(q-1)$.

M5. The set of all polar subspaces of a finite polar space of polar dimension n over $GF(q)$ is an M_n-design with $K_i = (q^i - 1)/(q-1)$. Several families of polar spaces are known (Tits [7], Buekenhout and Shult [1]).

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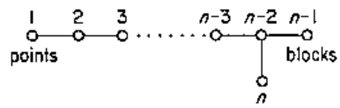
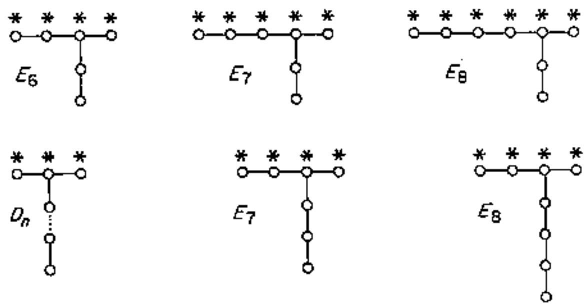
M6. For polar spaces of type D_n over $GF(q)$ (Tits [7]), the set of varieties belonging to the nodes $\neq n$ in the diagram



(which we shall call a *half polar space*) is an M_{n-1}-design with $K_i = (q^i - 1)/(q-1)$ for $i < n-1$, $K_{n-1} = (q^n - 1)/(q-1)$.

M7. In the same way, the starred nodes (*) of the diagrams for buildings of type D_n



and E_n give rise to M_i-designs (for i = number of stars). This follows from the transitivity properties of the automorphism group of the relevant buildings (Tits [7]), together with the fact that the residue of a block (= variety belonging to the rightmost starred node) is a truncated projective space, hence a matroid.

M8. Perfect matroid designs (Welsh [8]) of dimension n are M_n-designs; in fact a matroid (normalized such that $K_0 = 0$, $K_1 = 1$) is a perfect matroid design iff it is an M_n design for the appropriate n. The only known perfect matroid designs are projective spaces, affine spaces, Steiner systems, affine triple systems and their truncations, see [8].

M9. The regular semilattices of Type II over $GF(q)$ defined in Delsarte [3] are M_n-designs with $K_i = (q^i - 1)/(q-1)$.

M10. For $1 \leq i \leq n$, the set of $\leq i$-varieties of an M_n-design is an M_i-design, and the set of $\leq i$-varieties, together with the blocks, is an M_{i+1}-design. Thus, the previous examples give rise to many others.

In 1.1–1.7, we assume that an M_n-design is given.

LEMMA 1.1. *Suppose that $0 \leq i \leq j \leq k \leq n$, $i \leq l \leq k$, $l \leq n-1$. Then, for given $x \in X_i$, $y \in X_j$, $z \in X_k$ with $x \leq y \leq z$, the number of l-varieties u with $u \leq z$, $u \cap y = x$ is*

$$N_{li}^{kj} = \frac{(K_k - K_j)(K_k - K_{j+1}) \cdots (K_k - K_{j+l-1-i})}{(K_l - K_i)(K_l - K_{i+1}) \cdots (K_l - K_{l-1})}. \tag{1}$$

PROOF. Obviously, $N_{ii}^{kj} = 1$, and for $i = l$ the product (1) is empty. Hence we may assume by induction that $i < l$, and the formula holds for $N_{l,i+1}^{k,j+1}$. We count the number N of pairs $(p, u) \in X_1 \times X_l$ with $p \not\leq x$, $p \in u \leq z$, $u \cap y = x$. For each of the $K_k - K_j$ points $p \not\leq y$ with $p \in z$, there is a unique $(i+1)$-variety v containing x and p, and we have $v \leq u$ since $p, x \leq u$. By induction there are $N_{l,i+1}^{k,j+1}$ possible u, whence $N = (K_k - K_j) N_{l,i+1}^{k,j+1}$. On the other hand, given u, p can be chosen in $K_l - K_i$ ways, whence $N = N_{li}^{kj}(K_l - K_i)$, and (1) follows.

COROLLARY 1.2. *For given $x \in X_i$, $z \in X_k$ with $x \leq z$, the number of j-varieties y with $x \leq y \leq z$ is*

$$\mu^j_{ik} = \frac{(K_k - K_i)(K_k - K_{i+1}) \cdots (K_k - K_{j-1})}{(K_j - K_i)(K_j - K_{i+1}) \cdots (K_j - K_{j-1})}. \tag{2}$$

COROLLARY 1.3. *The number of i-varieties contained in a k-variety ($k \leq n$) is*

$$\begin{bmatrix} k \\ i \end{bmatrix} = \frac{K_k(K_k - K_1) \cdots (K_k - K_{i-1})}{K_i(K_i - K_1) \cdots (K_i - K_{i-1})}. \tag{3}$$

REMARK. For the M_n-designs of Examples M2 and M3, $\begin{bmatrix} k \\ i \end{bmatrix}$ is the ordinary binomial coefficient, and for the M_n-designs of Examples M4 and M5, $\begin{bmatrix} k \\ i \end{bmatrix}$ is the Gaussian binomial coefficient. This explains the notation used.

COROLLARY 1.4. *The set of subspaces of a block of an M_n-design is a perfect matroid design.*

PROOF. The set in question is a matroid, and by 1.2, (K), (R) are satisfied with $K'_i = K_i$, $R'_i = \mu^{n-1}_{in}$.

LEMMA 1.5. *For $i \leq n-1$, the number of i-varieties contained in two given blocks x and y is*

$$\begin{Bmatrix} \mu \\ i \end{Bmatrix} = \frac{\mu(\mu - K_1) \cdots (\mu - K_{i-1})}{K_i(K_i - K_1) \cdots (K_i - K_{i-1})}, \tag{4}$$

where μ is the number of points contained in x and y.

PROOF. Will be deleted (similar to the proof of 1.1).

Note that $\begin{Bmatrix} \mu \\ i \end{Bmatrix}$ is a polynomial of degree i in μ, and

$$\begin{bmatrix} k \\ i \end{bmatrix} = \begin{Bmatrix} K_k \\ i \end{Bmatrix}, \tag{5}$$

$$\mu^j_{ik} = \begin{bmatrix} k \\ j \end{bmatrix}\begin{bmatrix} j \\ i \end{bmatrix}\begin{bmatrix} k \\ i \end{bmatrix}^{-1}. \tag{6}$$

LEMMA 1.6. *The intersection of two blocks is either a subspace, or a union of $(n-1)$-varieties.*

PROOF. Let x, y be blocks. Every point of $x \cap y$ is in a maximal subspace of $x \cap y$ whence $x \cap y$ is the union of its maximal subspaces. Let z be such a maximal subspace. If z is an i-variety with $i \leq n-2$, and p a point of $x \cap y$ not in z then x and y contain the unique $(i+1)$-variety containing z and p, whence z is not maximal. Hence there is no such p, and $x \cap y = z$.

COROLLARY 1.7. *Two blocks with at most K_{n-1} common points intersect in a subspace.*

We restate some of our results in matrix form. Denote by I the identity matrix of any size, by J (resp. j) any all-one matrix (resp. vector), and by $f \circ A$ (where $A=(a_{xy})$ is a matrix and f a function) the matrix $(f(a_{xy}))$.

Let us define the *incidence matrices* $A_{ik}=(a_{ik}(x,y))_{x\in X_i, y\in X_k}$, where $a_{ik}(x,y)$ is 1 if $x \leq y$, and 0 otherwise, the *intersection matrices* $C_i = A_{in}^T A_{in}$ and the *distance matrix* $C = K_n J - C_1$. C and C_i are $b\times b$-matrices. The off-diagonal entries of C_1, i.e. the numbers of points in the intersections of two blocks, are called the *intersection numbers* of the design. The next result is an immediate consequence of the above results.

Lemma 1.8. *For an M_n-design, the following is true.*

(i) *For $i \leq j \leq k$, we have $A_{ij}A_{jk} = \mu^j_{ik} A_{ik}$.*

(ii) *For $i \leq n-1$, $C_i = f_i \circ C$, where $f_i(\xi) = \left\{ {K_n - \xi \atop i} \right\}$.*

(iii) $A_{0k} = j^T$, $C_0 = J$, $C_n = I$.

(iv) $A_{in}J = R_i J$, $A_{in}^T J = \left[{n \atop i} \right] J$, $C_i J = R_i \left[{n \atop i} \right] J$.

Theorem 1.9. *A short regular lattice of dimension n is an M_n-design iff there are polynomials $f_i(\xi)$ of degree i $(i=0,\ldots,n-1)$ such that $C_i = f_i \circ C$ for $i=0,\ldots,n-1$.*

Proof. By 1.8 (ii) every M_n-design has the stated property. Conversely, assume this property. Since $C_i = f_1 \circ C$ with linear $f_1(\xi)$, this is equivalent with the existence of polynomials $g_i(\xi)$ of degree i such that $C_i = g_i \circ C_1$. If we compare the (x, y)-entries we find that the number of i-varieties contained in two given blocks x and y is $g_i(\mu)$, where μ is the number of points contained in x and y.

Let x and y be blocks whose intersection is a j-variety. Then $\mu = K_j$. For $i > j$, there is no i-variety contained in x and y, and for $i=j$ there is exactly one. Hence K_j is a zero of g_i for $i > j$, and $g_i(K_i)=1$. Since g_i has degree i, $g_i(\xi = \text{const.}\ (\xi - K_0)\cdots(\xi - K_{i-1})$, and since $g_i(K_i)=1$, we have $g_i(\xi) = \left\{ {\xi \atop i} \right\}$, as defined in (4).

In every n-dimensional regular quasilattice, (M) holds for i-varieties with $i > n-2$. By induction, assume that (M) holds for i-varieties x with $i > j$, where $j \leq n-2$. For a given block z, we count the number N of triples $(p, x, y) \in X_1 \times X_j \times X_{j+1}$ with $p \not\leq x \leq y \leq z$, $p \in y$. Applying the hypothesis (with $x=y=z$) we find $g_{j+1}(K_n)$ possibilities for y, and for each y, $g_j(K_{j+1})$ possibilities for x (since y is the intersection of two blocks). For each x, y there are $K_{j+1} - K_j$ choices for p. Hence $N = (K_{j+1}-K_j) g_j(K_{j+1}) g_{j+1}(K_n) = (K_n - K_j) g_j(K_n)$, by the above formula for $g_i(\xi)$. On the other hand, there are $g_j(K_n)$ choices for x, then $K_n - K_j$ choices for p, and since we have a lattice, at most one choice for y. Since $N = (K_n - K_j) g_j(K_n)$, there is always such a y. Hence (M) holds for j-varieties x. This proves (M) for all x.

2. Delsarte Matrices and $1\frac{1}{2}$-designs

We start with a summary of some definitions and results of Neumaier [4]. A *distance matrix* is a real, symmetric matrix $C=(c_{xy})$ with the properties $c_{xx}=0$, $c_{xy}\geq 0$, $\sqrt{c_{xy}} + \sqrt{c_{yz}} \geq \sqrt{c_{xz}}$ (triangle inequality), for all rows x, y, z. A distance matrix has *strength* t if, for all non-negative integers i, k with $i+k \leq t$, there are polynomials $f_{ik}(\xi)$ of degree $\leq i$ such that

$$C^{(i)}C^{(k)} = f_{ik} \circ C; \tag{7}$$

here $C^{(i)} = (c^i_{xy})$. A *Delsarte matrix* is a distance matrix C without off-diagonal zeros such that C has strength t for all t.

If s is the number of distinct non-zero entries $\alpha_1, \ldots, \alpha_s$ of a distance matrix C without off-diagonal zeros then C is a Delsarte matrix iff C has strength $2s-2$, and in this case, the rows of C form an s-class association scheme, two rows x and y being ith associates iff $c_{xy}=\alpha_i$. From the proof of this result ([4], Theorem 3.4) we can easily deduce the following slightly stronger lemma.

LEMMA 2.1. *A distance matrix C without off-diagonal zeros is a Delsarte matrix iff, for $0 \leq i \leq k \leq s-1$, there are polynomials $f_{ik}(\xi)$ of degree $\leq i$ such that* (7) *holds; here s is the number of distinct non-zero entries of C.*

If A is the incidence matrix of a 1-design with block size k then $C=kJ-A^{\mathrm{T}}A$ is a distance matrix of strength 1. C has strength 2 iff the design is a $1\frac{1}{2}$-design, i.e. satisfies the following axiom:

(*) If x is a point and y is a block then the number of pairs (u, z) consisting of a point u and a block z with $x \leq z$, $u \leq z$, $u \leq y$ depends only on whether x is on y or not.

We want to give sufficient conditions for the distance matrix of a $1\frac{1}{2}$-design to be a Delsarte matrix. For $x \in X_k$, $y \in X_n$ we define $\alpha_{ik}(x, y)$ to be the number of pairs $(u, z) \in X_i \times X_n$ with $x \leq z$, $u \leq z$, $u \leq y$. We consider the following generalization of (*):

(S) If $x \in X_k$, $y \in X_n$, $x \cap y \in X_j$ then $\alpha_{ik}(x, y)=\alpha^j_{ik}$, for all non-negative integers i, j, k with $i<n$, $j \leq k<n$. An M_n-design with property (S) is called an *S_n-design.*

EXAMPLES

S1. Every $1\frac{1}{2}$-design is an S_2-design; in fact the two concepts are the same. Moreover, if we apply (S) for $i=k=1$, we see that the points and blocks of any S_n-design form a $1\frac{1}{2}$-design.

S2. The set of all $\leq n$-subsets of a v-set is an S_n-design (Cameron [2], Delsarte [3]).

S3. The set of all partial transversals of a partition of a v-set into $n<v$ classes of v/n points each is an S_n-design (Delsarte [3]).

S4. The set of all $\leq k$-subspaces of a projective space $PG(n, q)$ is an S_k-design (Cameron [2], Delsarte [3]).

S5. The set of all polar subspaces of a finite polar space or half polar space of polar dimension n and odd characteristic is an S_n-design (Stanton [5]; for the case $n=3$, see also Thas [6]).

S6. The regular semilattices of type II are M_n-designs (Delsarte [3]).

LEMMA 2.2. (S) *implies the existence of integers β^l_{ik} such that for $i<n$, $k<n$,*

$$A_{kn}C_i=\sum_{l=0}^{k} \beta^l_{ik}A^{\mathrm{T}}_{lk}A_{ln}, \tag{8}$$

$$C_iC_k=\sum_{l=0}^{k} \beta^l_{ik}\mu^k_{ln}C_l. \tag{9}$$

PROOF. The (x, y) entry of the matrix on the left of (8) is just the number counted in (S), whence it is α^j_{ik} if $x \cap y$ is a j-variety. On the other hand, the (x, y)-entry of $A^{\mathrm{T}}_{lk}A_{ln}$ is the number of l-varieties contained in $x \cap y$ which is $\begin{bmatrix} j \\ l \end{bmatrix}$. Hence (8) holds iff

$$\alpha^j_{ik}=\sum_{l=0}^{k}\begin{bmatrix} j \\ l \end{bmatrix}\beta^l_{ik}, \tag{10}$$

for all $i<n$, $j\leq k<n$. But this is a system of linear equations for β^j_{ik} with a triangular matrix which has ones on the diagonal. Hence there is an integral solution. (9) is obtained from (8) by left multiplication with A^{T}_{kn}, where we simplify with 1.8 (i).

THEOREM 2.3. *Let C be the distance matrix of an S_n-design. Then C has strength n, and if there are no repeated blocks and at most n distinct intersection numbers then C is even a Delsarte matrix.*

PROOF. For $i<n$, C_i is a polynomial of degree i in C, whence each $C^{(i)}$ is a linear combination of $C_0,\dots,C_i$. Hence (9) implies the existence of polynomials $f_{ik}(\xi)$ of degree $\leq k$ with (7) for $i<n$, $k<n$. Now $C^{(n)}C^{(0)}=C^{(n)}J$ has an (x,y)-entry $\sum_z c^n_{xz}=\sum_z c^{n-1}_{xz}c_{xz}$ which is the diagonal entry of $C^{(n-1)}C=f_{n-1,1}\circ C$. Therefore $C^{(n)}C^{(0)}=f_{n-1,1}(0)J=f_{n0}\circ C$ with $f_{n0}(\xi)=f_{n-1,1}(0)$ of degree 0. Hence C has strength n. Since the entries of C are K_n minus the intersection numbers, the second part follows from Lemma 2.1.

COROLLARY 2.4. *The blocks of an S_n-design without repeated blocks and with $s\leq n$ distinct intersection numbers form an association scheme with s classes.*

COROLLARY 2.5. *The blocks of a lattice which is an S_n-design form an association scheme.*

PROOF. In this case, the intersection numbers are contained in $\{K_0,K_1,\dots,K_{n-1}\}$, whence $s\leq n$.

For $n=2$, these are well known results, implying the existence of a strongly regular graph on the blocks for quasi-symmetric 2-designs, partial geometries and Steiner systems $S(2,k,v)$. For $n>2$, the corollary is related to Theorem 7 of Delsarte [3]. For, if a lattice is an S_n-design then it is a regular semilattice. In fact, Delsarte's $\Pi(j,r,s)$ equals α^j_{sr} if $r<n$, and $\begin{bmatrix} s\\ j\end{bmatrix}$ if $r=n$. Hence his Theorem 7 implies 2.5.

LEMMA 2.6. *Let $\mathcal{B}$ be an M_n-design with $s\leq n$ distinct intersection numbers. $\mathcal{B}$ is an S_n-design iff the following axiom holds:*

(S*) *The number of $z\in X_n$ with $x\leq z$, $|y\cap z|=\mu$ is a constant $\gamma^j_k(\mu)$, for every $x\in X_k$, $y\in X_n$ with $x\cap y\in X_j$, every intersection number μ, and all integers j, k with $j\leq k<n$.*

PROOF. Suppose first that (S*) holds. In the pairs admissible in axiom (S), there are $\gamma^j_k(\mu)$ choices for z with $|y\cap z|=\mu$, and then $\begin{Bmatrix}\mu\\ i\end{Bmatrix}$ choices for u. Hence

$$\alpha^j_{ik}=\sum_{\mu}\begin{Bmatrix}\mu\\ i\end{Bmatrix}\gamma^j_k(\mu), \tag{11}$$

independently of x and y. Hence (S) holds. Conversely, if (S) holds then we have (11) with $\gamma^j_k(\mu)$ possibly depending on x and y. Since (11) holds for all i, and since μ^i is a linear combination of $\begin{Bmatrix}\mu\\ 0\end{Bmatrix},\dots,\begin{Bmatrix}\mu\\ i\end{Bmatrix}$, we have, for $i=0,\dots,s-1$, $\sum_\mu \mu^i\gamma^j_k(\mu)=$ independent of x and y. Since these equations have a Vandermonde matrix, they have a unique solution, whence $\gamma^j_k(\mu)$ is independent of x and y.

REMARK. (S*) implies (S) for every M_n-design.

LEMMA 2.7. (S) *implies the existence of polynomials $q_i(\xi)$ of degree i (for all $i<n$) such that the matrices $J_i = q_i \circ C$ satisfy, for $i<n$, $k<n$,*

$$J_iJ_k = R_0\delta_{ik}J_k, \tag{12}$$

$$C_i = R_0^{-1} \sum_{l=0}^{i} \beta_{il}^l J_l. \tag{13}$$

PROOF. By (9), the vector space V_i generated by $C_0, \ldots, C_i$ is an algebra. Let J_i be a non-zero matrix in the orthogonal complement of V_{i-1} in V_i, or, if $V_{i-1} = V_i$, let $J_i = 0$. Then $J_i^2 = \text{const.}\, J_i$ and we may normalize J_i such that $J_i^2 = R_0J_i$. Then (12) holds and V_i is generated by $J_0, \ldots, J_i$. Hence $C_i = \sum_{k=0}^{i} d_{ik}J_k$ for appropriate d_{ik}, $d_{ii} \neq 0$. By (12) and (9), $\sum_{k=0}^{i} R_0 d_{ik}d_{jk}J_k = C_iC_j = \sum_{l=0}^{i} \beta_{ij}^l \mu_{ln}^i (\sum_{k=0}^{l} d_{lk}J_k)$. Comparing the coefficients of J_j we obtain $R_0 d_{ij}d_{jj} = \beta_{ij}^j \mu_{jn}^j d_{jj}$, and since $\mu_{jn}^j = 1$, $d_{ij} = R_0^{-1}\beta_{ij}^j$, which proves (13). Finally, V_i is also generated by $C^{(0)}, \ldots, C^{(i)}$, whence $J_i = q_i \circ C$ with a polynomial $q_i(\xi)$ of degree i.

REMARK. If (γ_{ik}) is the inverse of the triangular matrix (β_{ik}^k), with $\beta_{ik}^k = 0$ if $i<k$, then $J_k = \sum_{i=0}^{k} R_0\gamma_{ik}C_i$, whence $q_k(\xi) = \sum_{i=0}^{k} R_0\gamma_{ik}\begin{Bmatrix} K_n - \xi \\ i \end{Bmatrix}$.

LEMMA 2.8. *If there are s distinct intersection numbers then*

$$\beta_{ik}^l \mu_{in}^k = \beta_{ki}^l \mu_{ln}^i \text{ for } i, k \leq \min(s, n-1),\ l \leq k. \tag{14}$$

PROOF. $C_0, \ldots, C_s$ are linearly independent polynomials in C. Hence they commute with each other and, by comparing the coefficients of J_l in C_iC_k and C_kC_i in (9), the result follows.

LEMMA 2.9. *Let f_i be the rank of J_i. Then C_k and A_{kn} have rank $f_0 + \cdots + f_k$.*

PROOF. Denote by Hom(k) the row space of A_{kn}. Then Hom(k) is also the row space of $C_k = A_{kn}^{\mathrm{T}}A_{kn}$ and contains the rows of all matrices $C_i = (\mu_{in}^k)^{-1}A_i^{\mathrm{T}}A_{ik}A_k$ with $i \leq k$. Therefore Hom(k) contains the rows of all J_i, $i \leq k$, and hence its dimension is at least $f_0 + \cdots + f_k$. On the other hand, C_k is a linear combination of $J_0, \ldots, J_k$ whence we have equality.

REMARKS

1. By Lemma 2.9, we have

$$f_0 + \cdots + f_k \leq |X_k|. \tag{15}$$

There are already two-dimensional examples (partial geometries) with rank $A_{12} < |X_1|$, hence strict inequality in (15) is possible. On the other hand, we have equality for Examples S1 and S4 (see Cameron [2]).

2. By (13), the rank of C_i equals the sum of the f_k with $\beta_{ik}^k \neq 0$. Hence we have

$$\beta_{ik}^k \neq 0 \quad \text{for all } i \geq k. \tag{16}$$

THEOREM 2.10. *The distance matrix of an M_n-design $\mathcal{B}$ with exactly n distinct intersection numbers is a Delsarte matrix iff $\mathcal{B}$ satisfies the following axiom:*

(S_0) *$\alpha_{ik}(x, y)$ is a linear combination of $\alpha_{0k}(x, y), \ldots, \alpha_{k-1,k}(x, y)$ and i_{xy}, for all $i, k < n (x \in X_k,\ y \in X_n)$.*

Here $i_{xy} = 1$ if $x \leq y$, $i_{xy} = 0$ otherwise.

PROOF. First we show that (S_0) is equivalent to either of the following two statements:

(S_1) $A_{kn}C_i$ is a linear combination of $A_{kn}C_0, \ldots, A_{kn}C_{k-1}, A_{kn}$, for all $i, k < n$;
(S_2) C_kC_i is a linear combination of $C_kC_0, \ldots, C_kC_{k-1}, C_k$, for all $i, k < n$.

In fact, (S_0) and (S_1) are equivalent since $\alpha_{ik}(x, y)$ is the (x, y)-entry of $A_{kn}C_i$, and (S_2) follows from (S_1) by left multiplication with A_{kn}^T. If (S_2) holds, say $C_kC_i = \sum_{l=0}^{k-1} p_lC_kC_l + pC_k$ then define $X = A_{kn}(C_i - \sum_{l=0}^{k-1} p_lC_l - pI)$. Then $A_{kn}^TX = 0$, whence $X^TX = 0$, and so $X = 0$ which implies (S_1).

Now suppose that the distance matrix C of $\mathscr{B}$ is a Delsarte matrix. Then 1.8 (ii) implies that C_kC_i is a linear combination of $C_0, \ldots, C_k$ for $i, k < n$; and since $\mu_{ln}^k \neq 0$ for $l \leqslant k < n$, (9) holds with certain constants β_{ik}^l. Since the proofs of (12)–(16) depend only on (9) they are still valid. Hence

$$C_kC_i = \sum_{l=0}^{\min(i,k)} \beta_{kl}^l\beta_{il}^l J_l \tag{17}$$

and induction on i proves that for $i < k$, $C_kC_0, \ldots, C_kC_i$ generate V_i (use (16)). Hence $C_kC_0, \ldots, C_kC_{k-1}, C_k$ generate V_k and (17) implies (S_2).

Finally, suppose that (S_2) holds, and assume that we know already that, for all $l < k$ and all i, C_lC_i is a linear combination of $C_0, \ldots, C_l$ (this is true for $k = 1$). Since C_k and C_l commute as polynomials in C, (S_2) implies by our assumption that C_kC_i is a linear combination of $C_0, \ldots, C_k$. Hence this holds for all k, and C is shown to be a Delsarte matrix exactly as in 2.3.

THEOREM 2.11. *The distance matrix of a short regular lattice $\mathscr{B}$ is a Delsarte matrix iff $\mathscr{B}$ is an M_n-design satisfying* (S_0).

PROOF. If C is a Delsarte matrix then each C_i is a polynomial of degree i in C, and hence in C_1. This implies that the hypothesis of Theorem 1.9 is satisfied. Therefore, $\mathscr{B}$ is an M_n-design. Since $\mathscr{B}$ is a short regular lattice, the intersection numbers are $K_0, \ldots, K_{n-1}$. Hence 2.10 applies. The converse follows from Theorem 2.3.

REMARKS

1. In the situation of Theorem 2.11, must $\mathscr{B}$ be an S_n-design? For $n = 2$, the answer is yes: Since $\alpha_{0k}(x, y) = R_k$, (S_0) implies (S).

2. Axiom (S) states that $\alpha_{ik}(x, y) = \alpha_{ik}^j$ if $x \cap y$ is a j-variety. By (10), (14) and (16), α_{ik}^j is a polynomial of degree i in K_j, when (S_0) is a consequence of (S).

REFERENCES

1. F. Buekenhout and E. E. Shult, On the foundations of polar geometry, *Geom. Dedicata* **3** (1974), 155–170.
2. P. J. Cameron, Near regularity conditions for designs, *Geom. Dedicata* **2** (1973), 213–223.
3. Ph. Delsarte, Association schemes and t-designs in regular semilattices, *J. Combin. Theory Ser. A* **20** (1976), 230–243.
4. A. Neumaier, Distances, graphs and designs, *Eur. J. Combin.* **1** (1980), 163–174.
5. D. Stanton, Some q-Krawtchouk polynomials on Chevalley groups, *Amer. J. Math.* **102** (1980), 625–662.
6. J. A. Thas, Partial three-spaces in finite projective spaces, *Discrete Math.* **32** (1980), 299–322.
7. J. Tits, *Buildings of Spherical Type and Finite BN-pairs*. Lecture Notes in Mathematics 386, Springer-Verlag, Berlin–Heidelberg–New York, 1974.
8. D. J. A. Welsh, *Matroid Theory*. Academic Press, London, 1976.

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