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NEW INEQUALITIES FOR THE PARAMETERS OF
AN ASSOCIATION SCHEME

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New inequalities for the parameters of an association scheme

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Let X be a finite set with v objects. An s-class scheme on X is a partition of the set of all 2-subsets of X into $s \geq 2$ nonempty classes. Two objects x and y are i-th associates if $x \neq y$ and x,y is in the i-th class of the partition. We define $k_i(x)$ as the number of i-th associates of x, and $p_{ij}(x,y)$ as the number of $z \in X$ which are i-th associates of x, and j-th associates of y. If we write $D_i = (d_{xy}{}^i)$ with $d_{xy}{}^i = 1$ or 0 according as x and y are i-th associates or not then

$$D_i D_j = (p_{ij}(x,y)), \qquad (1)$$

$$p_{ij}(x,x) = k_i(x)\,\delta_{ij}, \qquad (1')$$

where δ_{ij} (= 1 or 0 according as i=j or not) is the Kronecker symbol. An association scheme is a scheme with $k_i(x) = k_i$ for all $x \in X$, and $p_{ij}(x,y) = p_{ij}{}^{\ell}$ whenever x and y are ℓ-th associates.

For the following well-known results see e.g. Cameron, et al [1]. The Bose-Mesner-algebra of an association scheme is the algebra V generated by the matrices I and $D_1,\dots,D_s$. There is a basis $E_0,\dots,E_s$ of V satisfying for i, j = 0,...,s

$$E_i E_j = \delta_{ij} E_j. \qquad (2)$$

Also, the Bose-Mesner-algebra is closed under pointwise multiplication $(a_{xy}) \circ (b_{xy}) = (a_{xy} b_{xy})$, whence

$$E_i \circ E_j = \sum_{\ell=0}^{s} q_{ij}^{\ell} E_\ell \qquad (3)$$

for appropriate numbers q_{ij}^{ℓ}, which can be calculated from the parameters. The Krein condition

$$q_{ij}^{\ell} \geqslant 0 \text{ for } i,j,\ell = 0,\ldots,s \qquad (4)$$

gives a restriction on the parameters.

The ranks f_i of E_i can also be calculated from the parameters; they satisfy $f_0+\ldots+f_s = v$, and the fact that all f_i must be integers places a severe restriction on the possible parameter sets. We prove here a new inequality for the ranks:

Theorem 1

The following inequalities hold:

$$\sum_{\ell : q_{ij}^{\ell} > 0} f_\ell \leqslant \begin{cases} f_i f_j & \text{for } i \neq j, \\ \frac{1}{2} f_i (f_i+1) & \text{for } i = j. \end{cases} \qquad (5)$$

Proof. By the following lemma, the rank of $E_i \circ E_j$ is at most $f_i f_j$ if $i \neq j$, and $\frac{1}{2}f_i(f_i+1)$ if $i = j$. On the other hand, since the E_i are mutually orthogonal, (3) implies that the rank of $E_i \circ E_j$ is given by the left hand side of (5).

Lemma

(i) Let A be a matrix of rank f. Then $A \circ A$ has rank $\leqslant \frac{1}{2}f(f+1)$.
(ii) Let A and B be matrices of the same size of rank f resp. g. Then $A \circ B$ has rank $\leqslant fg$.

Proof. Write $A = (a_{xy})$, and let $x_1,\dots,x_f$ be the labels of f independent rows. Then each a_{xy} is a linear combination of $a_{x_1y},\dots,a_{x_fy}$. Hence each entry a_{xy}^2 of $A\circ A$ is a linear combination of $a_{x_1y}^2,\dots,a_{x_fy}^2$, $a_{x_1y}a_{x_2y},\dots,a_{x_{f-1}y}a_{x_fy}$, of which there are $f + \binom{f}{2} = \frac{1}{2}f(f+1)$ terms. This proves (i), and the proof of (ii) is completely analogeously.

A 2-class association scheme is essentially the same as a strongly regular graph (see e.g. Seidel [2] for a definition). A strongly regular graph with $q_{22}^2 = 0$ is called a Smith graph. Cameron, et al [1] show that q_{22}^0 and q_{22}^1 are nonzero. Hence theorem 1 gives

Theorem 2

(i) The parameters of a strongly regular graph which is not a Smith graph satisfy

$$v \leq \tfrac{1}{2}f_2(f_2+1). \tag{6}$$

(ii) The parameters of a Smith graph satisfy

$$v \leq \tfrac{1}{2}f_2(f_2+3). \tag{7}$$

Proof. Apply theorem 1 with $i=j=2$, and observe that $f_0+f_1+f_2 = v$.

Example

The following parameter set for a strongly regular graph satisfies all previously known conditions for strongly regular graphs (as stated e.g. in Seidel [2]) but fails (6):

$$v=841,\ k=200,\ \lambda=87,\ \mu=35,\ f_2=40.$$

Problem. Characterize those graphs for which (6) is satisfied with equality.

Remarks. 1. More inequalities can be obtained similarly by looking at $E_i \circ E_j \circ E_k$, etc., but it is not known whether they are really more restrictive than those of theorems 1 and 2.

2. The special case of theorem 2, where the graph has a rank 3 automorphism group, has been proved already in Cameron, et al [1].

3. Theorem 2 improves the absolute bound (see e.g. Seidel [2]) for strongly regular graphs; it is not known how theorem 1 relates to the mor general absolute bound mentioned in [1], proposition 6.1.

References

1. P.J. Cameron, J.M. Goethals, and J.J. Seidel, The Krein condition, spherical designs, Norton algebras and permutation groups, Proc. Kon. Ned. Acad. Wet. A81 (1978), 196-206.

2. J.J. Seidel, Strongly regular graphs, an introduction, Proc. 7^{th} Brit. Comb. Conf. , Cambridge (LMS Lecture Notes Series 38) 1979, pp. 157-180.