Sonderabdruck aus
ARCHIV DER MATHEMATIK
Vol. 38, 1982 BIRKHÄUSER VERLAG, BASEL UND STUTTGART Fasc. 4

Completely regular twographs

By
A. Neumaier

Arch. Math., Vol. 38, 378—384 (1982)

0003-889X/82/3804-0007 $ 01.50 + 0.20/0

Completely regular twographs

By

A. Neumaier

1. Completely regular twographs. A twograph (cf. Taylor [10]) consists of a set of v points, together with a nonempty set $\mathscr{C}$ of unordered triples of points, called *coherent* triples, such that not all triples are coherent, and every set of four points contains an even number of coherent triples. A *clique* of $\mathscr{C}$ is a set C of points such that every triple contained in C is coherent. An *i-clique* is a clique of size i. A twograph is *regular* if every pair of points is in the same number of coherent triples. Taylor [10] associates to every twograph a class of similar $(-1, 1, 0)$-matrices; the twograph is regular, iff these matrices have just two distinct eigenvalues ϱ_1 and ϱ_2. In most cases, ϱ_1 and ϱ_2 will be odd integers, and we write $\varrho_1 = 2m - 1$, $\varrho_2 = 1 - 2s$. Then the information contained in 3.1—3.5 of [10] can be summarized as follows:

1.1. Proposition (Taylor [10]). *For every regular twograph, there are numbers s and m such that the number of points is*

$$v = (2m - 1)(2s - 1) + 1, \tag{1}$$

every pair of points is in exactly

$$a_2 = 2m(s - 1) \tag{2}$$

coherent triples, and every coherent triple is in exactly

$$a_3 = (m + 1)(s - 2) + 1 \tag{3}$$

4-cliques. Moreover, if s is rational, then s and m are integers satisfying

$$2 \leqq m \leqq (s - 1)^2 (2s + 1), \tag{4}$$

$$2 \leqq s \leqq (m - 1)^2 (2m + 1), \tag{5}$$

$$s + m - 1 \mid 2s(s - 1)(2s - 1), \tag{6}$$

whereas, if s is not rational then

$$v - 1 \quad \textit{is a sum of two squares}, \tag{7}$$

$$a_2 = (v - 2)/2, \quad a_3 = (v - 6)/4. \tag{8}$$

We call s and m the *parameters* of the regular twograph.

Let us call a clique C *regular* if, for every $x \notin C$, there is a partition of C into two nonempty subsets C_x^+ and C_x^- *of the same size* such that for $y, z \in C$, $\{xyz\}$ is coherent iff y and z are in the same class of the partition. This definition is motivated by the following proposition, which is an easy consequence of Taylor [10], 5.2 and 5.3.

1.2. Proposition (Taylor [10]).

(i) *If C_i is an i-clique of a regular twograph with parameters s and m then $i \leqq 2s$, with equality iff C_i is regular.*

(ii) *If a regular twograph contains a regular clique then the parameters s and m satisfy*

$$s \leqq (m-1)(2m-1). \tag{9}$$

A twograph is called *t-regular* if, for $i = 2, \ldots, t$, every i-clique is contained in a constant number $a_i > 0$ of $(i+1)$-cliques. Thus every regular twograph is 3-regular. By Proposition 1.2, the parameter s of a t-regular twograph satisfies $t \leqq 2s - 1$; a $(2s-1)$-regular twograph is called *completely regular*; in this case, s is an integer. Of course, every completely regular twograph contains regular cliques.

1.3. Theorem. *Let $\mathscr{C}$ be a t-regular twograph with parameters s, m, containing a regular $2s$-clique C. Then:*

(i) *For $i = 3, \ldots, t$, we have*

$$a_i = 2s - i + \frac{(s-2)\ldots(s-i+1)}{(2s-3)\ldots(2s-i+1)}(m-1). \tag{10}$$

(ii) *If $t \geqq s+2$, or $t = s+1$ and s odd, or $t \geqq s = 2$, then $\mathscr{C}$ is completely regular.*

(iii) *If $\mathscr{C}$ is completely regular then there is a positive integer p such that*

$$m = 1 + p\binom{2s-3}{s-2}. \tag{11}$$

Proof. (i) Let C be a regular $2s$-clique. Consider the incidence structure J whose points are the elements of C, and whose blocks are the formal expressions z^+ and z^- where z ranges over the points of $\mathscr{C}$ not in C. We define incidence by $x\,I\,z^+$ iff $x \in C_z^+$, and $x\,I\,z^-$ iff $x \in C_z^-$. We show that J is an i-design, for every $i \leqq t$. In fact choose $i \leqq t$, and let B be an i-subset of C. Then B is an i-clique, and the $(i+1)$-cliques containing B are the $2s - i$ sets $B \cup \{x\}$, $x \in C \setminus B$ and the $a_i - (2s - i) = a_i + i - 2s$ sets $B \cup \{z\}$, $z \notin C$, $B \subseteqq C_z^+$ or $B \subseteqq C_z^-$. Hence the points of B are incident with exactly $a_i + i - 2s$ blocks of J, whence J is a (self-complementary) $i - (2s, s, a_i + i - 2s)$-design with $b = 2(v - 2s) = 4(m-1)(2s-1)$ blocks. But an $i - (2s, s, \lambda)$-design has

$$\lambda = 0 \quad \text{if} \quad i \geqq s+1, \quad \text{and} \quad b = \frac{2s(2s-1)\ldots(2s-i+1)}{s(s-1)\ldots(s-i+1)}\,\lambda \quad \text{if} \quad i \leqq s$$

(see e.g. [5]). By substituting the values for b and λ, we obtain (10).

(ii) Suppose first that $t \geqq s+2$. For some $i \in \{1, \ldots, s-1\}$, let C be an $(s+i)$-clique of $\mathscr{C}$, and let C_s be an s-clique contained in C. Denote by Γ the graph whose vertices are the points $x \notin C_s$ such that $C_s \cup \{x\}$ is a clique, and whose edges are the pairs xy such that $C_s \cup \{x, y\}$ is a clique. By (i), Γ is edge-regular with a_s vertices, valency $a_{s+1} = s - 1$, and $a_{s+2} = s - 2$ triangles containing a given edge. Hence all neighbours of a vertex are adjacent, and Γ is a disjoint union of s-cliques. This implies that a given i-clique of Γ is in exactly $s - i$ $(i + 1)$-cliques. In particular, the number of $(s + i + 1)$-cliques containing C is $s - i$. Hence $a_{s+1} = s - i$, for $i = 1, \ldots, s - 1$, and $\mathscr{C}$ is completely regular. Now assume that $t = s + 1$ and $s = 2e - 1$ odd. By Taylor [11], a $2e$-regular twograph is in fact $(2e + 1)$-regular, and the preceding argument shows that $\mathscr{C}$ is completely regular. Finally, if $s = 2$ then $t \geqq s$ implies that $\mathscr{C}$ is regular, hence 3-regular, and therefore completely regular $(3 = 2s - 1)$.

(iii) This is obvious for $s = 2$. For $s > 2$, $2s - 1 \geqq s + 2$, so that by the proof of (ii), Γ is a disjoint union of s-cliques. In particular, the number a_s of vertices is a multiple of s, say $a_s = s(p + 1)$, with an integer p. Now (10) for $i = s$ implies (11), and we have $p > 0$ since $m > 1$.

Remark. By Proposition 1.1, part (i) of Theorem 1.3 holds for $i = 3$ even if no regular clique exists. But this becomes false for $i > 3$. For example, the regular twograph with $v = 276$, $s = 28$, $m = 3$ (this is the complement of the 276-twograph discussed below) is 5-regular and has $a_4 = 72$, $a_5 = 51$, whereas (10) would predict a nonintegral a_4. Indeed, the maximal clique size is $23 < 2s$ (see Taylor [10]).

1.4. Theorem. *A completely regular twograph has a parameter set from the following list (the parameter p is defined by* (11)*):*

No.	s	p	m	v	a_2	a_3	a_4	a_5
1	2	1	2	10	4	1		
2	2	2	3	16	6	1		
3	2	4	5	28	10	1		
4	3	1	4	36	16	6	2	1
5	3	3	10	96	40	12	2	1
6	3	4	13	126	52	15	2	1
7	3	9	28	276	112	30	2	1
8	4	1	11	148	66	25	8	3
9	4	2	21	288	126	45	12	3
10	4	8	81	1128	486	165	36	3
11	5	1	36	640	288	112	36	10
12	5	5	176	3160	1408	532	156	30
13	6	1	127	2784	1270	513	176	49

Proof. By Theorem 1.3(iii) and inequality (4) we have

$$\binom{2s-3}{s-2} \leqq m - 1 \leqq (s-1)^2(2s+1) - 1 = s^2(2s-3).$$

If $s \geqq 8$ then

$$\binom{2s-3}{s-2} = \frac{(2s-3)(2s-4)(2s-5)\ldots(s+1)s}{(s-2)(s-3)\ldots 3\cdot 2\cdot 1} = s(2s-3) \cdot \frac{2s-5}{s-3}\cdot\ldots\cdot\frac{s+1}{3} > s(2s-3)\,2^{s-5} \geqq (2s-3)s^2,$$

a contradiction. If $s \leqq 7$ then, after a short calculation, we find from (11) and the restrictions (4) and (6) just the indicated possibilities.

Remark. Concerning the existence of the thirteen cases in the list, we show in section 3:

(i) There are unique completely regular twographs with 10, 16, 28, 36, and 276 points;

(ii) There are no completely regular twographs with 126, 148, 640, and 2784 points;

(iii) The existence or nonexistence of completely regular twographs with 96, 288, 1128, and 3160 points remains open.

2. Related designs.

2.1. Theorem. *For a completely regular twograph with parameters s, m, and p defined by* (11), *there are integers $\Lambda_i > 0$ ($i = 1, \ldots, s+1$) such that for $i = 1, \ldots, s+1$, every i-clique is in exactly Λ_i regular cliques. Moreover,*

$$\Lambda_{s+1} = 1\,, \quad \Lambda_s = p + 1\,, \tag{12}$$

$$a_i \Lambda_{i+1} = \Lambda_i (2s - i) \quad \text{for} \quad i = 1, \ldots, s\,. \tag{13}$$

Proof. By Proposition 1.2 (i), the regular cliques are just the $2s$-cliques. We show first that $\Lambda_{s+1} = 1$, i.e. every $(s+1)$-clique is in a unique $2s$-clique. For $s = 2$ this follows from $a_3 = 1$, and for $s > 2$ it is due to the fact that the graph Γ constructed in the proof of Theorem 1.3 (ii) is a disjoint union of s-cliques. Now assume, by induction, that, for some $i \leqq s$, every $(i+1)$-clique is in Λ_{i+1} regular $2s$-cliques. For an i-clique C_i, the number of pairs (x, B) such that $x \notin C_i$, and B is a regular $2s$-clique containing $C_i \cup \{x\}$ is $a_i \Lambda_{i+1} = \Lambda_i (2s - i)$, where Λ_i is the number of $2s$-cliques containing C_i. Hence Λ_i is independent of C_i, and (13) holds. Finally, (13) for $i = s$ gives $s \Lambda_s = a_s \Lambda_{s+1} = a_s = s(p+1)$, by the proof of Theorem 1.3 (iii), whence $\Lambda_s = p + 1$.

We define a *special* $1^1/_2$-*design* (Neumaier [8]) with *nexus* e to be a 1-(V, K, R)-design with the properties that

(P1) every pair of distinct points is in 0 or Λ blocks, and both cases occur,

(P2) if B is a block then every point $x \notin B$ is adjacent with exactly e points of B.

Here two points are called *adjacent* if they are distinct and contained in some block. We shall need the following result:

2.2. Proposition (Neumaier [8]). *For any special $1^1/_2$-design with V points, block size K, and nexus e, there is an integer M such that*

$$V = K + e^{-1}(M-1) K (K-1)\,, \tag{14}$$

$$e(K + M - 1 - e) \mid M(M-1) K (K-1)\,. \tag{15}$$

Now let $\mathscr{C}$ be a completely regular twograph with parameters s, m, and let C_i be an i-clique of $\mathscr{C}$, $1 \leqq i \leqq s$. We define an incidence structure $\mathscr{S}(C_i)$ as follows: points are the elements $x \notin C_i$ such that $C_i \cup \{x\}$ is a clique, blocks are the sets $C \backslash C_i$, where C is a regular clique containing C_i, and incidence is inclusion. We call $\mathscr{S}(C_i)$ the *clique design* at C_i.

2.3. Theorem. *Let $\mathscr{C}$ be a completely regular twograph with parameter s, m. Then for $i = 1, \ldots, s-1$, the clique design at an i-clique is a special $1^1/_2$-design with a_i points, block size $2s - i$, and nexus $s - i$.*

Proof. Let C_i be an i-clique, and consider the clique design $\mathscr{S}(C_i)$. Obviously, the block size is $2s - i$, the number of points is a_i, every point is in Λ_{i+1} blocks, and every pair x, y of points is in 0 or Λ_{i+2} blocks, depending on whether $C_i \cup \{x, y\}$ is a clique, or not. To prove (P2), let B be a block and let $x \notin B$ be a point of $\mathscr{S}(C_i)$. Then $C = C_i \cup B$ is a regular $2s$-clique, and $C_i \cup \{x\}$ is a clique. By definition of a regular clique, two points $y, z \in C_i$ must be in the some class of the partition (C_x^+, C_x^-) of C, whence C_i is contained in C_x^+, say. Now it is easy to see that x is adjacent to $w \in B$ iff $w \in C_x^+ \backslash C_i$; hence (P2) holds with $e = |C_x^+ \backslash C_i| = s - i$.

2.4. Corollary. *The clique design at an $(s-1)$-clique of a completely regular twograph with parameters s, m, p related by* (11) *is a generalized quadrangle with block size $s + 1$, and $p + 1$ blocks through a point.*

Proof. The block size is $s + 1$, the number of blocks through a point is $\Lambda_s = p + 1$, and the nexus is $e = 1$, whence by [8], the clique design is a generalized quadrangle.

Remarks. 1. By [8], the point graph of a special $1\frac{1}{2}$-design is strongly regular; hence every completely regular twograph with $s \geqq 3$ gives rise to a strongly regular graph tower (defined in analogy to the rank 3 towers in Hubaut [7]); in fact the towers corresponding to No. 4 and No. 7 are rank 3 towers (cf. section 3).

2. If we translate the results of this section into Buekenhout's language of diagrams [2] then we see that completely regular twographs have a diagram $\circ\overset{C}{—}\circ \cdots \circ\overset{C}{—}\circ{=}\circ$ with $s + 1$ nodes. In particular, the clique design at an $(s-2)$-clique is a locally polar space of polar rank 2, in the sense of Buekenhout and Hubaut [3]. In fact, the double covers of twographs No. 1, 2, and 3 are their locally polar spaces D_2^+, A_2, and D_2^-, and the clique designs at a point of No. 4, resp. No. 7 are the locally polar spaces (2.3) resp. (2.1) of their Theorem 4. The space (1.2) of their Theorem 4 is a candidate for the residue of a 2-clique of a possible completely regular twograph No. 9.

Problem. Is it true that every strongly regular graph with $v = 165$, $k = 36$, $\lambda = 3$ is the point graph of a generalized quadrangle? If this holds it can be shown that a hypothetical regular twograph on 1128 points (with $s = 4$, $m = 81$) would have to be completely regular.

3. Existence questions.

3.1. Proposition. *There are unique completely regular twographs with* 10, 16, 28, 36, *and* 276 *points.*

Proof. By Seidel [9] (cf. Taylor [10]), there are unique regular twographs with $s = 2$ and 10, 16, or 28 points, and by 1.3 (ii), these are completely regular. Blokhuis and Haemers [1] show the existence and uniqueness of a completely regular twograph with 36 points. By Goethals and Seidel [6], there is a unique regular twograph with 276 points, and, again by [6], this twograph is completely regular.

Remarks. 1. The automorphism groups of the twographs in Proposition 3.1 are S_6, $2^4 S_6$, $Sp(6, 2)$, again $Sp(6, 2)$, and Con. 3, respectively; all of them are doubly transitive on points, and transitive on regular cliques.

2. Although there are at least 91 nonisomorphic regular twographs with 36 points (Bussemaker and Seidel [4]), only one of them is completely regular; in the notation of Taylor [9], this is the complement of the twograph Φ_+ (for $m = 3$) in example 6.4.

3.2. Proposition. *There are no completely regular twographs with* 126, 148, *or* 2784 *points.*

Proof. In the three cases (No. 6, No. 8, and No. 13 of Theorem 1.4), take the clique design at an appropriate i-clique. This is a special $1\frac{1}{2}$-design with the following parameters:

No.	i	$V = a_i$	$K = 2s - i$	$e = s - i$	M
6	2	52	4	1	5
8	2	66	6	2	5
13	3	513	9	3	22

But, in each case, (15) is violated. Hence there are no such completely regular twographs.

3.3. Proposition. *There is no completely regular twograph with* 640 *points.*

Proof. Blokhuis and Haemers [1].

References

[1] A. Blokhuis and W. Haemers, personal communication.

[2] F. Buekenhout, Diagrams for geometries and groups. J. Combin. Th. A **27**, 121–151 (1979).

[3] F. Buekenhout and X. Hubaut, Locally polar spaces and related rank 3 groups. J. Algebra **45**, 391–434 (1977).

[4] F. C. Bussemaker and J. J. Seidel, Symmetric Hadamard matrices of order 36. Report TU Eindhoven 70-WSK-02 (1970).

[5] P. J. Cameron and J. H. van Lint, Graphs, Codes, and Designs. LMS Lecture Notes **43**, Cambridge University Press, Cambridge 1980.

[6] J. M. Goethals and J. J. Seidel, The regular twograph on 276 vertices. Discr. Math. **12**, 143–158 (1975).

[7] X. Hubaut, Strongly regular graphs. Discr. Math. **13**, 357–381 (1975).

[8] A. Neumaier, Regular cliques in graphs and special $1^1/_2$-designs, in: Proc. Conf. Finite Geometries and Designs. LMS Lecture Notes **49**, Cambridge University Press, Cambridge, 244–259 (1981).

[9] J. J. SEIDEL, Strongly regular graphs of L_2-type and of triangular type. Indag. Math. 29, 188–196 (1967).
[10] D. E. TAYLOR, Regular twographs. Proc. London Math. Soc. (3) 35, 257–274 (1977).
[11] D. E. TAYLOR, Some topics in the theory of finite groups. Ph. D. Thesis. Oxford 1971.

Eingegangen am 24. 4. 1981

Anschrift des Autors:

A. Neumaier
Institut für angewandte Mathematik
der Universität Freiburg
D-7800 Freiburg