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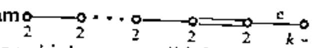
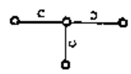
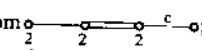
RECTAGRAPHS, DIAGRAMS, AND SUZUKI'S SPORADIC SIMPLE GROUP

A. NEUMAIER

Dedicated to N.S. Mendelsohn on the occasion of his 65th birthday

We investigate incidence structures with diagram  of rank $n+1$. For $n=2$, examples can be constructed from biplanes, semibiplanes, and strongly regular graphs with $\lambda=0$, $\mu=2$. For $n>2$, there are some examples related to the construction of a strongly regular graph on 1782 points by Suzuki. We also give some general theorems on diagrams which, e.g., imply that an incidence structure with diagram



gives rise to another incidence structure with diagram ; an example of this situation can be obtained from the rank 4 representation of $PSU_3(9)$ on 36 points.

1. Rectagraphs

All our graphs are finite, undirected, without loops or multiple edges. The *neighbourhood* of a set C of vertices (of a vertex x) of a graph Γ is the set $\Gamma(C)$ (resp. $\Gamma(x)$) of all vertices adjacent with all vertices of C (resp. with x). Γ is *regular of valency* k if every vertex is adjacent with exactly k other vertices. Often a vertex is simply called a *point*. An *s-claw* is a pair (x, S) where x is a point, and S is a set of s points adjacent with x such that no two points of S are adjacent. An *n-clique* is a set of n pairwise adjacent points.

A *rectagraph* (cf. [8]) is a connected, triangle-free graph with the property that each 2-claw is in a unique quadrangle. Perkel [8] shows that a rectagraph is always regular.

Examples. (1) Strongly regular graphs with parameters $\lambda=0$, $\mu=2$ are rectagraphs. By well-known parameter conditions (see, e.g., [7]) the valency must be of the form $k=s^2+1$, with an integer s not divisible by 4, and the number of vertices is then $v=\frac{1}{2}(s^2+s+2)(s^2-s+2)$. Examples are known for $s=1$ (quadrangle, $k=2$, $v=4$), for $s=2$ (Clebsch graph, $k=5$, $v=16$) and for $s=3$ (Gewirtz graph, $k=10$, $v=56$) (see, e.g., [7]). By considering the set of points at distance ≤ 2 from a given point it is easy to see that a rectagraph Γ of valency k contains $v \geq 1 + k + \binom{k}{2}$ points, with equality iff Γ has diameter 2 iff Γ is a strongly regular graph with parameters $\lambda = 0$, $\mu = 2$. These graphs also arise in connection with algebraic varieties in projective spaces (see [1]).

(2) The k-dimensional cubes (and, for $k \geq 5$, the half-cubes obtained by identifying antipodal vertices) are rectagraphs of valency k.

(3) A *biplane* (*semibiplane*) is a design with the property that any two distinct points are in 2 (0 or 2) blocks, and any two distinct blocks have 2 (0 or 2) common points. The incidence graph of a biplane or semibiplane $\mathscr{B}$, i.e., the graph whose vertices are the points and blocks of $\mathscr{B}$, adjacent iff they are incident, is a rectagraph whose valency equals the block size. It is easy to see that a rectagraph is bipartite iff it is the incidence graph of a semibiplane. Also, a bipartite rectagraph comes from a biplane iff its diameter is 3.

Proposition 1. *In a rectagraph, if x, y are points at distance i, then there are at least i points adjacent to y which have distance $i-1$ from x.*

Proof. This is true for $i = 1$, so assume that it is true for $i = 1, \ldots, j$. If x, y are at distance $j+1$ then there is a chain $x = x_0, x_1, \ldots, x_{j+1} = y$ of adjacent points. By induction, there are at least j points $s_1, \ldots, s_j$ adjacent to x_j which have distance $j-1$ from x.

Now define $t_0 = x_j$, and define t_l as the fourth point on the quadrangle containing the 2-claw $s_l x_j x_{j+1}$ $(l = 1, \ldots, j)$. Then $t_0, t_1, \ldots, t_j$ are $j+1$ distinct points adjacent to x_{j+1} and at distance j from x. This completes the induction.

Proposition 2. *A rectagraph Γ of valency k has diameter $\leq k$, and contains $v \leq 2^k$ points. Moreover, Γ has diameter k iff $v = 2^k$ iff Γ is a k-dimensional cube.*

Proof. By Proposition 1, the diameter is $\leq k$. If v_i is the number of points at distance i from a given vertex x then $v_0 = 1$. If we count in two ways the number N_i of edges yz with y at distance i, z at distance $i+1$ from x, we obtain $v_{i+1}(i+1) \leq N_i \leq v_i(k-i)$. Hence, by induction, $v_i \leq \binom{k}{i}$, and $v \leq \sum \binom{k}{i} = 2^k$.

Now suppose that the diameter of Γ is k. Choose two points a, b at distance k, and define Γ_0 as the set of points whose distances to a and b sum up to k. If $x \in \Gamma_0$ has distance i from a and $k-i$ from b then by Proposition 1, there are at least i vertices at distance $i-1$ from a, and at least $k-i$ vertices at distance $k-i-1$ from b, hence at distance $i+1$ from a, which are adjacent with x. But the valency is k, whence there are exactly i vertices at distance $i-1$, and $k-i$ vertices at distance $i+1$ from a in the neighbourhood of x, and so $\Gamma(x)$ is in Γ_0. Since $a \in \Gamma_0$ and Γ is connected we now have $\Gamma_0 = \Gamma$, and the above counting argument gives $v_{i+1}(i+1) = N_i = v_i(k-i)$, or $v_i = \binom{k}{i}$. Hence $v = 2^k$.

Finally, if $v = 2^k$ then $v_i = \binom{k}{i}$ for all i, in particular $v_k = 1$. Hence for every point a there is a point at distance k from a. In particular, by the above arguments, Proposition 1 holds with 'at least' replaced by 'exactly'. Now fix $a \in \Gamma$, and identify $x \in \Gamma$ (at distance i from a) with the characteristic vector $c(x)$ of the set of all $y \in \Gamma(a)$ at distance $i-1$ from x. Then $c(x)$ has i entries, and it is easy to show that c is an isomorphism from Γ into the k-dimensional cube.

Remark. This result has been obtained for the incidence graph of a semibiplane by Wild [10] with a similar proof. Wild also shows that a bipartite rectagraph Γ of valency k contains $v \geqslant k^2 - k + 2$ vertices, with equality iff Γ is the incidence graph of a biplane.

Proposition 3. *Let Γ be a rectagraph on X with valency k. Then the graph Γ_ε whose vertices are the pairs (x, α) with $x \in X$, $\alpha = \pm 1$, such that (x, α) and (x', α') are adjacent iff either $x = x'$ and $\alpha\alpha' = -1$, or x' is adjacent with x and $\alpha\alpha' = \varepsilon$, is a rectagraph of valency $k+1$, for $\varepsilon = \pm 1$.*

Proof. Straightforward.

Remarks. (1) If Γ is bipartite, with classes C^α, $\alpha = \pm 1$, then Γ_+ is bipartite, with classes $\{(x, \alpha) \mid x \in C^\alpha,\ \alpha = \pm 1\}$, $\{(x, -\alpha) \mid x \in C^\alpha,\ \alpha = \pm 1\}$. Hence this construction generalizes Wild's doubling construction [10] for semibiplanes. If Γ is a k-dimensional cube then Γ_+ is a $(k+1)$-dimensional cube.

(2) Γ_- is always bipartite, with classes $\{(x, \alpha) \mid x \in X\}$ for $\alpha = \pm 1$. If Γ is strongly regular, then Γ_- is the incidence graph of a biplane with a null polarity. This is a special case of a construction of symmetric 2-designs with a null polarity from strongly regular graphs with $\lambda = \mu - 2$ (see, e.g., [7]). If Γ is a k-dimensional cube, then, again, Γ_- is a $(k+1)$-dimensional cube. If Γ is a half-cube, then Γ_- is a cube with additional edges joining antipodal pairs of points.

(3) In the graph Γ_-' obtained from Γ_- by deleting the edges $(x, 1)$, $(x, -1)$, each 2-claw is in a unique quadrangle. Γ_-' is bipartite, and, unless Γ itself is bipartite, connected, whence a rectagraph. In particular, every rectagraph is contained in some bipartite rectagraph of the same valency, and a rectagraph of valency k which is not bipartite has at most 2^{k+1} vertices.

2. Diagrams

For the remainder, we assume the reader to be familiar with the paper "Diagrams for geometries and groups" by Buekenhout [2]. We take a slightly more general point of view, and call an incidence structure, which is connected under the incidence relation and satisfies Buekenhout's axiom (1), a *geometry*.

We refine the information contained in a diagram as follows. For a geometry (S, I, Δ, d), suppose that for all flags F of S of type $\Delta - \{i\}$, the cardinality of $R(F)$ is a constant p_i depending only on $i \in \Delta$. Then a *labelled diagram* of S consists of a diagram of S, together with the label p_i attached to node i of the diagram, for all $i \in \Delta$. This also makes clear what we understand by a partially labelled diagram. Since no confusion is possible we call a (partially) labelled diagram simply a diagram.

For example, $\underset{p}{\circ}$ is the diagram for a geometry of rank 1 with p points, $\underset{2}{\circ}\overset{(n)}{\text{——}}\underset{2}{\circ}$ is an ordinary n-gon, $\underset{2}{\circ}\overset{c}{\text{——}}\underset{p}{\circ}$ is the diagram for a complete graph with $p+1$ points, $\underset{k}{\circ}\overset{L}{\text{——}}\underset{r}{\circ}$ is the diagram for a 2-$(1+r(k-1),\ k,\ 1)$-design, $\underset{q+1}{\circ}\text{——}\underset{q+1}{\circ}=\!=\underset{s+1}{\circ}$ is the diagram for a 3-dimensional polar space over GF(q) with $s = 1,\ q^{1/2},\ q,\ q,\ q^{3/2},\ q^2$ for $Q_5^+(q)$, $H_5(q)$, $S_5(q)$, $Q_6(q)$, $H_6(q)$, and $Q_7^-(q)$, respectively (see, e.g., [3]). Thus the labelled diagram determines the geometry often almost upto isomorphy.

Labels for the diagrams of sporadic groups show at once which geometries are involved: F_{24} with diagram $\circ\text{——}\circ\text{——}\circ\overset{c}{\text{——}}\underset{5}{\circ}\text{——}\underset{5}{\circ}=\!=\underset{3}{\circ}$ involves $H_5(4)$, HS with diagram $\circ\overset{c}{\text{——}}\underset{5}{\circ}\text{——}\underset{5}{\circ}\overset{\text{ɔ}}{\text{——}}\circ$ involves PG$(2, 4)$, J_1 with diagram $\underset{2}{\circ}\overset{(6)}{\text{——}}\underset{2}{\circ}\overset{c}{\text{——}}\underset{11}{\circ}$ involves the ordinary hexagon, whereas HJ with diagram $\circ\overset{c}{\text{——}}\underset{3}{\circ}\overset{(6)}{\text{——}}\underset{3}{\circ}$ involves the generalized hexagon from $G_2(2)$.

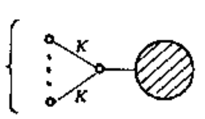
Sometimes, labels give information even for unlabelled diagrams. For example, since $\circ\overset{c}{\text{——}}\circ$ is always $\underset{2}{\circ}\overset{c}{\text{——}}\circ$, $\circ=\!=\circ\overset{c}{\text{——}}\circ$ must be a $\circ=\!=\underset{2}{\circ}\overset{c}{\text{——}}\circ$ and so the generalized quadrangle involved must be a rectangular grid (every point on two lines only); similarly, $\circ\text{——}\circ\text{——}\circ\overset{c}{\text{——}}\circ$ must be a $\underset{2}{\circ}\text{——}\underset{2}{\circ}\text{——}\underset{2}{\circ}\overset{c}{\text{——}}\circ$. Note also that $\underset{2}{\circ}\overset{c}{\text{——}}\underset{2}{\circ}$ is the same as $\underset{2}{\circ}\text{——}\underset{2}{\circ}$.

The ordinary quadrangle has diagram $\underset{2}{\circ}=\!=\underset{2}{\circ}$, hence rectagraphs of valency k belong to a diagram $\underset{2}{\circ}=\!=\underset{2}{\circ}\overset{c}{\text{——}}\underset{k-1}{\circ}$. So, for example, the Gewirtz graph yields a diagram $\underset{2}{\circ}=\!=\underset{2}{\circ}\overset{\text{ɔ}}{\text{——}}\underset{9}{\circ}$ for the simple group PSL$(3, 4)$. Semibiplanes with block size k have diagrams $\underset{2}{\circ}\overset{c}{\text{——}}\underset{k-1}{\circ}\overset{\text{ɔ}}{\text{——}}\underset{2}{\circ}$ (varieties are points, pairs of points on a block, and blocks), and the implication

$$\underset{2}{\circ}\overset{c}{\text{——}}\underset{k-1}{\circ}\overset{\text{ɔ}}{\text{——}}\underset{2}{\circ} \Rightarrow \underset{2}{\circ}=\!=\underset{2}{\circ}\overset{c}{\text{——}}\circ$$

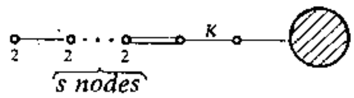
given in Example 3 of Section 1, is a special case of the following theorem.

Theorem 4. (i) *If there is a geometry with diagram*

s nodes 

(1)

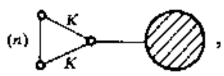
then there also is a geometry with diagram



(2)

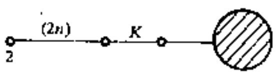
This geometry has an s-partite point graph, and conversely, if a geometry with diagram (2) *has an s-partite point graph, then there is a geometry with diagram* (1).

(ii) *If there is a geometry with diagram*



(3)

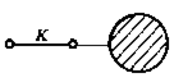
then there also is a geometry with diagram



(4)

This geometry has a bipartite point graph, and conversely, if a geometry with diagram (4) *has a bipartite point graph, then there is a geometry with diagram* (3).

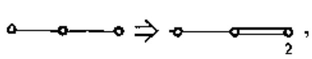
Proof. (i) Call the varieties at the s distinguished nodes points more specifically 1-points, ..., s-points. Replace these varieties by new varieties which we call 1-spaces, ..., s-spaces. An i-space is simply a flag consisting of i distinct points. Incidence between i-spaces is containment, and incidence between an i-space and an old variety is as before. The residue of an $(s-1)$-space has diagram

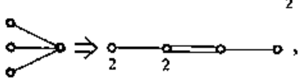


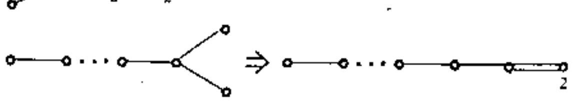
The residue of a variety belonging to the central node consists of the cliques of a complete s-partite graph, hence has diagram. Hence the new geometry has diagram (2). The converse (in the form stated) is straightforward.

(ii) is proved in the same way, using the observation that the incidence graph of a generalized n-gon is a generalized $2n$-gon with line size 2.

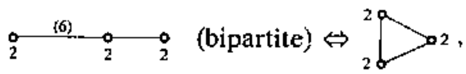
Theorem 4 contains various classical examples as special cases:

A_3 

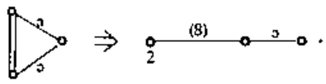
D_4 

D_n 

It also applies to the (bipartite) tesselation of a plane by regular hexagons:

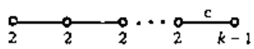


and to a diagram for Held's group (see [11]):



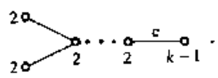
Here are two more constructions, the first of which generalizes Proposition 3.

Proposition 5. (i) *Every geometry with diagram*

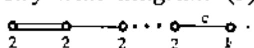


(5)

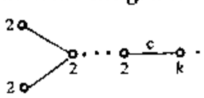
gives rise to a geometry with the same diagram and bipartite point graph, and hence to a geometry with diagram



(6)

(ii) *If there is a geometry with diagram* (5), *then there is a geometry of the same rank with diagram*  *.*

(iii) *If there is a geometry with diagram* (6), *then there is a geometry of the same rank with diagram*

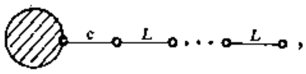


Proof. (i) Let S be a geometry with diagram (5) whose point graph is not bipartite. Take as new point set a red and a blue copy of the old point set, and for each i-variety ($i > 0$), which is an i-dimensional cube on the incident points, define two new i-varieties corresponding to a red–blue and a blue–red colouring of the natural bipartition of the cube. This defines naturally a new geometry with diagram (5). If we separate the red points and the blue points, then, by Theorem 4, we get a geometry with diagram (6).

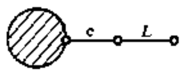
(ii) Define new points (x_0, α), where x_0 is an old point, $\alpha = \pm 1$. For $i > 0$, define new i-varieties to be either (x_i, α), where x_i is an old i-variety, $\alpha = \pm 1$, or (x_{i-1}), where x_{i-1} is an old $(i-1)$-variety. The incidence is given by (x_i, α) I (x_j, α') iff x_i I x_j, $\alpha = \alpha'$; (x_i, α) I (x_j) iff x_i I x_j, $i \leq j$; (x_i) I (x_j) iff x_i I x_j. The verification of the diagram is straightforward.

(iii) follows from (ii) by Remark 2 of Section 1.

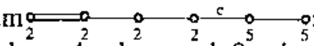
Proposition 6. *If there is a geometry with diagram*

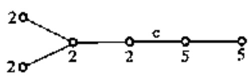
 (7)

then there also are geometries of smaller rank with diagrams

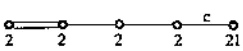
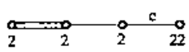
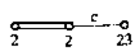
, , *etc*.

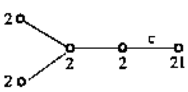
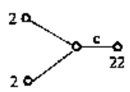
Proof. Delete the varieties belonging to the deleted nodes of diagram (7) and apply [2, Theorem 7].

Example. By Cameron [6] the group $2^{12}M_{24}$ acts on the 2^{12} cosets of the binary Golay code as the graph of a parallellism of the 4-subsets of a 24-set. This gives rise to a geometry with diagram ; the varieties are points, edges, quadrangles, 3-cubes, 4-cubes and 8-cubes corresponding to certain 0-, 1-, 2-, 3-, 4-, and 8-dimensional subspaces. Moreover, the point graph is bipartite, whence there is a geometry for the diagram

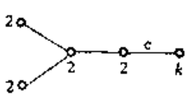


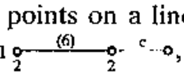
(the group is $2^{11}M_{24}$, and the diagram was mentioned in [2, Example 12.3]). Truncation according to Proposition 6 gives geometries for the diagrams

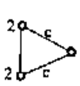
, , 

 and ,

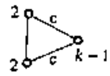
and then Proposition 5 shows the existence of geometries with diagram

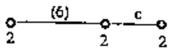
 for all $k \geqslant 21$.

Finally we mention that the incidence graph of a thick partial plane with the property that any two points on a line are on a unique triangle give rise to a geometry with diagram , hence to a geometry with diagram

.

If every point is in k lines and every line contains k points, then we have the diagrams



and 

. Unfortunately, I don't know any example of this situation.

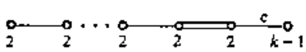
3. Geometries of type S_n

For the purpose of the next theorem, call a group H acting on a set X *fully 2-transitive* if it is 2-transitive and the pointwise stabilizer of two points fixes no other point.

Theorem 7. *Let Γ_n be a graph with the property that, for some $n \geq 2$,* (i) *there are n-cliques, and* (ii) *the neighbourhood of a $(n-2)$-clique is a rectagraph of valency k.*

Suppose that Γ_n possesses a group G_n of automorphisms such that (iii) *G_n is transitive on n-cliques, and the stabilizer of an n-clique induces on it the symmetric group S_n, and* (iv) *the stabilizer H of a $(n-1)$-clique C is fully 2-transitive on the neighbourhood of C.*

If Γ_n is connected, then it is the point graph of a geometry S of rank $n+1$ with diagram



, (8)

and G_n is transitive on the maximal flags of S.

Proof. Let us call an n-clique B of Γ_n a *mate* of another n-clique C of Γ_n if $B \cup C$ is a $K_{n\times 2}$, a complete multipartite graph with n classes of size 2. Let K be a group conjugate to a two-point stabilizer of H, and denote by $F_n(K)$ the fixpoint set of K in Γ_n. To facilitate the induction we also introduce the empty graph Γ_0, and the graph Γ_1 with k vertices and no edge. If Γ_n, G_n satisfy the hypothesis of the theorem, then so do the neighbourhood $\Gamma_{n-1} = \Gamma_n(x)$ together with the stabilizer $G_{n-1} = (G_n)_x$ of any point x, with the same group H. Hence the neighbourhood in $F_n(K)$ of a point of $F_n(K)$ is a $F_{n-1}(K)$. The essential argument is the following lemma.

Lemma 8. *For all integers $n \geq 0$, every n-clique contained in $F_n(K)$ has exactly one mate in $F_n(K)$.*

Proof. The lemma is trivial for $n=0$ and true for $n=1$ since, by (iv), $F_1(K)$ contains exactly two (nonadjacent) points. For $n=2$, if $F_2(K)$ contains an edge xy, then it contains, by definition of K, a 2-claw $(x, \{yy'\})$ which is in a quadrangle $xyx'y'$, and $x'y'$ is a mate of xy (x' is fixed by K since it is uniquely determined by x, y, y'). If $x''y''$ is another mate of xy then x' and x'' are mates of x in the neighbourhood of y, hence $x'=x''$, and similarly $y'=y''$. Hence the mate is unique. Now we proceed by induction, and assume Lemma 8 for 0, $1, \ldots, n-1$ in place of n.

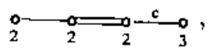
For $n \geqslant 3$, if $F_n(K)$ contains an n-clique C, choose three distinct points $x_i \in C$, $i=1, 2, 3$. Let B be the mate of $C' = C - \{x_1, x_2, x_3\}$ in the neighbourhood $F_{n-3}(K)$ of x_1, x_2, x_3. Let B_i be the mate of $C-\{x_i\}$ in the neighbourhood of x_i ($i=1, 2, 3$), and let x_{ji} be the opposite of x_j in B_i ($j \neq i$). Then $B_1 - \{x_{21}, x_{31}\}$ is a mate of C' in $F_{n-3}(K)$ whence $B_1 = B \cup \{x_{21}, x_{31}\}$, and similarly $B_2 = B \cup \{x_{32}, x_{12}\}$, $B_3 = B \cup \{x_{13}, x_{23}\}$. Therefore, $B \cup \{x_{12}\}$ and $B \cup \{x_{13}\}$ are mates of $C - \{x_2, x_3\}$ in the neighbourhood of x_2 and x_3, whence $x_{12} = x_{13} = y_1$, say, and similarly $x_{23} = x_{21} = y_2$, $x_{31} = x_{32} = y_3$. Now it is easy to see that $B \cup \{y_1, y_2, y_3\}$ is an n-clique, hence a mate of C, and that this mate is unique.

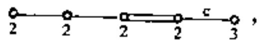
Proof of Theorem 7 (continued). Let C and C' be n-cliques intersecting in a $(n-1)$-clique C_0, so that $C = C_0 \cup \{a\}$, $C' = C_0 \cup \{b\}$, with two nonadjacent points a, b in the neighbourhood of C_0. By (iv) the pointwise stabilizer K of $C \cup C'$ is conjugate to a two-point stabilizer of H. By Lemma 8, C has a unique mate B in $F_n(K)$. For $x \in C$, denote by x' the opposite of x in $B \cup C$. An application of Lemma 8 to $\Gamma_1 = \Gamma_n(C_0)$, with $n=1$, shows that $b = a'$ whence B contains b. Now let $\bar{B}$ be any mate of C in Γ_n containing b. For $x \in C$, denote by $\bar{x}$ the opposite of x in $\bar{B} \cup C$. Then $\bar{a} = b = a'$. If $x \in C - \{a\}$, then in the neighbourhood of the $(n-2)$-clique $C - \{a, x\}$, $xa\bar{x}a'$ and $xax'a'$ are quadrangles containing the 2-claw $(x, \{aa'\})$, whence by (ii), $\bar{x} = x'$. Hence $\bar{B} = B$, and $C \cup C'$ is in the unique $K_{n\times 2}$ $B \cup C$. Hence we have the following:

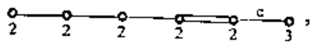
(P_n) Two n-cliques which intersect in a $(n-1)$-clique are in a unique $K_{n\times 2}$.

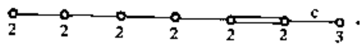
Now it is easy to check that we obtain a geometry S of rank $n+1$ with the required properties, if we call the $(i+1)$-cliques i-varieties (for $i = 0, \ldots, n-1$), and the $K_{n\times 2}$ n-varieties.

Theorem 9. *There are geometries with the following groups and diagrams:*

Aut $G_2(2)$ 

Aut HJ 

Aut $G_2(4)$ 

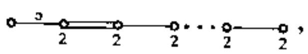
Aut Sz 

Proof. Suzuki [9] constructs strongly regular graphs Γ_3, Γ_4, Γ_5, Γ_6 with 36, 100, 416, and 1782 points and transitive automorphism groups $G_3 = \text{Aut } G_2(2)$, $G_4 = \text{Aut HJ}$, $G_5 = \text{Aut } G_2(4)$, and $G_6 = \text{Aut Sz}$, such that the neighbourhood (stabilizer) of a point of Γ_i is isomorphic to Γ_{i-1} (G_{i-1}), for $i = 6, 5, 4, 3$; here Γ_2 is a rectagraph with 14 points, namely the incidence graph of the unique biplane with $k = 4$, and $G_2 = \text{Aut PSL}_3(2)$. Hence these graphs satisfy the conditions of Theorem 7 with $k = 4$, $H = S_4$.

Now we consider, in a slightly more general setting, the dual geometries. Let Γ_n be a graph satisfying (P_n) and let the following hold:

(Q_n) Every maximal clique has n points, and every nonmaximal clique is in at least 3 maximal cliques.

Define a geometry S by calling the $K_{n\times 2}$ points (0-varieties) and the i-cliques $(n+1-i)$-varieties $(i = 1, \ldots, n)$, with natural incidence. This gives a geometry of rank $n+1$ with diagram



(9)

and Γ_n can be recovered as the graph on the blocks (n-varieties) defined by calling two blocks adjacent if they are distinct and incident with a common line (1-variety). A geometry arising in this way from a graph Γ_n with (P_n) and (Q_n) is said to be *of type S_n*. It can be shown that a geometry with diagram (9) is of type S_n iff the block graph satisfies (P_n) and (Q_n). On the other hand, the dual of the half-cube [2, p. 128] has rank 3 and diagram (9) with 3 nodes but a complete block graph, hence is not of type S_n.

Note that the residue of a block in a geometry of type S_n is a geometry of type S_{n-1}.

Let S be a geometry of type S_n. For a variety $v \in S$, denote by $P(v)$, $B(v)$ the set of points resp. blocks of S incident with v. Then $B(v)$ is a $K_{n\times 2}$ if v is a point, an $(n+1-i)$-clique if v is an i-variety, $i > 0$, and for every i-clique B of blocks there is a unique $(n+1-i)$-variety v with $B(V) = B$, and every $K_{n\times 2}$ is of the form $B(x)$ with a point x.

Proposition 10. *Let S be a geometry of type S_n, and let v, w be varieties of S. Then*

(i) $P(v) \subseteq P(w)$ *iff* $v \leqslant w$ *iff* $B(v) \supseteq B(w)$,
(ii) $P(v) = P(w)$ *iff* $v = w$ *iff* $B(v) = B(v)$.

Proof. Obviously, $v \leqslant w$ iff $B(v) \supseteq B(w)$, and $v \leqslant w$ implies $P(v) \subseteq P(w)$. Assume that $v \not\leqslant w$ and v is not a point. Then $B(v)$ is a clique not containing $B(w)$. Hence there is a block $b \in B(w)$ with $b \not\in B(v)$. By (Q_n) there is a n-clique $B \supseteq B(v)$ with $b \not\in B$. By (P_n) and (Q_n) there are at least 3 $K_{n\times 2}$ containing B, but at most one containing B and b. Hence there is a $K_{n\times 2}$ containing B but not b, and a corresponding point $x \in S$ such that $b \not\in B(x) \supseteq B$. Then $B(v) \subseteq B(x)$, $B(w) \not\subseteq B(x)$, whence $x \leqslant v$ but $x \not\leqslant w$. Hence $P(v) \not\subseteq P(w)$, which proves (i). (ii) is a consequence of (i).

By Proposition 10 we may identify a variety v with the point set $P(v)$, and we may talk about the intersection of varieties. For the next proposition, call two points *adjacent* if they are on a line, and call a *triangle* a set of 3 mutually adjacent points not contained in a line.

Proposition 11. *In a geometry of type S_n, any two adjacent points are on a unique line, and every triangle is in a unique plane.*

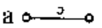
Proof. Let l, l' be two distinct intersecting lines. Then $B(l)$, $B(l')$ are distinct n-subcliques of a $K_{n\times 2}$, and there are nonadjacent blocks $b \in B(l)$, $b' \in B(l')$. Let π be the plane with $B(\pi) = B(l) - \{b\}$, and let l'' be the line with $B(l'') = B(\pi) \cup \{b'\}$. Then $B(\pi) \subseteq B(l) \cap B(l'')$ whence l and l'' are in π. Since π is a $\circ\overset{5}{\text{———}}\circ$, l and l'' intersect in a unique point. Now if x is a point incident with l and l' then $B(x) \supseteq B(l) \cup B(l') \supseteq B(l'')$ whence x is also incident with l''. Hence l and l' have a unique point in common, and so two adjacent points xy are on a unique line $\overline{xy}$.

Now let xyz be a triangle not in a block. Define $B_1 = B(\overline{xy})$, $B_2 = B(\overline{xz})$, $B_3 = B(\overline{yz})$. Then B_1, B_2, B_3 are disjoint n-cliques. Since $B(x) \supseteq B_1 \cup B_2$, and contains $2n$ blocks, $B(x) = B_1 \cup B_2$, and similarly $B(y) = B_1 \cup B_3$, $B(z) = B_2 \cup B_3$. Now $B(x)$, $B(y)$, $B(z)$ are $K_{n\times 2}$ whence $B = B_1 \cup B_2 \cup B_3$ contains $3n$ blocks, and every block is nonadjacent to exactly two other blocks. If B contains a triple (b_0, b_1, b_2) such that b_0 is not adjacent with b_1 and b_2, but b_1 and b_2 are adjacent, then $C_2 = \{b_1, b_2\}$ may be extended to a 3-clique C_3 in at least $3n - 5$ ways, C_3 to a 4-clique C_4 in at least $3n - 8$ ways, etc., and we see that there is a $(n+1)$-clique, violating (Q_n). Hence, if b_0 is nonadjacent with b_1 and b_2, then b_1 and b_2 are nonadjacent. Therefore, B is a $K_{n\times 3}$. But this contradicts (P_n). Hence every triangle is in a block. Repetition of the argument in the residue of this block, etc., shows now that every triangle is in a plane. By part (i) of the next proposition, this plane is unique.

Proposition 12. *Let S be a geometry of type S_m, and suppose that, for some $m \leqslant n$, if two varieties are contained in some m-variety, then their intersection is a variety or empty. (This holds, e.g., for $m = 2$.) Then the following holds:*

(i) *Two varieties containing a common line intersect in a variety.*

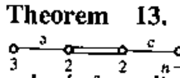
(ii) *A variety intersects an i-variety with $i < m$ in a subspace or empty.*

Proof. The hypothesis holds for $m = 2$ since every plane is a 

, hence a partial plane.

(i) Let v, w be varieties containing the line l. Then $B(v) \cup B(w)$ is a subset of $B(l)$, hence a clique. Therefore, there is a variety u such that $B(v) \cup B(w) = B(u)$, and a point x is in $v \cap w$ iff $B(x) \supseteq B(v) \cup B(w) = B(u)$ iff x is in u. Hence $v \cap w = u$.

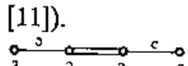
(ii) Let v, w be varieties such that $v \cap w$ is neither empty nor a variety. We may choose v, w such that the rank of w is minimal, say, w is an i-variety. Then neither of v, w is a point, and by (i), v and w have no common line. Hence $B(v)$, $B(w)$ are cliques, and $B(v) \cup B(w)$ is not a clique. So there are nonadjacent blocks $b \in B(v)$, $c \in B(w)$. Let u be the $(i+1)$-variety with $B(u) = B(w) - \{c\}$, and let v' be the i-variety with $B(v') = B(u) \cup \{b\}$. If $x \in v \cap w$, then $B(x) \supseteq B(v) \cup B(w) \supseteq B(v') \cup B(w)$, whence $x \in v' \cap w$. Therefore $v \cap w \subseteq v' \cap w$. If $v' \cap w = w'$ is a variety, then $v \cap w' \subseteq v \cap w = v \cap (v \cap w) \subseteq v \cap (v' \cap w) = v \cap w'$ and so $v \cap w' = v \cap w$ is not a variety. By the minimality of the rank of w, $w' = w$, i.e., $w \leqslant v'$ contrary to our construction. Hence $v' \cap w$ is not a variety. But v' and w are contained in the $(i+1)$-variety u, hence, for $i < m$, in some m-variety, contradiction.

Theorem 13. *For $n = 3, 4, 5, 6$, there are geometries with diagram* 

satisfying axiom (3) *of* [2]. *The lines have* 3 *points, and if abc, ade, bef are lines, then cdf is a line.*

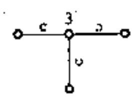
Proof. By [2, Example 12.1 and Theorem 6] the duals of the geometries of Theorem 9 satisfy the assumptions of Proposition 12 with $m = 4$. Now apply Proposition 6 to get the diagram, [2, Theorem 6] to get axiom (3), and Proposition 11 to get the closure property for the lines.

Remarks. (1) In case of rank $n + 1 \geqslant 5$, the geometries of Theorem 9 do not satisfy axiom (3) (see [11]).

(2) The geometry 

for Aut HJ is not self-dual since the residue of a block has 21 points whereas the residue of a block of the dual has 24 points.

There is another realization of the diagram $\underset{2}{\circ}\!-\!\underset{2}{\circ}\!=\!\underset{2}{\circ}\overset{c}{-}\underset{3}{\circ}$, namely a triple cover of the geometry given in Theorem 3 for Aut $G_2(2)$. This is due to the fact that $G_2(2)$ has a simple subgroup $PSU_3(9)$ of index 2. $PSU_3(9)$ has a rank 4 representation on 36 points over $PSL_3(2)$, and an associated directed graph Γ on 36 vertices with in-valency 7 and out-valency 7 (e.g., take as vertices the points, lines, and flags of PG(2, 2), and another vertex 0, and take as edges $0 \to p$, $p \to l$ $[p \notin l]$, $p \to (p, l)$ $[p \in l]$, $l \to 0$, $l \to (p, l')$ $[p = l \cap l']$, $(p, l) \to p'$ $[l = pp']$, $(p, l) \to l$ $[p \in l]$, $(p, l) \to (p', l')$ $[p \in l \ni p' \in l', p \neq p', l \neq l']$). Now take three copies X_1, X_2, X_3 of the vertex set of Γ, and define the 3-partite graph Γ_4^* on $X_1 \cup X_2 \cup X_3$ with edges $x_i y_{i+1}$ iff $x \to y$, i mod 3. This graph Γ_4^* satisfies the conditions of Theorem 7. Hence it leads to a geometry with diagram $\underset{2}{\circ}\!-\!\underset{2}{\circ}\!=\!\underset{2}{\circ}\overset{c}{-}\underset{3}{\circ}$, and since it has a 3-partite point graph, also to a geometry with diagram



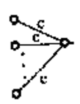
by Theorem 4.

Suzuki's graph for Aut $G_2(2)$ is obtained by identifying all x_1, x_2, x_3. Notice the similarity with Proposition 5!

Problems. (1) Is there a geometry extending the sequence of geometries in Theorem 9 (resp. Theorem 13)?

(2) Classify all partial planes with lines containing 3 points such that if *abc*, *ade*, *bef* are lines, then *cdf* is a line.

(3) Are there geometries with diagram



related to HJ, $G_2(4)$ and Sz?

(4) Are there any other applications of Theorem 7, with $n \geqslant 3$?

(5) Are there geometries with diagram $\underset{2}{\circ}\!-\!\underset{2}{\circ}\!=\!\underset{q}{\circ}\overset{Af}{-}\underset{q+1}{\circ}$? For $q = 2$, $\underset{2}{\circ}\overset{Af}{-}\underset{3}{\circ}$ is the same as $\underset{2}{\circ}\overset{c}{-}\underset{3}{\circ}$, and we have the above problem and examples. For q a prime power we have at least examples of $\underset{2}{\circ}\!=\!\underset{q}{\circ}\overset{Af}{-}\underset{q+1}{\circ}$, whose varieties are points and lines (rank 0), non-flags (rank 1), and flags (rank 2) of a projective plane of order q, with a suitable incidence defined on the varieties. The details are left to the reader.

We close with some remarks on Problem 4. If Γ_n satisfies the conditions of Theorem 7, then the neighbourhood of a $(n-2)$-clique is a rectagraph with a point- and edge-transitive group G of automorphisms such that G_x is fully 2-transitive on $\Gamma(x)$. It would be interesting to characterize such rectagraphs. Beside the n-dimensional cubes and the half-cubes (obtained by identifying opposite vertices of an n-dimensional cube with $n \geq 5$), which realize $H = A_n$ or S_n, I know of a number of sporadic examples: The Clebsch and Gewirtz graph (and their bipartite 2-cover, from Remark 3 of Section 1) realize $H = A_5$ ($k = 5$, $v = 16$ or 32) and $H = \mathrm{PSL}_2(9)$ ($k = 10$, $v = 56$ or 112), and the incidence graphs of the biplanes $B(4)$, $B(5)$, and $B(6)$ of [4] realize $H = S_4$ ($k = 4$, $v = 14$), $H = A_5$ ($k = 5$, $v = 22$), and $H = A_6$ or S_6 ($k = 6$, $v = 32$). Except for the case $H = S_4$, discussed above, it is not known whether any of these extend to a $\circ\!-\!\!-\!\circ\!=\!=\!\circ\!\overset{c}{-\!-}\!\circ$.

Institut für Angewandte Mathematik
Universität Freiburg
Freiburg, W-Germany

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