REGULAR SETS AND QUASI-SYMMETRIC 2-DESIGNS

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Abstract. The paper presents a classification of quasi-symmetric 2-designs, and sufficient parameter information to generate a list of all feasible "exceptional" parameter sets for such designs with at most 40 points. The main tool is the concept of a regular set in a strongly regular graph.

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1. Regular sets in strongly regular graphs

Throughout the paper, all graphs are finite, undirected, without loops or multiple edges. A graph Γ is strongly regular (see e.g. [9], [11], [16]) if (i) every vertex is adjacent with exactly k other vertices, and (ii) the number of vertices adjacent with two distinct vertices x and y is λ or μ, depending on whether x and y are adjacent or not. Related to a graph is its adjacency matrix $M = (m_{xy})$, indexed by the vertices, with $m_{xy} = 1$ if $\overline{xy}$ is an edge, $m_{xy} = 0$ otherwise. If I,J denote the identity and the all-one matrix (of suitable size) then a graph is strongly regular iff its adjacency matrix satisfies

$$MJ = kJ, \quad M^2 = (\lambda-\mu)M+(k-\mu)I+\mu J. \tag{1}$$

The adjacency matrix of a connected strongly regular graph has just three distinct eigenvalues k (valency), r (≥ 0), s (≤ -1); the eigenvalue k is simple and has the all-one vector $\mathbf{j}$ as an associated eigenvector. In terms of r, s, and μ, the other parameters of a strongly regular graph can be expressed by

$$v = (k-r)(k-s)/\mu, \quad k = \mu - rs, \quad \lambda = \mu + r + s, \tag{2}$$

where v denotes the total number of vertices. The multiplicity of the eigenvalue r is given by

$$f = \frac{k(k-s)(-s-1)}{\mu(r-s)} . \tag{3}$$

Now let Γ be a strongly regular graph with parameters (2). A nonempty set B of vertices of Γ is a regular set with valency d and nexus e if the number of vertices of B adjacent with a point $x \in \Gamma$ is d ($< n$) or e (> 0), depending on whether $x \in B$ or not. We call a regular set positive if $d \geq e$, and negative if $d < e$. It is easy to see that the complement of a regular set is also regular, with same sign, valency d' and nexus e', where

$$d' = k - e, \quad e' = k - d. \tag{4}$$

Also, a subset B of Γ is regular iff the subgraphs induced on B and its complement are both regular. In the terminology of Delsarte [6], a regular set is a 1-design in Γ, and the pair $(B, \Gamma \setminus B)$ is a regular bipartition of Γ.

Denote by M_1 the adjacency matrix of the graph induced on a regular set B of Γ. Then the adjacency matrix of Γ can be written as

$$M = \begin{pmatrix} M_1 & N \\ N^T & M_2 \end{pmatrix},$$

and the properties of a regular set imply

$$M_1\mathbf{j} = d\mathbf{j}, \qquad M_2\mathbf{j} = (k-e)\mathbf{j},$$

$$N\mathbf{j} = (k-d)\mathbf{j}, \quad N^T\mathbf{j} = e\mathbf{j} .$$

These relations imply that the vector

$$\begin{pmatrix} (k-d)\mathbf{j} \\ -e\mathbf{j} \end{pmatrix}$$

is an eigenvector of M for the eigenvalue $d-e < k$. Hence $d-e \in \{r,s\}$, and we have

<u>Proposition 1</u>

The parameters of a regular set B satisfy the relation

$e = d-r$ if B is positive,

$e = d-s$ if B is negative. □

In particular, if a strongly regular graph contains a regular set then the eigenvalues are integers.

<u>Proposition 2</u>

The number of vertices of a regular set B of valency d is

$K = (k-s)(d-r)/\mu$ if B is positive,

$K = (k-r)(d-s)/\mu$ if B is negative.

<u>Proof</u>. We count in two ways the number of edges $\overline{xy}$ with $x \notin B$, $y \in B$ and get $(v-K)e = K(k-d)$, whence

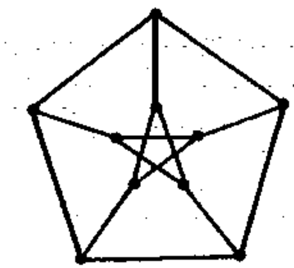
$$K = ve/(k-d+e). \tag{5}$$

Now use Proposition 1 and equation (2) and simplify. □

<u>Examples</u>. 1. If Γ is a disjoint union of cliques, a positive regular set is a union of classes ($e=0$, $d=k$), and a negative regular set is a set with e points from every class ($d=e-1$).

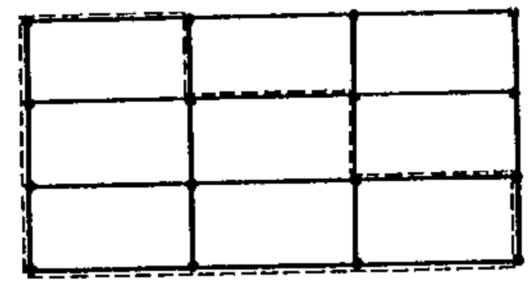
2. If Γ is a complete multipartite graph, a positive regular set is a set with i points from every class ($d=e=K-i$), and a negative regular set is a union of classes ($e=K$, $d=K-m$).

3. In the Petersen graph, the 12 pentagons are positive regular sets



with $K=5$, $d=2$, $e=1$, and the 5 cocliques of size 4 are negative regular sets with $K=4$, $d=0$, $e=2$.

4. In the lattice graph $L_2(n)$, the union of e parallel lines form



n=4

positive regular sets with $k=en$, $d=n+2-e$, and the union of t disjoint transversals form examples of negative regular sets with $K=tn$, $d=2t$, $e=2t+2$. For $t=2$, the polygon indicated in the figure is one of several possibilities.

5. If B is a positive (negative) regular set of size K, valency d, and nexus e in Γ, then, in the complementary graph $\overline{\Gamma}$, B is a negative (positive) regular set with valency $\overline{d}$ and nexus $\overline{e}$ given by

$$\overline{d} = K-1-d, \quad \overline{e} = K-e.$$

This explains the similarity in the first two examples.

6. Many examples of regular cliques ($d=K-1$) are given in Neumaier [13]. Regular cliques are always positive. Complementarily, regular cocliques ($d=0$) are always negative.

Regular sets can be viewed as extremal cases of induced regular subgraphs:

Proposition 3

Let B be a set of vertices such that the graph induced on B is regular of valency d. Then the number K of vertices of B satisfies the inequality

$$(k-s)(d-r)/\mu \le K \le (k-r)(d-s)/\mu. \tag{6}$$

The lower (upper) bound is attained iff B is a positive (negative) regular set.

Proof. For $x \notin B$, denote by e_x the number of vertices of B adjacent with x. Counting in two ways the number of edges $\overline{xy}$ with $x \notin B$, $y \in B$ gives

$$\sum e_x = K(k-d), \tag{7}$$

and counting in two ways the number of paths $\overline{zxy}$ of length 2 with $x \notin B$, $y,z \in B$, $y \neq z$ gives

$$\sum e_x(e_x-1) = Kd(\lambda+1-d) + K(K-1-d)\mu. \tag{8}$$

Here the sum is over all $x \notin B$. Using (2), (7), and (8) gives

$$\sum e_x^2 = K^2\mu - K(d-r)(d-s). \tag{9}$$

From (7), the averave value of e_x is

$$e := \frac{K(k-d)}{v-K}, \tag{10}$$

and we compute

$$\sum (e_x-e)^2 = K^2\mu - K(d-r)(d-s) - 2eK(k-d) + e^2(v-K)$$

$$= K^2\mu - K(d-r)(d-s) - K^2(k-d)^2/(v-K)$$

$$= -K(\mu K-(k-r)(d-s))(\mu K-(k-s)(d-r))/(\mu(v-K)), \tag{11}$$

where we simplified with (2). Now the sum of squares is nonnegative, whence μK must lie between $(k-r)(d-s)$ and $(k-s)(d-r)$. But $(k-r)(d-s) - (k-s)(d-r) = (r-s)(k-d) > 0$, whence (6) holds. If equality holds in (6) then $\sum(e_x-e)^2 = 0$, whence $e_x = e$ for all $x \notin B$. Therefore, B is a regular set, and from (2) and (10) we find $e = d-r$ (resp. $e = d-s$) if the lower (resp. upper) bound is attained. □

Note that this proof contains a matrix-free proof of Proposition 1 and 2.

2. Semiregular partially balanced designs

A partially balanced design (with two associate classes) is a pair $(\Gamma, \mathfrak{B})$ consisting of a connected strongly regular graph Γ (whose v vertices are now called points) and a collection $\mathfrak{B}$ of subsets of Γ (called blocks) such that (i) every block contains K points ($2 \le K \le v-1$), (ii) every point is in R (> 0) blocks, and (iii) two distinct points x,y are in q or p common blocks ($p \neq q$) depending on whether x,y are adjacent or not. For other, equivalent definitions see e.g. [2], [15]. Associated with a partially balanced design is its incidence matrix $A = (a_{xB})$ indexed by points and blocks, with $a_{xB} = 1$ if $x \in B$, $a_{xB} = 0$ otherwise. The v×n-matrix $N = AA^T$ has three nonnegative eigenvalues, among them the simple eigenvalue $\lambda = RK$. A partially balanced design is called semiregular (in [2]: special) if $\det(N) = 0$, i.e. if $\lambda = 0$ is an eigenvalue of N. The results of Neumaier [12; Section 3] imply that every $1\frac{1}{2}$-design with two connection number p and q (< R) is a semiregular partially balanced design; the converse follows easily from the following result of Bridges and Shrikhande [2]:

Proposition 4

A partially balanced design is semiregular iff there are numbers d and e such that every block is a regular set with valency d and nexus e. □

Proposition 5

If Γ is a rank 3 graph then the orbit of every regular set is a semiregular partially balanced design.

Proof. The automorphism group of Γ is transitive on vertices, edges, and nonedges. This implies (ii) and (iii) in the definition of a partially balanced design. Obviously, automorphic images of a regular set are regular sets with the same parameters; this provides (1) and semiregularity. □

Proposition 6

The parameters of a semiregular partially balanced design can be written in terms of d and e as

$$b = \frac{R(k-r)}{d-r}, \quad t = \frac{R(k-d)}{k(-s-1)}, \; p = R+st, \quad q = R+(s+1)t \tag{12}$$

if all blocks are positive, and as

$$b = \frac{R(k-s)}{d-s}, \quad t = \frac{R(k-d)}{k(r+1)}, \; p = R-rt, \quad q = R-(r+1)t \tag{13}$$

if all blocks are negative. In particular,

$$\begin{aligned} &p < q \quad \text{iff the blocks are positive,} \\ &p > q \quad \text{iff the blocks are negative.} \end{aligned} \tag{14}$$

Proof. For fixed $z \in \Gamma$, we count in two ways the number of pairs (x,B) with $x \in B$, resp. with $x,z \in B$, x adjacent with z, resp. with $x,z \in B$, x not adjacent with z, and obtain

$$\begin{aligned} &Kb = Rv, \\ &kq = Rd, \\ &(v-1-k)p = R(K-1-d). \end{aligned} \tag{15}$$

Now assume that the blocks are positive. Then $e = d-r$, $K = (k-s)(d-r)/\mu$ by Proposition 1 and 2, and with (2) we find $K-1-d = -(r+1)(k+ds)/\mu$, $v/K = (k-r)/(d-r)$. From (2), we also find $v-1-k = k(r+1)(-s-1)/\mu$, whence by (15),

$$b = \frac{R(k-r)}{d-r}, \quad q-R = \frac{Rd}{k} - R = \frac{-R(k-d)}{k} = (s+1)t,$$

$$p-R = \frac{-R(k+ds)}{k(-s-1)} - R = \frac{Rs(k-d)}{k(-s-1)} = st.$$

This implies (12). Since $d < k$, $s \leq -1$, and $R > 0$ we have $t > 0$, hence $p < q$. The case of negative blocks follows by interchanging the eigenvalues r and s, and replacing t with −t. □

Remarks. 1. Since $t = \pm(p-q)$, the number t in (12) resp. (13) must be a positive integer.

2. If $(\Gamma, \mathfrak{B})$ is a partially balanced design and Γ is not complete multipartite then $\overline{\Gamma}$ is connected, whence $(\overline{\Gamma}; \mathfrak{B})$ is a partially balanced design with p and q interchanged. Hence for the proper choice of Γ we will have $p < q$, and all blocks are positive.

Proposition 7

In a semiregular partially balanced design with positive blocks, the number b of blocks satisfies

$$b \geq f+1, \tag{16}$$

where f is given by (3). Equality holds iff any two blocks intersect in the same number of points.

Proof. This is a special case of a theorem for $1\frac{1}{2}$-designs given in Neumaier [12]. □

In the terminology of statisticians, $b = f+1$ characterizes the linked designs. If we dualize a linked design we obtain a 2-design with only two intersection numbers p and q, i.e. a quasi-symmetric 2-design. This is the topic of the next section.

3. Quasi-symmetric 2-designs

A 2-(v^*,k^*,λ^*)-design consists of a set P of v^* points and a collection $\mathfrak{B}$ of b^* blocks such that each block consists of k^* points and every pair of points is in λ^* blocks. Then every point is in a constant number r^* of points, and the relations

$$b^*k^* = r^*v^*, \quad r^*(k^*-1) = \lambda^*(v^*-1) \tag{17}$$

hold (see e.g. Raghavarao [15]). A 2-design is called quasi-symmetric if any two blocks have either p or q common points, $p < q$, and if both possibility occur. Goethals and Seidel [7] showed that the graph Γ whose vertices are the blocks, adjacent if they have q common points (the block graph) is strongly regular. We denote its parameters as in Section 1.

Proposition 8

For each point $x \in P$, the set $S(x) := \{B \in \mathcal{B} \mid x \in B\}$ is a positive regular set of the block graph with valency d and nexus e given by

$$d=((k^*-1)(\lambda^*-1)-(r^*-1)(p-1))/(q-p), \quad e=(k^*\lambda^*-r^*p)/(q-p). \qquad (18)$$

Proof. Fix $x \in P$. For each block B, denote by e_B the number of blocks through x adjacent with B. We count in two ways the number $s(x,B)$ of pairs (y,C) such that $x,y \in C$, $y \in B$, $y \neq x$, $C \neq B$. If $B \in S(x)$ then $x \in B$ and $e_B(q-1)+(r^*-1-e_B)(p-1) = s(x,B) = (k^*-1)(\lambda^*-1)$ whence $e_B = (k^*-1)(\lambda^*-1)-(r^*-1)(p-1)/(q-p)$. If $B \notin S(x)$ then $x \notin B$ and $e_Bq+(r^*-e_B)p = s(x,B) = k^*\lambda^*$ whence $e_B = (k^*\lambda^*-r^*p)/(q-p)$. Hence each set $S(x)$ is a regular set with valency and nexus given by (18). By Proposition 4, the dual of a quasi-symmetric 2-design is a semiregular partially balanced design. Hence, since $p < q$ by definition, $S(x)$ is positive by Proposition 6. □

Proposition 9

The parameters of a quasi-symmetric 2-designs can be expressed in terms of the parameters of the block graph as follows:

$$v^* = f+1, \quad k^* = (f+1)e/(k-r), \quad p = k^*+st, \quad q = k^*+(s+1)t, \qquad (19)$$

$$b^* = v \quad , \quad r^* = ve/(k-r) \quad , \lambda^* = r^*-(r-s)t, \quad d = e+r \quad , \qquad (20)$$

with a positive integer

$$t = \frac{k^*(k-r-e)}{k(-s-1)} \quad . \qquad (21)$$

Proof. The results of the last section apply with

$$v = b^*, \quad b = v^*, \quad K = r^*, \quad R = k^*. \qquad (22)$$

By Proposition 7, $v^* = b = f+1$ since the dual of a quasi-symmetric 2-design satisfies the equality condition. If we solve the first equation of (12) for R and substitute (22) we find $k^* = ve/(k-r) = (f+1)e/(k-r)$ and obtain (19).
From Proposition 1 we have $d = e+r$. From (17) we find $r^* = b^*k^*/v^* = vk^*/(f+1) = ve/(k-r)$ and $(v^*-1)(r^*-\lambda^*) = r^*(v^*-k^*) = ve(f+1)(k-e-r)/(k-r)^2 = tvk(-s-1)/(k-r) = tk(k-s)(-s-1)/\mu = t(r-s)f = t(r-s)(v^*-1)$, using (2) and (3), whence $r^*-\lambda^* = t(r-s)$, $\lambda^* = r^*-t(r-s)$. Therefore (20) holds.

Substitution of $R = k^*$ and $d = e+r$ into the second equation of (12) gives (21). Finally, $t = q-p$ is a positive integer. □

For further reference we note the formula

$$v^* = f+1 = -\frac{(k-r)(\mu+s(k-s))}{(r-s)\mu} \qquad (23)$$

which follows from (3) by a simple calculation.

Proposition 10

For a quasi-symmetric 2-design with connected block graph,

$$b^* \leq \frac{1}{2}v^*(v^*-1).$$

Proof. For connected strongly regular graphs $s < -1$ whence $q < k^*$, so a result of Cameron and van Lint [5; Prop. 3.4] applies. □

Proposition 11

The complement of a quasi-symmetric 2-design is again a quasi-symmetric 2-design; the corresponding block graphs are isomorphic.

Proof. The new blocks are the complements of the old blocks. Two adjacent old blocks have complements intersecting $\overline{p} = v^*-2k^*+p$ points two nonadjacent old blocks have complements intersecting in $\overline{q} = v^*-2k^*+q$ points. Since $\overline{p} < \overline{q}$ the two block graphs are isomorphic.

We now consider some particular classes of quasi-symmetric 2-designs.

Class 1. Multiples of symmetric 2-designs. In a symmetric 2-design, every block contains k^* points and any two blocks intersect in $\lambda < k^*$ points. The design consisting of $m > 1$ copies of the blocks has intersection numbers $p = \lambda$ and $q = k^*$, hence is quasi-symmetric; the block graph is a disjoint union of cliques.

Class 2. Strongly resolvable 2-designs. A 2-design with v^* points and b^* blocks is strongly resolvable if the blocks can be partitioned into (the minimal number of) b^*-v^*+1 classes such that every point occurs in the same number of blocks of each class. By a theorem of Hughes and Piper [10], strongly resolvable 2-designs are quasi-symmetric, and the block graph is a complete multipartite graph.

Class 3. Steiner systems with $v^* > k^{*2}$. A Steiner system $S(2,k^*,v^*)$ is the same as a 2-(v^*,k^*,λ^*)-design with $\lambda^* = 1$. Since two points are on a unique block, two blocks intersect in 0 or 1 point. Hence Steiner systems are quasi-symmetric. Their block graphs are the Steiner graphs, cf. [11]. The excluded Steiner systems with $v^* \le k^{*2}$ are affine planes ($v^* = k^{*2}$) belonging to class 2, projective planes ($v^* = k^{*2}-k^*+1$) with only one intersection number, and the designs with only one block ($v^* = k^*$) with no intersection number.

Class 4. Residuals of biplanes. By results of Hall and Connor [8; Lemma 4.1, Thm. 3.2], every 2-design with parameters $v^* = \binom{n}{2}$, $k^* = n-1$, $\lambda^* = 2$, $r^* = n+1$, $b^* = \binom{n+1}{2}$ is quasi-symmetric with intersection numbers $p = 1$, $q = 2$, and is the residual design of a unique biplane (= symmetric 2-designs with $\lambda = 2$). The block graph is the complement of a triangular graph $T(n+1)$. The known biplanes (see Cameron [4]) realize the cases $n = 3,4,5,6,7,10,12,14$, sometimes with several non-isomorphic solutions. The Bruck-Ryser-condition for biplanes excludes infinitely many values of n, starting with $n = 8,9,11,13,\ldots$.

Theorem Q

(i) A quasi-symmetric 2-design with disconnected block graph is of class 1.

(ii) A quasi-symmetric 2-design with complete multipartite block graph is of class 2.

(iii) A quasi-symmetric 2-design with $p = 0$, $q = 1$ is of class 3.

(iv) A quasi-symmetric 2-design with $p = 1$, $q = 2$ is of class 4, or a 2-$(5,3,3)$-design.

Proof. (i) A disconnected strongly regular graph is a disjoint union of ≥ 2 cliques of the same size m. By Example 1 of Section 2, positive regular sets have $d = k$. By Proposition 8 and equations (14), (20) we hence have $kq = Rd = k^*k$, or $q = k^*$. Therefore, adjacent blocks contain the same points, and the blocks of the design form copies of another 2-design $\mathcal{B}'$. Since two nonadjacent blocks intersect in the same number p of points, $\mathcal{B}'$ must be a symmetric 2-design.

(ii) This is part of Theorem 5.3 of Beker and Haemers [1].

(iii) $p = 0$, $q = 1$ implies that two blocks have at most one common point. But two distinct points are in $\lambda^* \ge 1$ blocks whence $\lambda^* = 1$.

(iv) If $p = 1$, $q = 2$ then (19) implies that $t = 1$, $k^* = 1-s$, and using (22),

$$e = \frac{(k-r)(1-s)}{f+1} = \frac{(s-1)(r-s)\mu}{\mu+s(k-s)} . \tag{24}$$

Now (20), (2) and (24) imply that $e+sr^* = e+sve/(k-r) = e+s(k-s)e/\mu = (\mu+s(k-s))e/\mu = (s-1)(r-s)$ whence $e \equiv -r \bmod s$. Hence for a suitable integer i,

$$e = -si-r, \quad d = -si. \tag{25}$$

Equation (21) implies $1 = (1-s)(k-d)/k(-s-1)$ whence $k(-s-1) = (1-s)k-(1-s)d$, $2k = d(1-s)$, and by (25),

$$k = \frac{1}{2}\, is(s-1). \tag{26}$$

If we insert (25) and (26) into (24), observe that $\mu = k+rs$ (by (2)), and simplify, we find the relation

$$(2r+s(i-1))^2 = (i+1)(2i-s^2(i-1)). \tag{27}$$

Now (26) implies that $i > 0$.
If $i = 1$ then by (27), (25), (26), (23), and (2) we find

$$r = 1,\ s = -d,\ e = d-1,\ k = \binom{d+1}{2},\ \mu = \binom{d}{2},\ f+1 = \binom{d+2}{2},\ v = \binom{d+3}{2}$$

whence by (19) and (20),

$$v^* = \binom{d+2}{2},\quad k^* = d+1,\ \lambda^* = 2.$$

Therefore, the design is of class 4.
If $i > 1$ then (27) implies $0 \le 2i-s^2(i-1)$ whence $(s^2-2)(i-1) \le 2$. This is only possible if $i = 2$, $s = -2$. In this case we obtain as before

$$r = 1,\ s = -2,\ e = 3,\ k = 6,\ \mu = 4,\ f+1 = 5,\ v = 10,$$

$$v^* = 5,\ k^* = 3,\ \lambda^* = 3,$$

which is the second alternative in the statement. □

Note that there is a unique 2-$(5,3,3)$-design, consisting of 5 points and the 10 possible point triples. Its complement is of class (iii).

4. Exceptional quasi-symmetric 2-designs with few points

We call a quasi-symmetric 2-design $\mathcal{B}$ exceptional if neither $\mathcal{B}$ nor its complement is in class 1, 2, 3, or 4. There are fairly many feasible exceptional parameter sets with a small number of points. By Proposition 11 it is sufficient to consider designs with $k^* \leq \frac{1}{2}v^*$, and a list of all possibilities with $2k^* \leq v^* \leq 40$ was compiled as follows. Using the necessary conditions given in Neumaier [11], [14], we calculated the possible parameter sets for strongly regular graphs with f $(= v^*-1) \leq 39$ which were connected and not complete multipartite. For each "graph" obtained we checked whether there are one or more values of e such that the parameters resulting from Proposition 9 are integral, and $2k^* \leq v^*$. Then the designs belonging to class 3 and class 4 were deleted. To rule out some of the remaining 36 "designs" two further existence tests were applied; they can be considered as analogues of the Krein condition [16] and the improved absolute bound [14] for strongly regular graphs.

Proposition 12

The parameters of a quasi-symmetric 2-designs satisfy the inequality

$$B(B-A) \leq AC, \tag{28}$$

where

$$A = (v^*-1)(v^*-2), \quad B = r^*(k^*-1)(k^*-2), \tag{29}$$

$$C = r^*d(q-1)(q-2)+r^*(r^*-1-d)(p-1)(p-2). \tag{29}$$

Equality holds in (28) iff any three distinct points are in a constant number of blocks.

Proof. For distinct points x,y,z, denote by λ_{xyz} the number of blocks containing x,y, and z. Now fix a point x, and take the following sums over all pairs (y,z) with $x \neq y \neq z \neq x$. By counting suitable configurations in two ways we find $\sum 1 = A$, $\sum \lambda_{xyz} = B$, $\sum \lambda_{xyz}(\lambda_{xyz}-1) = C$, given by (29). Hence the average value of λ_{xyz} is $\bar\lambda = B/A$, and $0 \leq \sum(\lambda_{xyz}-\bar\lambda)^2 = (C+B)-2\bar\lambda B+\bar\lambda^2 A = C+B-B^2/A = (AC-B(B-A))/A$, from which the assertion follows. □

Proposition 13

If for a quasi-symmetric 2-design

$$b^* = \frac{1}{2}v^*(v^*-1) \tag{30}$$

then (28) holds with equality.

Proof. By a result of Cameron and van Lint [5; Prop. 3.6], equation (30) implies that the design is a 4-design. In particular, the equality condition of Proposition 12 is satisfied. □

Proposition 12 is quite powerful, and eliminates 12 of the 36 cases. As an example, for the parameter sets

$$v^* = 27, \quad k^* = 7, \quad \lambda^* = 21, \quad r^* = 91, \quad b^* = 351,$$
$$p = 1, \quad q = 3, \quad d = 60, \quad e = 28,$$

equation (30) holds but (28) is satisfied with strict inequality. Unfortunately, all parameter sets with $b^* < \frac{1}{2}v^*(v^*-1)$ pass Proposition 12. But one of them,

$$v^* = 19, \quad k^* = 7, \quad \lambda^* = 7, \quad r^* = 21, \quad b^* = 57,$$
$$p = 1, \quad q = 3, \quad d = 18, \quad e = 14$$

is impossible since no strongly regular graph with corresponding parameters

$$v = 57, \quad k = 42, \quad \lambda = 31, \quad \mu = 30, \quad r = 4, \quad s = -3$$

exists (see Wilbrink and Brouwer [17]). There remained 23 parameter sets are listed in Table 1. The entry 'yes' under the heading 'Ex ?' indicates that a quasi-symmetric 2-design with the stated parameters is known.

The designs No. 6, 7, 9, 10, 11 are well-known classical designs, related to the binary Golay code (see Goethals and Seidel [7]). Examples 9 and 11 must be the unique Steiner systems S(3, 6, 22) and S(4, 7, 23) constructed by Witt [18]; indeed for No. 9, 10, and 11, relation (28) is satisfied with equality, whence we have 3-designs, and a counting argument similar to that of Proposition 12 shows that No. 11 must be a 4-design.

Designs No. 14 and 21 were constructed by Peter Cameron (personal communication) from the symplectic group Sp(6,2), and design No. 16 was realized by Andries Brouwer (personal communication) as the set of all planes in the projective space PG(4,2); in fact, these are the first members of 3 infinite families of quasi-symmetric designs.

For No. 4, 17, and 23, no designs are known, but the block graphs of Steiner triple systems with 21, 33, and 39 points, respectively, have the parameters needed for the block graphs of No. 4, 17, and 23. Perhaps this can be used for a construction.

It is hoped that Table 1 will challenge some readers to construct a few more quasi-symmetric 2-designs, or to devise new existence tests which eliminate some of the undecided cases.

Finally, we mention one more interesting feasible parameter set:

$$v^* = 56,\quad k^* = 16,\quad \lambda^* = 6,\quad r^* = 32,\quad b^* = 77,$$

$$p = 4,\quad q = 6,\quad d = 6,\quad e = 4.$$

No.	Ex?	v*	k*	λ*	p	q	v	k	λ	μ	d	e
1	?	19	9	16	3	5	76	45	28	24	25	18
2	?	20	10	18	4	6	76	35	18	14	21	14
3	?	20	8	14	2	4	95	54	33	27	27	18
4	?	21	9	12	3	5	70	27	12	9	15	9
5	?	21	8	14	2	4	105	52	29	22	26	16
6	yes	21	6	4	0	2	56	45	36	36	15	12
7	yes	21	7	12	1	3	120	77	52	44	33	22
8	?	22	8	12	2	4	99	42	21	15	21	12
9	yes	22	6	5	0	2	77	60	47	45	20	15
10	yes	22	7	16	1	3	176	105	68	54	45	28
11	yes	23	7	21	1	3	253	140	87	65	60	35
12	?	24	8	7	2	4	69	20	7	5	10	5
13	?	28	7	16	1	3	288	105	52	30	45	20
14	yes	28	12	11	4	6	63	32	16	16	16	12
15	?	29	7	12	1	3	232	77	36	20	33	14
16	yes	31	7	7	1	3	155	42	17	9	18	7
17	?	33	15	35	6	9	176	45	18	9	27	15
18	?	33	9	6	1	3	88	60	41	40	20	15
19	?	35	7	3	1	3	85	14	3	2	6	2
20	?	35	14	13	5	8	85	14	3	2	8	4
21	yes	36	16	12	6	8	63	30	13	15	15	12
22	?	37	9	8	1	3	148	84	50	44	28	18
23	?	39	12	22	3	6	247	54	21	9	27	12

Table 1. Quasi-symmetric 2-(v^*,k^*,λ^*)-designs with intersection numbers p,q and block graph parameters v,k,λ,μ; subgraphs induced by a point have valency d and nexus e in the block graph.
The list covers all designs with $2k^* \leq v^* \leq 40$ not characterized by Theorem Ω.

The block graph has parameters

$$v^* = 77, \quad k^* = 16, \quad \lambda^* = 0, \quad \mu^* = 4.$$

These are the complementary parameters of the block graph of $S(3, 6, 22)$, which might be a good start for a construction.

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