SIAM J. ALG. DISC. METH.
Vol. 4, No. 2, June 1983

0196-5212/83/0402-0001 $01.25/0

LATTICES OF SIMPLEX TYPE*

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Abstract. Lattices of simplex type provide a common setting for the Leech lattice, a lattice related to Shult and Yanushka's [Geom. Dedicata, 9 (1980), pp. 1–72] very regular line system with 256 lines and Du Val's [Proc. London Math. Soc., 42 (1937), pp. 18–51] hyperbolic version of the root system E_8. Many other examples are given.

Introduction. The present paper discusses some observations on the borderline between a few famous topics, reflection groups (Coxeter), extremal lattices (Leech, Sloane), spherical designs (Goethals, Seidel), line systems (Shult, Yanushka) and primitive permutation groups.

After a definition of lattices of simplex type, we present a normal form, and discuss the fact that any lattice of simplex type is a refinement of a "trivial" lattice. Among the trivial lattices we find the root lattices A_p. Then a number of nontrivial examples are given, among them the Leech lattice [3], some extremal lattices and a lattice in $\mathbb{R}^{16}$ whose set of 256 lines through the minimal norm vectors forms one of the nice tetrahedrally closed line systems of Shult and Yanushka [15].

In § 2 we define the hyperbolic transform of a lattice of simplex type. It is a generalization of a construction of Du Val [19] who studied the root system E_8 in a hyperbolic setting (see also Manin [11]). This leads to lattices of hyperbolic simplex type, which are slightly more general than hyperbolic transforms. By forming sections we obtain sequences of lattices, and some examples show that sections give a natural geometric interpretation to some well-known sequences of combinatorial configurations and associated permutation groups.

Section 3 partly explains the sporadic nature of the examples. We study integral lattices of simplex type and divide them into two classes: standard and exceptional lattices. Standard lattices turn out to be related to certain self-orthogonal codes, and they seem to be abundant. On the other hand, if the minimal norm n is given, there are only finitely many exceptional lattices generated by norm n vectors. The cases $n=2$ and $n=3$ are discussed in some detail, and relations to star-closed and tetrahedrally-closed line systems become apparent.

1. The Euclidean normal form. We motivate our investigation with a property of the Leech lattice, i.e., the unique even unimodular lattice Λ_{24} in $\mathbb{R}^{24}$ with minimal norm 4 (Conway [3]). In Leech and Sloane [10], there are two constructions for the Leech lattice. The first [10, § 4.4] exhibits a set of 24 vectors in $\sqrt{2}\,\Lambda_{24}$ of shape $(-1\frac{1}{2}, (\frac{1}{2})^{23})$—with respect to a suitable basis—of norm 8 and mutual inner product 4, and the second [10, § 5.7] exhibits a set of 24 vectors in $\sqrt{3}\,\Lambda_{24}$ of shape $(-2\frac{1}{2}, (\frac{1}{2})^{23})$—with respect to another basis—of norm 12 and mutual inner product 3. After scaling, this leads to vectors $z_1, \cdots, z_{24} \in \Lambda_{24}$ with

$$(z_i, z_j) = \begin{cases} 4 & \text{if } i=j, \\ 2 & \text{if } i \neq j \end{cases}$$

and to vectors $z'_1, \cdots, z'_{24} \in \Lambda_{24}$ with

$$(z'_i, z'_j) = \begin{cases} 4 & \text{if } i=j, \\ 1 & \text{if } i \neq j. \end{cases}$$

We also note that any two distinct vectors of norm 4 in Λ_{24} have inner product $(x, y) \leqq 2$. Thus Λ_{24} contains (in two essentially distinct ways) a regular simplex consisting of 24 minimal norm vectors.

Let us say that a lattice E of dimension $p \geqq 2$ is of *strict (Euclidean) simplex type* if there are numbers $n > m > 0$ such that (with the inner product associated with E):

(L1) There are vectors $z_1, \cdots, z_p \in E$ such that for $i, j = 1, \cdots, p$,

$$(1) \qquad (z_i, z_j) = \begin{cases} n & \text{if } i=j, \\ m & \text{if } i \neq j. \end{cases}$$

(L2) The minimal norm is n; i.e., $(x, x) \geqq n$ for all $x \in E \setminus \{0\}$.

(L3) If x, y are distinct norm n vectors of E then $(x, y) \leqq m$.

If only (L1) and (L2) hold we say that E is of *simplex type.* (Actually, we should speak of (strict) simplex type with respect to $z_1, \cdots, z_p$ since, as shown above, Λ_{24} is of simplex type in two essentially different ways. But in order to keep the language simple we delete the reference to $z_1, \cdots, z_p$.) Clearly, (L2) implies that E is a Euclidean lattice. Also, for $i \neq j$, the norm of $z_i - z_j$ is $2n - 2m$; hence (L2) implies the relation

$$(2) \qquad n \geqq 2m.$$

In the extremal case $n = 2m$, we say that E is of *strong* simplex type.

John Leech (private communication) observed the following geometric interpretation. A p-dimensional lattice is of simplex type if the vertex figure (the set of minimal norm vertices) contains a set of p equidistant points forming a regular simplex. If the lattice is of strict simplex type, this simplex forms a cell of the vertex figure. If the lattice is of strong simplex type then the origin can be joined to give a regular $(p+1)$-simplex, a fundamental cell of the honeycomb of the lattice. In particular, *every lattice of strong simplex type is strict.*

Note that a scalar multiple $cE = \{cx \mid x \in E\}$ of a lattice E of simplex type is again of simplex type, with $n' = c^2 n$, $m' = c^2 m$. The quotient

$$(3) \qquad d = \frac{n-m}{m} \geqq 1$$

is independent of scaling, and hence a useful invariant. In their construction of the Leech lattice, Leech and Sloane use a particular standard basis with respect to which the z_i take a simple form. Motivated by this, we say that a lattice E is in *Euclidean normal form* if there are numbers p and t such that in terms of the standard basis $a_1 = (1, 0, \cdots, 0)^T, \cdots, a_p = (0, 0, \cdots, 1)^T$ of $\mathbb{R}^p$, the following statements hold:

(E1) E is a Euclidean lattice of dimension p.

(E2) $z_i = \sum_{l=1}^{p} a_l - t a_i \in E$ for $i = 1, \cdots, p$.

(E3) $0 < 2t < p \leqq t^2 + 2t$.

Condition (E2) implies that (1) holds with

$$(4) \qquad n = p - 2t + t^2, \quad m = p - 2t, \quad d = \frac{t^2}{p - 2t},$$

and condition (E3) guarantees that $n \geqq 2m > 0$, in particular $n > m > 0$.

THEOREM 1.1. (i) *A lattice in Euclidean normal form is of simplex type if and only if its minimal norm is $n = p - 2t + t^2$.*

(ii) *Every lattice of simplex type is isomorphic to a multiple of a lattice in Euclidean normal form.*

Proof. (i) This part is clear from the preceding.

* Received by the editors August 26, 1981, and in revised form May 21, 1982.
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(ii) Let E be a lattice of simplex type, with parameters p, n, m and d, defined by (3). The equation $d=t^2/(p-2t)$ has a positive solution $t=-d+\sqrt{d(d+p)}$, and (E3) holds since $d\geqq 1$. Moreover, with $c=t^{-1}\sqrt{n-m}$, we have $n-m=c^2t^2=dm$, whence $m=c^2t^2d^{-1}=c^2(p-2t)$, $n=c^2t^2+m=c^2(p-2t+t^2)$. Now put $z=\sum_{i=1}^p z_i$. Then $(z,z_i)=(p-1)m+n=c^2(p-t)^2$ and $(z,z)=\sum(z,z_i)=pc^2(p-t)^2$. Now it is easy to show that the vectors a_i defined by

$$a_i=\frac{1}{ct}\left(\frac{1}{p-t}z-z_i\right),\qquad i=1,\cdots,p,$$

form an orthonormal basis of $\mathbb{R}^p$, and we have

$$z_i=c\left(\sum_{l=1}^p a_l-ta_i\right),\qquad i=1,\cdots,p. \tag{5}$$

If we identify $a_1,\cdots,a_p$ with the standard basis of $\mathbb{R}^p$ (which amounts to an isomorphism) we find that the lattice $c^{-1}E$ is in Euclidean normal form. □

If E is a lattice of simplex type then the sublattice E_0 generated by $z_1,\cdots,z_p$ is also of simplex type, with the same parameters. We call E_0 the *trivial part* of E, and say that E is *trivial* if $E=E_0$, i.e., if E is generated by $z_1,\cdots,z_p$. Since every lattice of simplex type is the refinement of a trivial lattice, it is important to determine all trivial lattices.

THEOREM 1.2. (i) *For any dimension $p\geqq 2$, and any pair (m,n) with $n\geqq 2m>0$ there is a trivial lattice of strict simplex type and parameters p,n,m.*

(ii) *Two trivial lattices are isomorphic if and only if they have the same parameters p,n,m.*

Proof. (i) As in the proof of Theorem 1.1, find $c>0$ and $d\geqq 1$ such that $m=c^2(p-2t)$, $n=c^2(p-2t+t^2)$. For the standard basis (a_i) of $\mathbb{R}^p$, the vectors (5) satisfy (1); so we have to show that $E=\langle z_1,\cdots,z_p\rangle$ is of strict simplex type. Now for $x=\sum\alpha_iz_i$ we have $(x,x)=\sum_{i,j}\alpha_i\alpha_j(z_i,z_j)=\sum_{i,j}\alpha_i\alpha_j(m+(n-m)\delta_{ij})=(\sum\alpha_i)^2m+(\sum\alpha_i^2)(n-m)$. This implies that the z_i are linearly independent ($x=0\to(x,x)=0\to\sum\alpha_i^2=0\to$ all $\alpha_i=0$), hence $\dim E=p$. Moreover, if $x\in E\backslash\{0\}$ has norm $\leqq n$ then the α_i are integers not all zero, and $\sum\alpha_i^2\leqq 2$ since $2(n-m)\geqq n$. Hence x is one of $\pm z_i$, $\pm(z_i-z_j)$, $\pm(z_i+z_j)$, where $j\neq i$. In the last case, $(x,x)=4m+2(n-m)>n$, in the second case $(x,x)=2(n-m)>n$ unless $n=2m$ when $(x,x)=n$, and in the first case $(x,x)=n$. Hence E has minimal norm n, i.e., (L2) holds, and the vectors of minimal norm are

$$\pm z_i\quad\text{if } n>2m,\qquad \pm z_i,\pm(z_i-z_j)\quad\text{if } n=2m.$$

In both cases, (L3) is satisfied, whence E is of strict simplex type.

(ii) By Theorem 1.1, a trivial lattice is isomorphic to one of the lattices just constructed. □

We write $T_p^{n,m}$ for a p-dimensional trivial lattice of simplex type with parameters n,m. The lattices $T_p^{2,1}$ arise in connection with extreme forms. Coxeter's [5] extreme form A_p is the symmetric bilinear form corresponding to the lattice A_p consisting of all $x\in\mathbb{Z}^{p+1}$ with $\sum x_i=0$. If we denote by $a_1,\cdots,a_{p+1}$ the standard basis of $\mathbb{Z}^{p+1}$ then A_p is generated by the vectors $z_i=a_i-a_{p+1}$ $(i=1,\cdots,p)$ which satisfy (1) with $n=2$, $m=1$. Since the minimal norm of A_p is $n=2$, A_p is a trivial lattice of simplex type, hence isomorphic to $T_p^{2,1}$. Note also that the vectors of minimal norm of A_p form a root system and a spherical 3-design (see [2], [9]). The lattices $T_2^{n,m}$ arise in a classification problem. It is not difficult to show that every two-dimensional lattice generated by its vectors of minimal norm is either a trivial lattice of simplex type (i.e., one of the $T_2^{n,m}$), or a multiple of $\mathbb{Z}^2$. In dimension $p>2$, not every lattice generated by its vectors of minimal norm is of simplex type; for $p\geqq 4$, counterexamples are the lattices D_p consisting of all $x\in\mathbb{Z}^p$ with $\sum x_i\equiv 0 \bmod 2$. On the other hand, there are lattices of simplex type not generated by their vectors of minimal norm; see Example 7 below.

Now we give some examples of nontrivial lattices of simplex type. In each case, there is a t such that the vectors of shape $(1^{p-1},1-t)$ are minimal norm vectors of the lattice; hence all examples are in Euclidean normal form (we use $a_1,\cdots,a_p$ as standard basis of $\mathbb{R}^p$). We construct examples with the parameters listed in Table 1 (τ is the number of vectors of minimal norm).

TABLE 1

p	d	τ	strict?	t	n	m	
6	2	32	yes	2	6	2	C_{16}
8	1	240	yes	2	8	4	E_8
16	2	512	yes	4	24	8	S_{256}
24	1	196,560	yes	4	32	16	Λ_{24}
24	3	196,560	no	6	48	12	Λ_{24}
48	1	52,416,000	yes	6	72	36	$P48p$, $P48q$

Example 1. Let E consist of all vectors $\sum x_ia_i\in\mathbb{R}^6$ with

(i) all x_i are integers congruent to the same value γ mod 2;

(ii) the coordinate sum $\sum x_i$ is divisible by 4.

The minimal norm of E is $n=6$; there are 32 vectors of norm 6 (forming the polytope $h\gamma_6$, cf. Coxeter [5]), namely

$$2\times 6 \text{ of shape } \pm(1^5,-1),$$

$$20 \text{ of shape } (1^3,(-1)^3);$$

they form a spherical 3-design. Moreover the corresponding set of 16 lines is the line system C_{16} of Shult and Yanushka [15].

Example 2. Let E consist of all vectors $\sum x_ia_i\in\mathbb{R}^8$ with

(i) all x_i are integers congruent to the same value γ mod 2;

(ii) the coordinate sum $\sum x_i$ is congruent to 2γ mod 4.

This is the lattice $2E_8$ defined, e.g., in Leech and Sloane [10]. The minimal norm is 8; there are 240 vectors of norm 8, namely

$$2\times 8 \text{ of shape } \pm(1^7,-1),$$

$$2\times 56 \text{ of shape } \pm(1^5,(-1)^3),$$

$$4\times 28 \text{ of shape } (\pm2,\pm2,0^6);$$

they form the root systems E_8 [2]. This root system is a tight spherical 7-design [8].

Example 3. Let E consist of all vectors $\sum x_ia_i\in\mathbb{R}^{16}$ with

(i) all x_i are integers congruent to the same value γ mod 2;

(ii) the set of indices i for which x_i mod 4 takes a given value is a $\mathscr{C}$-set;

(iii) the coordinate sum $\sum x_i$ is congruent to 4γ mod 8.

Here a $\mathscr{C}$-set is either $\varnothing$, or $\{1,\cdots,16\}$, or a hyperplane in an affine space $AG(4,2)$ whose points are labelled $1,\cdots,16$. (There are 30 such hyperplanes, and the $\mathscr{C}$-sets form a self-orthogonal linear code with respect to the symmetric difference.) The minimal norm is 24; there are 512 vectors of norm 24, namely

$$2\times 16 \text{ of shape } \pm(1^{15}, -3),$$

$$2\times 240 \text{ of shape } \pm(1^8, 3, (-1)^7),\ 1^8 \text{ on a hyperplane.}$$

These vectors form a spherical 5-design (Neumaier [12]). The corresponding set of 256 lines can be shown to be isomorphic to the line system S_{256} of Shult and Yanushka [15].

Example 4. *Let E consist of all vectors* $\sum x_i a_i \in \mathbb{R}^{24}$ with

(i) all x_i are integers congruent to the same value γ mod 2;

(ii) the set of indices i for which x_i mod 4 takes a given value is a $\mathscr{C}$-set;

(iii) the coordinate sum $\sum x_i$ is congruent to 4γ mod 8;

but this time a $\mathscr{C}$-set is a subset of $\{1, \cdots, 24\}$ whose characteristic vector belongs to the binary Golay code. This is Conway's [3] description of the Leech lattice $\sqrt{8}\,\Lambda_{24}$. The minimal norm of E is $n = 32$; there are 196,560 vectors of norm 32; they form a tight spherical 11-design [8].

Example 5. Let E consist of all vectors $\sum x_i a_i \in \mathbb{R}^{24}$ with

(i) all x_i are integers congruent to the same value γ mod 2;

(ii) $(x_1, \cdots, x_{24})$ mod 3 is a codeword of $\mathscr{C}$;

(iii) the coordinate sum $\sum x_i$ is congruent to 2γ mod 4.

Now $\mathscr{C}$ is either the ternary $(24, 3^{12}, 9)$ quadratic residue code, or the $(24, 3^{12}, 9)$ symmetry code; they both contain the words $\pm(1^{24})$. Hence by Leech and Sloane [10, § 5.7], we get in both cases $\sqrt{12}\,\Lambda_{24}$, with minimal norm 48. As remarked in the introduction, E is not of strict Leech type.

Example 6. Let E consist of all vectors $\sum x_i a_i \in \mathbb{R}^{48}$ with

(i) all x_i are integers congruent to the same value γ mod 2;

(ii) $(x_1, \cdots, x_{48})$ mod 3 is a codeword of $\mathscr{C}$;

(iii) the coordinate sum $\sum x_i$ is congruent to 2γ mod 4;

but $\mathscr{C}$ is the $(48, 3^{24}, 15)$ ternary quadratic residue code or symmetry code. By Leech and Sloane [10, § 5.7], we get the extremal lattices $P48q$ and $P48p$ with minimal norm 72, and 52,416,000 minimal norm vectors, cf. also Sloane [16].

Note that in all the examples given so far, the vectors of minimal norm generate the lattice. In the next example, the situation is different.

Example 7. Let E consist of all vectors $\sum x_i a_i \in \mathbb{R}^{15}$ with

(i) all x_i are integers congruent to the same value γ mod 3;

(ii) the coordinate sum $\sum x_i$ is divisible by 12.

The minimal norm is 18; there are 240 vectors of norm 18, namely

$$2\times 15 \text{ of shape } \pm(1^{14}, -2),$$

$$210 \text{ of shape } (3, -3, 0^{13}).$$

But $(3, -3, 0^{13}) = (1, -2, 1^{13}) - (-2, 1, 1^{13})$; hence the 240 vectors only generate the trivial part of E, which does not contain, say, $12a_1$.

We close this section with some examples whose simplest presentation is not the Euclidean normal form.

Example 8. Let E consist of all vectors $\sum x_i a_i \in \mathbb{R}^7$ such that:

(i) all x_i are integers;

(ii) the set of indices i for which x_i is odd is a $\mathscr{C}$-set;

where a $\mathscr{C}$-set is either $\varnothing$, or the complement of a line in the Fano plane $PG(2, 2)$ whose points are labelled $1, 2, \cdots, 7$ (cf. Example 19; the $\mathscr{C}$-sets form a linear code with respect to symmetric difference). This is the lattice $\sqrt{2}\,E_7$. The minimal norm is 4, and E contains 126 norm 4 vectors,

$$14 \text{ of shape } (\pm 2, 0^6),$$

$$2^4\times 7 \text{ of shape } ((\pm 1)^4, 0^3), \text{ zeros on a line.}$$

The lattice is of strong simplex type with respect to the simplex consisting of the 7 vertices of shape $(1^4, 0^3)$.

Example 9. Let E consist of all vectors $\sum x_i a_i \in \mathbb{R}^{15}$ such that

(i) all x_i are integers;

(ii) the set of indices i for which x_i is odd is a $\mathscr{C}$-set;

(iii) the coordinate sum $\sum x_i$ is divisible by 4,

where now a $\mathscr{C}$-set is either $\varnothing$, or the complement of a plane in the projective space $PG(3, 2)$ whose points are labelled $1, 2, \cdots, 15$ (the planes can be taken as the 15 sets $\{i, i+1, i+2, i+4, i+5, i+8, i+10\}$ mod 15; the $\mathscr{C}$-sets again form a linear code). This is the lattice Λ_{15} of Leech and Sloane [10, § 3.4]. The minimal norm is 8, and E contains 2,340 norm 8 vectors,

$$4\times\binom{15}{2} \text{ of shape } ((\pm 2)^2, 0^{13}),$$

$$15\times 2^7 \text{ of shape } ((\pm 1)^8, 0^7), \text{ zeros on a plane, even times } +.$$

The lattice is of strong simplex type with respect to the simplex consisting of the 15 vertices of shape $(1^8, 0^7)$.

Example 10. Similarly, the lattice obtained from Λ_{32} in Leech and Sloane [10] by equating a coordinate to zero gives a lattice of strong simplex type in $\mathbb{R}^{31}$ with respect to the simplex whose 31 vertices are those of shape $(1^{16}, 0^{15})$ with zeros on the coordinates of a hyperplane of $PG(4, 2)$.

2. Hyperbolic transforms. Let E be a lattice in $\mathbb{R}^p$ of simplex type, with parameters $n, m, d = (n-m)/m$. We adjoin to $\mathbb{R}^p$ an element w orthogonal to $\mathbb{R}^p$ and of norm $-d$. In this way we get a hyperbolic space $\mathbb{R}^p \oplus \mathbb{R}w$. This space contains the *hyperbolic transform* of E which we define as the hyperbolic lattice

$$H = c^{-1}E \oplus d^{-1}\mathbb{Z}w \quad (c = \sqrt{n-m});$$

the hyperbolic transform of an individual element $z \in E$ is defined as the element

$$x = c^{-1}z - d^{-1}w \in H.$$

Proposition 2.1. *Let E be a p-dimensional lattice of simplex type. The hyperbolic transform H of E has the following properties:*

(M1) *H contains vectors $e_1, \cdots, e_p$ such that for $i, j = 1, \cdots, p$,*

$$(e_i, w) = (e_i, e_i) = 1, \qquad (e_i, e_j) = 0 \quad \text{if } i \neq j.$$

(M2) *For all $x \in H$ linearly independent from w,*

$$(x, w) = 1 \Rightarrow (x, x) \geqq 1.$$

Moreover, if E is of strict simplex type, then H also satisfies:

(M3) *If x, y are distinct vectors from the set*

$$H_{1,1} = \{x \in H \mid (x, w) = (x, x) = 1\}$$

then $(x, y) \leqq 0$.

Proof. (i) The vectors $e_i = c^{-1}z_i - d^{-1}w$ $(i = 1, \cdots, p)$ belong to H and satisfy $(e_i, w) = -d^{-1}(w, w) = 1$, $(e_i, e_j) = c^{-2}(z_i, z_j) + d^{-2}(w, w) = (n-m)^{-1}(m + (n-m)\delta_{ij}) - d^{-1} = \delta_{ij}$. Hence (M1) holds.

(ii) If $x \in H\backslash\langle w\rangle$ and $(x, w) = 1$ then $x = c^{-1}z - d^{-1}w$ with $z \in E\backslash\{0\}$, and by (L2), $(x, x) = c^{-2}(z, z) + d^{-2}(w, w) \geqq c^{-2}n - d^{-1} = 1$. Hence (M2) holds.

(iii) If $x \in H_{1,1}$ then $(x, w) = 1$ implies that $x = c^{-1}z - d^{-1}w$ with $z \in E$. Now $c^{-2}(z, z) = (x + d^{-1}w, x + d^{-1}w) = (x, x) + 2d^{-1}(x, w) + d^{-2}(w, w) = 1 + 2d^{-1} - d^{-1} = (n-m)^{-1}n$, hence $(z, z) = n$. If $y \in H_{1,1}$ is distinct from x then similarly $y = c^{-1}z' - d^{-1}w$ with $z' \in E$, $(z', z') = n$, and $z' \neq z$. If E is of strict Leech type then by (L3), $(x, y) = c^{-2}(z, z') + d^{-2}(w, w) \leqq c^{-2}m - d^{-1} = 0$, whence (M3) holds. □

We denote by $\mathbb{R}^{p,1}$ the standard hyperbolic space, i.e., the real linear space of dimension $p+1$ equipped with the indefinite inner product

$$(x, y) = -x_0y_0 + x_1y_1 + \cdots + x_py_p.$$

Let us say that a lattice $H \subseteq \mathbb{R}^{p,1}$ is of *hyperbolic simplex type* with respect to $w \in \mathbb{R}^{p,1}$ if $(w, w) = -d \leqq 1$ and (M1) and (M2) hold; we say that H is *strict* if also (M3) holds. Trivial examples of lattices of strict hyperbolic simplex type are the lattices $\langle e_1, \cdots, e_p, d^{-1}w\rangle$, where

$$w = -\sqrt{N}e_0 + e_1 + \cdots + e_p, \qquad N \geqq p + 1 \tag{6}$$

and $e_0 = (1; 0, \cdots, 0)^T$, $e_1 = (0; 1, \cdots, 0)^T, \cdots, e_p = (0; 0, \cdots, 1)^T$ is the standard basis of $\mathbb{R}^{p,1}$. These lattices are just the hyperbolic transforms of trivial lattices of simplex type, with $d = N - p$.

THEOREM 2.2. (i) *The hyperbolic transform of a lattice of (strict) simplex type is a lattice of (strict) hyperbolic simplex type.*

(ii) *Let H be a lattice of (strict) hyperbolic simplex type with respect to a vector w satisfying $(w, w) = -d \leqq -1$. If $d^{-1}w \in H$ then $E = w^\perp \cap H = \{x \in H \mid (x, w) = 0\}$ is a lattice of (strict) simplex type with parameters $n = 1 + d^{-1}$, $m = d^{-1}$.*

Proof. Part (i) follows directly from the definition and Proposition 2.1.

(ii) The vectors $z_i = e_i + d^{-1}w$ $(i = 1, \cdots, p)$ satisfy $(z_i, z_j) = (e_i, e_j) + d^{-1}(e_i, w) + d^{-1}(e_j, w) + d^{-2}(w, w) = \delta_{ij} + d^{-1}$, hence E satisfies (L1) with $n = 1 + d^{-1}$, $m = d^{-1}$. Further, if $x \in E\backslash\{0\}$ then $x' = x - d^{-1}w$ is in $H\backslash\langle w\rangle$ and satisfies $(x', w) = 1$, whence by (M2), $1 \leqq (x', x') = (x, x) + d^{-2}(w, w) = (x, x) - d^{-1}$, and so $(x, x) \geqq 1 + d^{-1} = n$. Hence (L2) holds. Finally, if x, y are distinct norm $1 + d^{-1}$ vectors of E then $x' = x - d^{-1}w$ and $y' = y - d^{-1}w$ are distinct vectors in $H_{1,1}$. Hence, if H is strict, $0 \geqq (x', y') = (x, y) + d^{-2}(w, w) = (x, y) - d^{-1}$, whence $(x, y) \leqq d^{-1} = m$. So (M3) holds if H is strict. □

Lattices of hyperbolic simplex type have the following obvious hereditary property:

PROPOSITION 2.3. *If H is a $(p+1)$-dimensional lattice of (strict) hyperbolic simplex type with respect to w, then every p-dimensional section $H^{(i)} = e_i^\perp \cap H = \{x \in H \mid (x, e_i) = 0\}$ $(i = 1, \cdots, p)$ is of (strict) hyperbolic simplex type with respect to $w^{(i)} = w - e_i$.*

Note, that if $(w, w) = -d$, then $(w^{(i)}, w^{(i)}) = -d - 1$. Hence the number

$$N = p + d \tag{7}$$

remains invariant under forming sections. In fact, the vector $e_0 = N^{-1/2}(-w + e_1 + \cdots + e_p)$ satisfies $(e_0, e_0) = -1$, $(e_0, e_i) = 0$ for $i = 1, \cdots, p$; hence $e_0; e_1, \cdots, e_p$ can be identified with the standard basis of $\mathbb{R}^{p,1}$. In the following, we shall do this. Then w is given by formula (6) above.

Remarks. 1. Even if H is a hyperbolic transform, i.e., $d^{-1}w \in H$, then, in general, $H^{(i)}$ will not contain $(d+1)^{-1}w^{(i)}$, and hence will not be a hyperbolic transform. In particular, lattices of hyperbolic simplex type form a richer class than lattices of (Euclidean) simplex type. But it can be shown that if H is a lattice of hyperbolic simplex type with respect to w and H contains a multiple of $d^{-1}w$ then the projection

$$E = \left\{x - \frac{(x, w)}{(w, w)}w \,\middle|\, x \in H\right\}$$

of H onto $w^\perp$ is a Euclidean lattice satisfying (L1) and:

(L2*) If $z = \sum \alpha_i z_i \in E$ then $\sum \alpha_i = 1$ implies $\sum \alpha_i^2 \geqq 1$.

Conversely, if a Euclidean lattice E satisfies (L1) and (L2*) then there is a lattice H of hyperbolic simplex type (with respect to w) such that E is isomorphic to a multiple of the projection of H onto $w^\perp$.

2. If H is a lattice of hyperbolic simplex type with respect to w and if H contains a nonzero, integral multiple of $d^{-1}w$, then *the set $H_{1,1}$ is finite.* Indeed, the vectors of $H_{1,1}$ are projected to vectors $\sum \alpha_i z_i \in w^\perp$ with

$$d(\textstyle\sum \alpha_i^2) + (\sum \alpha_i)^2 = d + 1.$$

Since this equation describes a bounded domain, the projection of $H_{1,1}$—being a bounded part of a Euclidean lattice—is finite. But each vector of $H_{1,1}$ is determined by its projection.

3. If H is of strict hyperbolic simplex type then, by Andreev's lemma (cf. Vinberg [18, p. 19]), the set $H_{1,1}$ determines a hyperbolic polyhedron with the property that hyperplanes corresponding to nonadjacent faces do not intersect.

4. If H is of (strict) hyperbolic simplex type then the sublattice $H' = \{x \in H \mid (x, w)$ integral$\}$ is also of strict hyperbolic simplex type and $H'_{1,1} = H_{1,1}$. Hence the following axiom can always be forced to hold:

(M4) For all $x \in H$, (x, w) is integral.

Note that (M4) is trivially satisfied if H is a hyperbolic transform, or if H is generated by $H_{1,1}$.

Example 11. The following example, considered first by Du Val [19], is studied extensively in Manin [11, Chapt. 4], in connection with cubic forms, and led me to the study of hyperbolic transforms. Let $e_0, e_1, \cdots, e_p$ be the standard basis of $\mathbb{R}^{p,1}$, and let $H = \mathbb{Z}^{p,1}$ be the lattice generated by $e_0, e_1, \cdots, e_p$. For $p \leqq 8$, H is of strict hyperbolic simplex type with respect to $w = -3e_0 + e_1 + \cdots + e_p$. In fact, $(w, w) = 9 - p \leqq -1$, and (M1) is obvious. Further if $x = \alpha e_0 - \sum_{i=1}^p \alpha_i e_i \in H$ and $(x, w) = 1$ then α, α_i are integers and $\sum \alpha_i = 3\alpha - 1$. Now consider $(x, x) = -\alpha^2 + \sum \alpha_i^2$. Modulo 2 we get $(x, x) \equiv -\alpha + \sum \alpha_i \equiv 2\alpha - 1 \equiv 1$, whence (x, x) is odd. By the Cauchy–Schwarz inequality $\sum \alpha_i^2 \geqq (\sum \alpha_i)^2/p \geqq (3\alpha - 1)^2/8$, whence $8(x, x) \geqq -8\alpha^2 + (3\alpha - 1)^2 = (\alpha - 3)^2 - 8 \geqq -8$. But if $(x, x) = -1$ then we have equality throughout, whence $\alpha = 3$, $p = 8$, $\sum \alpha_i^2 = \sum \alpha_i = 8$, which implies $x = -w$. Since (x, x) is an odd integer, $(x, x) \geqq 1$ for $x \neq -w$, so (M2) holds. Finally, if $x \in H_{1,1}$ then $1 = (x, x) \geqq (\alpha - 3)^2 - 1$ or $3 - \sqrt{2} \leqq \alpha \leqq 3 + \sqrt{2}$, and since α is an integer, $-1 \leqq \alpha \leqq 4$. Now the equations $\sum \alpha_i = 3\alpha - 1$, $\sum \alpha_i^2 = \alpha^2 + 1$, $-1 \leqq \alpha \leqq 4$ have only finitely many integral solutions which lead to the list of vectors in $H_{1,1}$, given by Manin [11, Prop. 26.1] and shown in Table 2.

From this list, (M3) is easily verified. In fact, for $p = 8$, $H = \mathbb{Z}^{p,1}$ is the hyperbolic transform of the root lattice E_8; this can be seen from the fact that both E_8 (as defined above) and $E = w^\perp \cap H$ are generated by 8 special norm 2 vectors whose mutual inner products determine the Dynkin diagrams for E_8 (i.e., the inner product of two vectors is -1 if they are adjacent in the diagram, and 0 otherwise). See Figs. 1 and 2, and Coxeter [5].

TABLE 2

Type of vector	$\mathbb{R}^{3,1}$	$\mathbb{R}^{4,1}$	$\mathbb{R}^{5,1}$	$\mathbb{R}^{6,1}$	$\mathbb{R}^{7,1}$	$\mathbb{R}^{8,1}$
e_1	3	4	5	6	7	8
$e_0-e_1-e_2$	3	6	10	15	21	28
$2e_0-e_1-e_2-e_3-e_4-e_5$			1	6	21	56
$3e_0-2e_1-e_2-e_3-e_4-e_5-e_6-e_7$					7	56
$4e_0-2e_1-2e_2-2e_3-e_4-e_5-e_6-e_7-e_8$						56
$5e_0-2e_1-2e_2-2e_3-2e_4-2e_5-2e_6-e_7-e_8$						28
$6e_0-3e_1-2e_2-2e_3-2e_4-2e_5-2e_6-2e_7-2e_8$						8
Total number of vectors	6	10	16	27	56	240
Automorphism group	D_6	S_5	2^4S_5	$O_6^-(2)$	$2Sp(6,2)$	$2O_8^+(2)$
Configuration name	prism	$t\alpha_4$	$h\gamma_5$	2_{21}	3_{21}	4_{21}

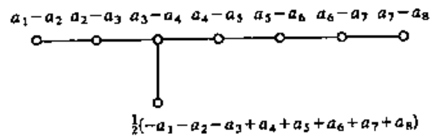

FIG. 1

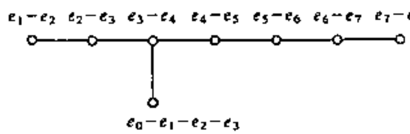

FIG. 2

The sets $H_{1,1}$ are related to interesting classical configurations. Their Euclidean projections are the successive vertex figures of the honeycomb 5_{21} $(=E_8)$, the Gosset polytopes $4_{21}, 3_{21}, 2_{21}$ and the polytopes $h\gamma_5$ and $t\alpha_4$ (cf. Coxeter [5]). The sets $H_{1,1}$ are also related to certain famous graphs. For $p \leqq 6$, the only occurring inner products of distinct vectors are 0 and -1, and the graphs obtained by calling two vectors adjacent if they have inner product -1 turn out to be the hexagon ($p=3$), the Peterson graph ($p=4$), the Clebsch graph ($p=5$) and the Schläfli graph ($p=6$) (Seidel [13]); the latter is related to the 27 lines on a cubic surface (Baker [1]). For $p=7$, the same construction yields a graph with 56 vertices, related to the regular twograph on 28 vertices (Taylor [17]) and to the 28 bitangents of a plane quartic curve (Dickson [6]). For $p=8$, $H_{1,1}$ is also related to the set of 240 Cayley units (Coxeter [4]).

Example 12. Let H be the lattice generated by $\sqrt{2}e_0, e_1, \cdots, e_p$. For $p \leqq 7$, H is of strict hyperbolic simplex type with respect to $w=-2\sqrt{2}e_0+e_1+\cdots+e_p$. This is proved as above and for $H_{1,1}$ we get Table 3.

For $p=7$, we get the hyperbolic transform of the root lattice E_7, using Fig. 3 as Dynkin diagram. The Euclidean normal form for E_7 would have $t=2\sqrt{2}-1$, and does not lead to a nice description (but cf. Example 8). For $p=6$, we get the hyperbolic transform of the lattice defined in Example 1. For $p=4, 5$, the graphs corresponding to $H_{1,1}$ are the cube and the complement of the triangular graph $T(6)$.

TABLE 3

Type of vector	$\mathbb{R}^{4,1}$	$\mathbb{R}^{5,1}$	$\mathbb{R}^{6,1}$	$\mathbb{R}^{7,1}$
e_1	4	5	6	7
$\sqrt{2}e_0-e_1-e_2-e_3$	4	10	20	35
$2\sqrt{2}e_0-2e_1-e_2-e_3-e_4-e_5-e_6$			6	42
$3\sqrt{2}e_0-2e_1-2e_2-2e_3-2e_4-e_5-e_6-e_7$				35
$4\sqrt{2}e_0-3e_1-2e_2-2e_3-2e_4-2e_5-2e_6-2e_7$				7
Total number of vectors	8	15	32	126
Automorphism group	2^3S_3	S_6	2^5S_6	$2Sp(6,2)$

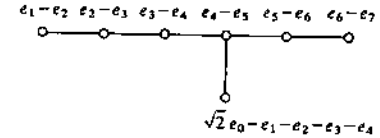

FIG. 3

Remark. The lattice E_6 is not of simplex type; cf. Theorems 3.5 and 3.6.

Example 13. Let H be the lattice generated by $\sqrt{N}e_0, e_1, \cdots, e_p$. For $p \leqq N-1$, H is of strict hyperbolic simplex type with respect to $w=-\sqrt{N}e_0+e_1+\cdots+e_p$. The table for $H_{1,1}$ is Table 4. For $p=N-1$, $d=1$, and H is also generated by w and $e_1, \cdots, e_p$; by a remark above, H is the hyperbolic transform of the trivial lattice $T_p^{2,1}=A_p$. Since A_p is a root lattice, we have again a Dynkin diagram (see Fig. 4). For $p=N-2$, H is the hyperbolic transform of $\mathbb{Z}^p$.

TABLE 4

Type of vector	$\mathbb{R}^{N-2,1}$	$\mathbb{R}^{N-1,1}$
e_1	$N-2$	$N-1$
$\sqrt{N}e_0-2e_1-e_2-\cdots-e_{N-2}$	$N-2$	$(N-1)(N-2)$
$2\sqrt{N}e_0-3e_1-2e_2-\cdots-2e_{N-1}$		$N-1$
Total number of vectors	$2(N-2)$	$N(N-1)$

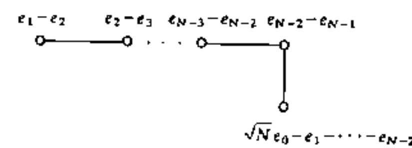

FIG. 4

Example 14. Let H be the lattice consisting of all vectors $\alpha_0\sqrt{3}e_0-\alpha_1e_1-\cdots-\alpha_{23}e_{23}\in\mathbb{R}^{23,1}$ such that:

(i) for $i=0,\cdots,23$, the number $2\alpha_i$ is integral;

(ii) the set of indices $i\in\{0,\cdots,23\}$ such that α_i is nonintegral is a $\mathscr{C}$-set;

(iii) $\alpha_0+\cdots+\alpha_{23}$ is integral,

where $\mathscr{C}$ is the binary Golay code (cf. Goethals and Seidel [7], also for the numbers quoted below). H is of strict hyperbolic simplex type with respect to $w=-3\sqrt{3}e_0+e_1+\cdots+e_{23}$, and $H_{1,1}$ (resp. their sections) contain the vectors listed in Table 5.

Example 15. Let H be the hyperbolic transform of the Leech lattice Λ_{24}. From the work of Conway [3], it can be shown that the table for $H_{1,1}$ and its sections is Table 6.

Example 16. Similarly, Shult and Yanushka's calculations [15] for their line system S_{256} imply that the hyperbolic transform of the lattice constructed in Example 3 has Table 7 as table for $H_{1,1}$.

Remark. The projections of the sets $H_{1,1}$ given in the examples form interesting spherical t-designs; cf. Delsarte, Goethals and Seidel [8].

TABLE 5

Type of vector	$\mathbb{R}^{19,1}$	$\mathbb{R}^{20,1}$	$\mathbb{R}^{21,1}$	$\mathbb{R}^{22,1}$	$\mathbb{R}^{23,1}$
e_1	19	20	21	22	23
$\frac{1}{2}\sqrt{3}e_0-\frac{1}{2}(e_1+\cdots+e_7)$	52	80	120	176	253
$\sqrt{3}e_0-\frac{1}{2}(e_1+\cdots+e_{16})$	1	5	21	77	253
$\frac{3}{2}\sqrt{3}e_0-\frac{3}{2}e_1-\frac{1}{2}(e_2+\cdots+e_{23})$					23
Total number of vectors	72	105	162	275	$2\cdot 276$
Automorphism group	?	$L_3(4)$	$U_4(3)$	McL	2 Con. 3

TABLE 6

	$\mathbb{R}^{21,1}$	$\mathbb{R}^{22,1}$	$\mathbb{R}^{23,1}$	$\mathbb{R}^{24,1}$
Total number of vectors	336	891	4,600	196,560
Automorphism group	$2^9L_3(4)$	$U_6(2)$	2 Con. 2	Con. 0

TABLE 7

	$\mathbb{R}^{14,1}$	$\mathbb{R}^{15,1}$	$\mathbb{R}^{16,1}$
Total number of vectors	56	135	512
Automorphism group	$2^3L_3(2)$?	$Sp(6,2)$	$2^{8+1}Sp(6,2)$

3. Integral lattices of simplex type. In this section we classify the *integral* lattices of simplex type into standard and exceptional lattices. The aim is to determine all integral lattices of simplex type; but we are very far from achieving this.

For $(m,n)=1$, we relate the standard lattices of simplex type to certain linear

self-orthogonal codes. From this it appears that standard lattices of simplex type are very abundant. On the other hand, restricting ourselves to integral lattices generated by their minimal norm vectors, we are able to show that for given minimal norm n there are only finitely many such *exceptional* lattices. For $n=2$ and $n=3$ we push the analysis further, and find close relations to star-closed and tetrahedrally-closed line systems (Cameron et al. [2]. Shult, Yanushka [15]).

Let E be an integral lattice of simplex type with (integral) parameters n, m. We call E *standard* if $(n-m)E$ is contained in the trivial part E_0 of E, and *exceptional* otherwise. Note that this definition is no longer scaling invariant. In particular, a suitable multiple cE of any integral lattice E of simplex type is always standard (take, e.g., for c the discriminant of E). We also mention that an exceptional lattice of simplex type contains a unique maximal standard sublattice, namely $E'=\{x\in E\mid(n-m)x\in E_0\}$.

We treat the standard case first. If $n-m=1$ then E must be trivial. But since n, m are integers with $n\geqq 2m>0$, this happens only for $n=2$. In particular, we have:

PROPOSITION 3.1. *A standard integral lattice of simplex type with minimal norm 2 is isomorphic to some* A_p.

If $n-m\geqq 2$ then let us define a code C over the integers mod $n-m$, consisting of those p-tuples $(\beta_1,\cdots,\beta_p)$ mod $n-m$ such that

$$x=\frac{1}{n-m}\sum\beta_iz_i,\qquad \beta_i\text{ integral}\tag{8}$$

is in E. Since E is standard and $z_1,\cdots,z_p\in E$, this code describes E completely.

PROPOSITION 3.2. *C is a linear code. Moreover, if $(m,n)=1$ then C is self-orthogonal, and orthogonal to* $(1,\cdots,1)$.

Proof. If $\beta=(\beta_1,\cdots,\beta_p)$, $\beta'=(\beta_1',\cdots,\beta_p')\in C$, and if x (as in (8)) and $x'=(n-m)^{-1}\sum\beta_i'z_i$ are corresponding vectors of E then $\beta+\beta'=(\beta_1+\beta_1',\cdots,\beta_p+\beta_p')$ is the codeword belonging to $x+x'$. Hence C is linear. Since $z_i\in E$, the inner product

$$(x,z_i)=\frac{1}{n-m}\sum\beta_l(z_l,z_i)=\frac{m}{n-m}\sum\beta_l+\beta_i$$

is integral, say, $=\alpha+\beta_i$, and we have

$$\sum\beta_l=\frac{n-m}{m}\alpha,\qquad \alpha\text{ integral.}\tag{9}$$

If $(m,n)=1$, (9) implies $\sum\beta_l\equiv 0 \bmod n-m$, whence C is orthogonal to $(1,\cdots,1)$. Finally, if x and x' are as above then their inner product is given by

$$(x,x')=\frac{m}{(n-m)^2}(\textstyle\sum\beta_l)(\sum\beta_l')+\frac{1}{n-m}\sum\beta_l\beta_l'.\tag{10}$$

Now (x,x') is integral, and for $(m,n)=1$, the first term on the right-hand side of (10) is integral. Hence the second term is also integral, i.e., $\sum\beta_l\beta_l'\equiv 0 \bmod n-m$, and C is self-orthogonal. □

We are now ready to describe all standard integral lattices of simplex type with minimal norm 3.

THEOREM 3.3. *The standard integral lattices of simplex type with minimal norm 3 are just the lattices E consisting of all vectors $\frac{1}{2}(\beta_1z_1+\cdots+\beta_pz_p)$ with integers β_i such that $(\beta_i,\cdots,\beta_p)$ mod 2 is in a given binary even self-orthogonal code C of minimum weight at least 6. Such a lattice is generated by its norm 3 vectors if and only if C is generated by its words of weight 6.*

Proof. Since $n \geqq 2m > 0$ and m is integral we have $n = 3$, $m = 1$. Since E is standard, Proposition 3.2 implies that E consists of all vectors $x = \frac{1}{2}(\beta_1 z_1 + \cdots + \beta_p z_p)$ with $\beta = (\beta_1, \cdots, \beta_p) \bmod 2 \in C$, where C is binary ($n - m = 2$), even (here = orthogonal to $(1, \cdots, 1)$), and self-orthogonal. Clearly any such lattice L contains $z_1, \cdots, z_p$, and it is easy to see from (10) that L is an integral lattice. So the only question is whether E contains vectors of norm smaller than $n = 3$. Now x has norm $\mu = \frac{1}{4}(\sum \beta_i)^2 + \frac{1}{2}(\sum \beta_i^2)$. A straightforward calculation shows that $\mu \leqq 2$ if and only if β is of type $\pm(1^2, 0^{p-2})$, $(1, -1, 0^{p-2})$ or $(1^2, -1^2, 0^{p-4})$. Hence E has minimal norm 3 (and is of simplex type) if and only if C has minimal weight $\geqq 6$. Finally, $\mu = 3$ if and only if β is of type $(2, 0^{p-1})$ or $(1^3, -1^3, 0^{p-6})$; two other possibilities $\pm(1^3, -1, 0^{p-4})$ and $\pm(1^2, -2, 0^{p-3})$ cannot occur if C has minimal weight $\geqq 6$. Hence E is generated by norm 3 vectors if and only if C is generated by words of weight 6. □

Example 17. Let $p = 6$, $C = \{(0^6), (1^6)\}$. This gives us again the lattice of Example 1.

Example 18. Replace in the Pasch configuration shown in Fig. 5 each point by two, and let C be the code of length 12 generated by the characteristic vectors of the resulting 4 sets of size 6. The corresponding lattice is of simplex type, has dimension 12, and contains 104 norm 3 vectors, namely

$$2 \times 12 \text{ from } \beta_i = \pm(2, 0^{11}),$$

$$20 \times 4 \text{ from } \beta_i = (1^3, -1^3, 0^6).$$

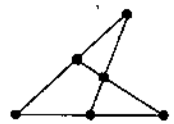

FIG. 5

Example 19. In the same way, the Fano plane in Fig. 6 gives a 14-dimensional lattice of simplex type with $2 \times 14 + 20 \times 7 = 168$ minimal norm vectors.

Example 20. Let $\mathscr{C}$ be the code generated by the 16 characteristic vectors of sets of the shape shown in the 4×4-grid in Fig. 7 (this is a well-known biplane). The corresponding 16-dimensional lattice is of simplex type and there are $2 \times 16 + 20 \times 16 = 352$ minimal norm vectors.

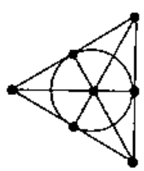

FIG. 6

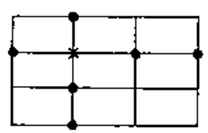

FIG. 7

Now let us consider exceptional integral lattices of simplex type.

THEOREM 3.4. *For given minimal norm n, there are only finitely many exceptional integral lattices of simplex type generated by norm n vectors.*

Proof. Suppose that E is exceptional, and generated by norm n vectors. Then there is a norm n vector $x \in E$ such that $(n - m)x \notin \langle z_1, \cdots, z_p \rangle$. Put

$$x = \frac{1}{n-m} \sum \alpha_i z_i. \tag{11}$$

Then

$$(x, z_i) = \frac{m}{n-m} \sum \alpha_i + \alpha_i \in \mathbb{Z}, \tag{12}$$

$$(x, x) = \frac{m}{(n-m)^2} (\sum \alpha_i)^2 + \frac{1}{n-m} (\sum \alpha_i^2) = n. \tag{13}$$

Now not all α_i are integral, hence (12) implies that

$$\frac{m}{n-m} \sum \alpha_i = \alpha + \varepsilon, \qquad \alpha \text{ integral}, \quad 0 < \varepsilon < 1. \tag{14}$$

From this, we get

$$\alpha_i = \beta_i - \varepsilon, \qquad \beta_i \text{ integral}. \tag{15}$$

Substituting (15) into (14) and (13) gives, after some simplification,

$$\sum \beta_i = \frac{1}{m} \{\alpha(n-m) + \varepsilon(n + m(p-1))\}, \tag{16}$$

$$\sum \beta_i(\beta_i - 1) = n(n-m) - \frac{1}{m} \{\alpha(\alpha+1)(n-m) + \varepsilon(1-\varepsilon)(n + m(p-1))\}. \tag{17}$$

Now the left-hand side of (17) is a nonnegative integer, and $\alpha(\alpha+1)(n-m) \geq 0$. Hence we find

$$\varepsilon(1-\varepsilon)(n + m(p-1)) \text{ is an integer} \leqq mn(n-m) < n^3. \tag{$*$}$$

Also, (16) implies that ε is rational, hence $\varepsilon = u/v$ with coprime integers u, v and $0 < u < v$ since $0 < \varepsilon < 1$. Now $(*)$ implies that $n + m(p-1) = v^2 w$ with an integer $w > 0$, and the resulting inequality is $u(v-u)w < n^3$. Since u, $v - u$ and w are positive integers, this leaves only finitely many choices for u, v and w. Also m is bounded by $0 < m \leqq n/2$. Hence p takes only finitely many values. So the theorem is proved if we show that for given m, n, p there are only finitely many integral lattices of simplex type with these parameters. But, in fact, m, n, p determine a unique trivial lattice of simplex type $T_p^{n,m}$, and *any* integral lattice has only finitely many integral refinements. Since every lattice of simplex type is a refinement of its trivial part, the proof is completed. □

Remark. If $d = (n-m)/m$ is integral then (16), (17) can be written as

$$\sum \beta_i - d\alpha = \varepsilon(p + d), \tag{16'}$$

$$\sum \binom{\beta_i}{2} + d\binom{\alpha+1}{2} = \tfrac{1}{2}(m^2 d(d+1) - \varepsilon(1-\varepsilon)(p+d)). \tag{17'}$$

Since the left-hand side is a nonnegative integer and $d(d+1)$ is even, we have

$$\varepsilon(1-\varepsilon)(p+d) \text{ is an even integer} \leqq m^2 d(d+1). \tag{$**$}$$

Proceeding as before, we find that $\varepsilon = u/v$, $p + d = v^2 w$ where u, v, w are positive integers such that $u < v$ and $u(v-u)w$ is an even integer $\leqq m^2 d(d+1)$. We shall use equations (16′) and (17′) to determine the exceptional integral lattices of simplex type with $n = 2$.

THEOREM 3.5. *The only exceptional integral lattices of simplex type with minimal norm 2 are the root lattices $E_7 = \langle z_1, \cdots, z_7, \frac{1}{2}(z_1 + \cdots + z_7) \rangle$ in $\mathbb{R}^7$ and $E_8 =$*

$\langle z_1, \cdots, z_8, \frac{1}{3}(z_1+\cdots+z_8)\rangle$ *in* $\mathbb{R}^8$; *here*

$$(z_i, z_j) = \begin{cases} 2 & \text{if } i = j, \\ 1 & \text{if } i \neq j. \end{cases}$$

Proof. We have $n = 2$, $m = 1$, hence $d = 1$. The numbers u, v, w of the remark above must satisfy $u(v-u)w = 2$, leaving $(u, v, w) = (1, 2, 2)$, $(1, 3, 1)$ or $(2, 3, 1)$. Hence either $p = 7$, $\varepsilon = \frac{1}{2}$ or $p = 8$, $\varepsilon \in \{\frac{1}{3}, \frac{2}{3}\}$. To find the exceptional vectors it is sufficient to solve (16'), (17') for $\alpha \geqq 0$ (otherwise replace x by $-x$). For $p = 7$, (16') and (17') hold if and only if $\alpha = 0$, $\beta = (1^4, 0^3)$, or equivalently if and only if $\alpha_i = \frac{1}{2}(1^4, (-1)^3)$; hence the shape of exceptional norm 2 vectors is $\pm\frac{1}{2}(z_1 + z_2 + z_3 + z_4 - z_5 - z_6 - z_7)$. Each such vector with $z_1, \cdots, z_7$ generates E_7; therefore $E_7 \subseteq E$. Similarly, for $p = 8$ we find $\alpha = 0$, $\beta = (1^3, 0^5)$ or $(1^6, 0^2)$, whence $\alpha_i = \frac{1}{3}(2^3, (-1)^5)$ or $\frac{1}{3}(1^6, (-2)^2)$. Therefore the shape of an exceptional vector is $\pm\frac{1}{3}(2z_1 + 2z_2 + 2z_3 - z_4 - z_5 - z_6 - z_7 - z_8)$ or $\pm\frac{1}{3}(z_1 + z_2 + z_3 + z_4 + z_5 + z_6 - 2z_7 - 2z_8)$, and $E_8 \subseteq E$. Now any vector of integral norm in $\mathbb{R}^7$ (resp. $\mathbb{R}^8$) which has integral inner product with all vectors of E_7 (resp. E_8) is itself in E_7 (resp. E_8); this can be shown in a similar way as we found the norm 2 vectors. Hence $E = E_7$ or $E = E_8$. $\square$

Proposition 3.1 and Theorem 3.5 are related to the following theorem which is implicitly in Cameron, et al. [2].

THEOREM 3.6 (Cameron, Goethals, Seidel, Shult). *An integral lattice generated by norm 2 vectors is isomorphic to one of the root lattices* $A_p = \{x \in \mathbb{Z}^{p+1} | \sum x_i = 0\}$, $D_p = \{x \in \mathbb{Z}^p | \sum x_i \text{ even}\}$, $E_8 = A_8 + \frac{1}{3}\mathbb{Z}(1^6, (-2)^3)$, $E_7 = A_7 + \frac{1}{2}\mathbb{Z}(1^4, -1^4)$, *or* $E_6 = E_7 \cap (0^6, 1^2)^\perp$.

Proof. Any two norm 2 vectors x, y ($y \neq \pm x$) have inner product $\in \{0, \pm 1\}$, whence the corresponding set S of lines has angles 60° or 90°. Since $(x, y) = -1$ implies that $z = -x - y$ also has norm 2, S is star-closed in the sense of Cameron et al. [2]. Hence by [2, Thm. 3.5], the norm 2 vectors generate one of the lattices mentioned. $\square$

For minimal norm 3, we give only a partial result:

THEOREM 3.7. *An exceptional integral lattice of simplex type with minimal norm 3, generated by norm 3 vectors, can exist only in dimensions* 6, 7, 14, 16, 22, 23, 25, 30 *or* 47.

Proof. We have $n = 3$, $m = 1$, $d = 2$, and (**) requires that $\varepsilon(1-\varepsilon)(p+2)$ be an even integer $\leqq 6$. With ε rational, $0 < \varepsilon < 1$, this leaves the values in Table 8 for p and ε:

TABLE 8

p	6	7	14	16	22	23	25	30	47
ε	$\frac{1}{2}$	$\frac{1}{3}, \frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{3}, \frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$	$\frac{1}{3}, \frac{2}{3}$	$\frac{1}{4}, \frac{3}{4}$	$\frac{1}{7}, \frac{6}{7}$

Remark. Let E be an arbitrary integral lattice of minimal norm 3. If $x, y \in E$ have norm 3, $y \neq \pm x$ then $(x, y)^2 < (x, x)(y, y) = 9$, whence $-2 \leqq (x, y) \leqq 2$. But if $(x, y) = \pm 2$ then $(x \mp y, x \mp y) = 3 - 2 - 2 + 3 = 2$, a contradiction. Hence $(x, y) \in \{0, \pm 1\}$, and the set of norm 3 vectors of E form a set Σ of vectors of type $\{0, \frac{1}{3}\}$ in the sense of Shult and Yanushka [15]. Moreover, if x, y, z are norm 3 vectors with $(x, y) = (x, z) = (y, z) = -1$ then $w = -x - y - z$ has also norm 3, and $(x, w) = (y, w) = (z, w) = -1$. Hence Σ is tetrahedrally-closed.

This explains the occurrence of some of Shult and Yanushka's line systems in the present context. In fact, at least four of their examples occur: C_{16} ($p = 6$; Example 1), C_{28} (case $p = 7$ of Example 11; projection to $w^\perp$), S_{256} ($p = 16$; Example 3) and a system of 2300 lines (case $p = 23$ of Example 15; projection to $w^\perp$). In fact, the last three examples come from exceptional lattices.

Acknowledgments. I am indebted to Prof. J. Seidel who interested me in the relations between combinatorial configurations and hyperbolic spaces. I recall with pleasure the discussions I have had with him on this subject. Further thanks go to John Leech and the referee for useful comments.

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