Sonderdruck aus
Arch. Math., Vol. 43, 137–141 (1984)

0003-889 X/84/4302-0137 $ 2.50/0

The extremal case of some matrix inequalities

By

Arnold Neumaier

Introduction. The paper discusses the problem of characterizing the matrices satisfying one of the equalities $|AB| = |A||B|$, $|(I-A)^{-1}| = (I-|A|)^{-1}$, or $|A^{-1}| = \langle A \rangle^{-1}$, where $|\cdot|$ denotes componentwise absolute value, and $\langle A \rangle$ is Ostrowski's comparison matrix for A. Some open problems remain.

1. The equation $|AB| = |A||B|$. Denote by $\mathbb{R}^{n\times m}$ and $\mathbb{C}^{n\times m}$ the set of real and complex $n \times m$-matrices respectively, and denote the (i, k)-entry of $A \in \mathbb{C}^{n\times m}$ by A_{ik}. The identity matrix (of any size) is written as I. We define the relations $\varrho \in \{<, \leqq, >, \geqq\}$ on $\mathbb{R}^{n\times m}$ by the convention

$$A \,\varrho\, B :\Leftrightarrow A_{ik} \,\varrho\, B_{ik} \quad \text{for all} \quad i, k.$$

We consider the *absolute value* $|\cdot|: \mathbb{C}^{n\times m} \to \mathbb{R}^{n\times m}$ defined by

$$|A|_{ik} := |A_{ik}| \quad \text{for all} \quad i, k,$$

and Ostrowski's [3] *comparison operator* $\langle\cdot\rangle: \mathbb{C}^{n\times n} \to \mathbb{R}^{n\times n}$ defined by

$$\langle A \rangle_{ii} := |A|_{ii}, \quad \langle A \rangle_{ik} := -|A_{ik}| \quad \text{for} \quad i \neq k.$$

We do not use the more familiar notation $\mathfrak{M}(A)$ for the comparison matrix of A in order to emphasise the close relationship to the absolute value apparent from the following easily established relations (cf. [2]).

$$|A \pm B| \leqq |A| + |B|, \qquad \langle A \pm B \rangle \geqq \langle A \rangle - |B|,$$
$$|AB| \leqq |A||B|, \qquad |AB| \geqq \langle A \rangle |B|.$$

In this section we consider the case of equality in one of these inequailities: the equation $|AB| = |A||B|$. As a tool we use the complex sign function $\operatorname{sgn}: \mathbb{C} \to \mathbb{C}$ defined by

$$\operatorname{sgn} 0 = 0, \qquad \operatorname{sgn} x = x/|x| \quad \text{if} \quad x \neq 0,$$

and the following obvious observation.

Lemma. *If $|\sum x_i| = \sum |x_i|$ then there is a nonzero $\gamma \in \mathbb{C}$ such that $\operatorname{sgn} x_i \in \{0, \gamma\}$ for all i.* □

Theorem 1. *Let $A \in \mathbb{C}^{p\times n}$, $B \in \mathbb{C}^{n\times m}$ satisfy the equation*

$$(1) \qquad |AB| = |A||B|.$$

Then one of the following holds:

(i) *Every row of A contains some zero entry.*

(ii) *Every column of B contains some zero entry.*

(iii) *There are nonsingular diagonal matrices D, D', D'' such that $D'AD$ and $D^{-1}BD''$ are real and nonnegative.*

Proof. The assumption implies that

$$\left|\sum_{j=1}^{n} A_{ij} B_{jk}\right| = |(AB)_{ik}| = |AB|_{ik} = (|A||B|)_{ik} = \sum_{j=1}^{n} |A_{ij}||B_{jk}|, \tag{2}$$

for $i = 1, \ldots, p$ and $k = 1, \ldots, m$. With

$$\alpha_{ij} := \operatorname{sgn} A_{ij}, \quad \beta_{jk} := \operatorname{sgn} B_{jk}, \tag{3}$$

(2) and the lemma imply the existence of nonzero numbers $\gamma_{ik} \in \mathbb{C}$ with

$$\alpha_{ij}\beta_{jk} \in \{0, \gamma_{ik}\} \quad \text{for all} \quad i, j, k. \tag{4}$$

Now suppose that neither (i) nor (ii) holds. Then there are indices r, s such that $|A_{rj}||B_{js}| \neq 0$ for $j = 1, \ldots, n$, and by (4),

$$\alpha_{rj}\beta_{js} = \gamma_{rs} \neq 0. \tag{5}$$

Therefore, the diagonal matrices

$$D' := \operatorname{Diag}\left(\frac{1}{\gamma_{is}}\right), \quad D := \operatorname{Diag}(\beta_{js}), \quad D'' := \operatorname{Diag}\left(\frac{\gamma_{rs}}{\gamma_{rk}}\right)$$

are nonsingular, and by (3) and (5) we have

$$(D'AD)_{ij} = \frac{A_{ij}\beta_{js}}{\gamma_{is}} = |A_{ij}|\frac{\alpha_{ij}\beta_{js}}{\gamma_{is}} \in \{0, |A_{ij}|\},$$

$$(D^{-1}BD'')_{jk} = \frac{B_{jk}\gamma_{rs}}{\beta_{js}\gamma_{rk}} = \frac{\alpha_{rj}B_{jk}}{\gamma_{rk}} = \frac{\alpha_{rj}\beta_{jk}}{\gamma_{rk}}|B_{jk}| \in \{0, |B_{jk}|\}.$$

Hence (iii) holds. □

Conversely, it is obvious that $|AB| = |A||B|$ for all pairs (A, B) satisfying (iii). On the other hand, the characterization of those pairs (A, B) with $|AB| = |A||B|$ satisfying (i) or (ii) depends on the zero structure of A and B and seems to be a nontrivial combinatorial problem. Simple examples which may occur here are the matrix pairs of type

$$A = (C \;\; 0), \quad B = \begin{pmatrix} 0 \\ D \end{pmatrix}, \quad \text{or} \quad A = (C \;\; I), \quad B = \begin{pmatrix} I \\ C \end{pmatrix},$$

with arbitrary C, D of the proper size. Moreover, with (A, B), the pairs $(M'AM, M^{-1}BM'')$ occur, where M, M', M'' are nonsingular monomial matrices, i.e. have precisely one nonzero entry in each row and column. A particular case, used in the next section, is

$$|AB| = |A||B| \quad \text{if } A \text{ or } B \text{ is diagonal}. \tag{6}$$

2. The equation $|A^{-1}| = \langle A \rangle^{-1}$. A square matrix $A \in \mathbb{C}^{n\times n}$ is called an *H-matrix* if there is a positive vector $u \in \mathbb{R}^n$ such that $\langle A \rangle u > 0$, and it is called an *M-matrix* if, in addition, $A = \langle A \rangle$. For equivalent definitions, see e.g. Bermann and Plemmons [1]. The spectral radius of a square matrix $A \in \mathbb{C}^{n\times n}$ is denoted by $\sigma(A)$. In this section we consider equality in Ostrowski's [3] relation $|A^{-1}| \leqq \langle A \rangle^{-1}$ for H-matrices, and prove as a preparation the following result of independent interest.

Theorem 2. *Let $A \in \mathbb{C}^{n\times n}$ be a square matrix with $\sigma(|A|) < 1$. Then*

$$|(I - A)^{-1}| \leqq (I - |A|)^{-1}.$$

Moreover, if A is irreducible then the following assertions are equivalent:

(i) $|(I - A)^{-1}| = (I - |A|)^{-1}$,

(ii) $|(I - A)^{-1}|$ *and* $(I - |A|)^{-1}$ *have a common entry.*

(iii) *A is a diagonally similar to a real nonnegative matrix, i.e. a nonsingular diagonal matrix D exists such that $D^{-1}AD$ is real and nonnegative.*

Proof. Since $\sigma(A) \leqq \sigma(|A|) < 1$, the Neumann series

$$(I - A)^{-1} = I + A + A^2 + \ldots + A^l + \ldots,$$
$$(I - |A|)^{-1} = I + |A| + |A|^2 + \ldots + |A|^l + \ldots,$$

converge, and by the triangle inequality and repeated application of $|AB| \leqq |A|\,|B|$ we get (7). – To prove the equivalence of (i), (ii), and (iii), we remark first that (i) $\Rightarrow$ (ii) obviously.

Now suppose that (ii) holds, where the common entry is in place (i, k). The (i, k) entry of C^l for a matrix $C \in \mathbb{C}^{n\times n}$ is $\sum c_{ii_1} c_{i_1 i_2} \ldots c_{i_{l-1}k}$, summed over all combinations of indices $i_1, \ldots, i_{l-1}$. Hence the (i, k)-entry of $(I - A)^{-1}$ and $(I - |A|)^{-1}$ is

$$1 + \sum A_{ii_1} A_{i_1 i_2} \ldots A_{i_l k} \quad \text{and} \quad 1 + \sum |A_{ii_1} \ldots A_{i_l k}|,$$

respectively, so that by the lemma, $A_{ii_1} A_{i_1 i_2} \ldots A_{i_l k} \geqq 0$ for all index sequences $i, \ldots, i_l$ $(l \geqq 0)$. Now let $i_0, \ldots, i_s$ $(s \geqq 0)$ be an arbitrary index sequence. Since A is irreducible, there are sequences $j_1, \ldots, j_p$ and $k_1, \ldots, k_q$ such that

$$A_{ij_1} \ldots A_{j_p i_0} \neq 0, \quad A_{i_0 k_1} \ldots A_{k_q k} \neq 0.$$

Therefore

$$A_{ij_1} \ldots A_{j_p i_0} A_{i_0 k_1} \ldots A_{k_q k} > 0,$$
$$A_{ij_1} \ldots A_{j_p i_0} A_{i_0 i_1} \ldots A_{i_s i_0} A_{i_0 k_1} \ldots A_{k_q k} \geqq 0,$$

and therefore, the quotient of these expressions satisfies

$$A_{i_0 i_1} A_{i_1 i_2} \ldots A_{i_s i_0} \geqq 0 \tag{8}$$

for all index sequences $i_0, i, \ldots, i_s$ $(s \geqq 0)$. Let S be a maximal set of indices such that there exist nonzero numbers d_i $(i \in S)$ such that

$$d_i^{-1} A_{ii_1} A_{i_1 i_2} \ldots A_{i_s k} d_k \geqq 0$$

for all index sequences $i_i, \ldots, i_s$ $(s \geqq 0)$ and all $i, k \in S$. By (8), S is nonempty. If $S \neq \{1, \ldots, n\}$ then, since A is irreducible, one of the following two cases occurs.

Case 1. There is $i \notin S$ such that $A_{li} \neq 0$ for some $l \in S$. Fix one such $l \in S$ and put $d_i := d_l / A_{li}$. Then for $k \in S$, we have

$$d_i^{-1} A_{ii_1} \dots A_{i_s k} d_k = d_l^{-1} A_{li} A_{ii_1} \dots A_{i_s k} d_k \geqq 0.$$

Moreover, since A is irreducible, there are indices $l_1, \dots, l_s$ such that $A_{il_1}, \dots, A_{l_s l} \neq 0$ so that by (8),

$$d_k^{-1} A_{ki_1} \dots A_{i_s i} d_i = \frac{d_k^{-1} A_{ki_1} \dots A_{i_s i} A_{il_1} \dots A_{l_s l} d_l}{A_{li} A_{il_1} \dots A_{l_s l}} \geqq 0.$$

Together with (8) for $i_0 = i$, this contradicts the maximality of S.

Case 2. There is $i \notin S$ such that $A_{il} \neq 0$ for some $l \in S$. This leads to a contradiction by the dual argument.

Therefore, $S = \{1, \dots, n\}$, and with $D := \mathrm{Diag}(d_i)$ we have $(D^{-1} AD)_{ik} = d_i^{-1} A_{ik} d_k \geqq 0$ for all i, k so that (iii) holds. Finally, if (iii) holds then the nonnegative matrix $B := D^{-1} AD$ is similar to A, hence $\sigma(B) < 1$, $(I - B)^{-1} \geqq 0$, and $|(I - A)^{-1}| = |D(I - B)^{-1} D^{-1}| = |D| |(I - B)^{-1}| |D|^{-1} = |D| (I - B)^{-1} |D|^{-1} = (I - |D| B |D|^{-1})^{-1} = (I - |A|)^{-1}$ by (6). Therefore (i) holds. □

Corollary. *If* $A \in \mathbb{C}^{n \times n}$ *is an irreducible matrix with* $\sigma(|A|) < 1$ *not satisfying* (iii) *then*

(7a) $$|(I - A)^{-1}| < (I - |A|)^{-1}. \quad \square$$

If A is reducible then the three conditions are no longer equivalent; we only have the implications (iii) ⇒ (i) ⇒ (ii). Counterexamples to (i) ⇒ (iii) are the matrices of shape $A = \left(\begin{smallmatrix} 0 & c \\ 0 & 0 \end{smallmatrix}\right)$ which have

$$A^2 = |A|^2 = 0, \quad |(I - A)^{-1}| = |I + A| = I + |A| = (I - |A|)^{-1},$$

and counterexamples to (ii) ⇒ (i) are the strictly upper triangular matrices A, where $|(I - A)^{-1}|$ and $(I - |A|)^{-1}$ agree on and below the diagonal. The determination of all reducible A satisfying (i) or (ii) again seems to be a nontrivial problem.

Theorem 3. *Let A be an H-matrix. Then A and $\langle A \rangle$ are nonsingular, and*

(9) $$|A^{-1}| \leqq \langle A \rangle^{-1}.$$

Moreover, if A is irreducible then one of the following holds:

(i) (9) *holds with strict* (componentwise) *inequality:* $|A^{-1}| < \langle A \rangle^{-1}$.
(ii) (9) *holds with equality:* $|A^{-1}| = \langle A \rangle^{-1}$, *and A is diagonally equivalent to an M-matrix, i.e. nonsingular diagonal matrices D and D' exist such that $D'AD$ is an M-matrix.*

Proof. (9) is due to Ostrowski [3], but we reprove it in the process of obtaining the second part of the theorem. Write $A = M - N$ where $M = \mathrm{Diag}(A_{ii})$. Then $\langle A \rangle = |M| - |N|$. Since A is an H-matrix there is a real vector $u > 0$ such that $\langle A \rangle u > 0$. Now $\sigma(M^{-1} N) \leqq \sigma(|M^{-1} N|) = \sigma(|M|^{-1} |N|) < 1$ by (6) and since

$|M|^{-1}|N|u = u - |M|^{-1}\langle A\rangle u < u$. Hence $A = M - N$ and $\langle A\rangle = |M| - |N|$ are nonsingular, and by (6) and Theorem 2, we have

$$\begin{aligned}|A^{-1}| &= |(M-N)^{-1}| = |(I - M^{-1}N)^{-1}M^{-1}| = |(I - M^{-1}N)^{-1}||M|^{-1}\\ &\leqq (I - |M^{-1}N|)^{-1}|M|^{-1} = (I - |M|^{-1}|N|)^{-1}|M|^{-1}\\ &= (|M| - |N|)^{-1} = \langle A\rangle^{-1},\end{aligned}$$

so that (9) holds. Moreover, if $|A^{-1}|$ and $\langle A\rangle^{-1}$ have a common entry, so do $|(I - M^{-1}N)^{-1}|$ and $(I - |M^{-1}N|)^{-1}$. Now if A is irreducible then $M^{-1}N$ is irreducible, and by Theorem 2 there is a nonsingular diagonal matrix D such that $C := D^{-1}M^{-1}ND$ is real and nonnegative. The vector $v := |D|^{-1}u$ is positive and satisfies

$$Cv = |C|v = |D^{-1}M^{-1}ND||D|^{-1}u = |D|^{-1}|M|^{-1}|N|u < |D|^{-1}u = v.$$

Hence the matrix $B := I - C$ satisfies $\langle B\rangle = B$, $\langle B\rangle v = v - Cv > 0$ and is therefore an M-matrix. But $B = I - C = D^{-1}M^{-1}(M - N)D = D'AD$ with the nonsingular diagonal matrix $D' = D^{-1}M^{-1}$.

Conversely if D, D' are nonsingular diagonal matrices such that $B := D'AD$ is an M-matrix then $B^{-1} \geqq 0$ so that $|B^{-1}| = B^{-1} = \langle B\rangle^{-1}$ and

$$\begin{aligned}|A^{-1}| &= |DB^{-1}D'| = |D||B^{-1}||D'| = |D|\langle B\rangle^{-1}|D'|\\ &= (|D'|^{-1}\langle B\rangle|D|^{-1})^{-1} = \langle A\rangle^{-1}. \quad \square\end{aligned}$$

References

[1] A. Berman and R. J. Plemmons, Nonnegative Matrices in the Mathematical Sciences. New York 1979.

[2] A. Neumaier, New techniques for the analysis of linear interval equations. Linear Alg. Appl., to appear.

[3] A. M. Ostrowski, Über die Determinanten mit überwiegender Hauptdiagonale. Comment. Math. Helv. **10**, 69–96 (1937).

Eingegangen am 24. 10. 1983

Anschrift des Autors:

Arnold Neumaier
Institut für Angewandte Mathematik
Universität Freiburg
Hermann-Herder-Str. 10
D-7800 Freiburg