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INTERVAL SLOPES FOR RATIONAL FUNCTIONS AND ASSOCIATED CENTERED FORMS*

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Abstract. For an arithmetic expression $f(x)$ involving N rational operations, an $O(N)$ algorithm is given which computes an interval enclosure for the set of slopes $f[x, z]$ where x ranges over an interval X. Applications to real and complex centered forms are given, resulting in improvements over previous results by Ratschek [SIAM J. Numer. Anal., 17 (1980), pp. 656–662] and Petković [Freiburger Intervall-Berichte, 83(2) (1983), pp. 33–50].

1. Introduction. A standard tool in the interval analysis of nonlinear functions is the mean value theorem, stated quantitatively in the form

$$f(x)-f(z)\in F'(X)\cdot(x-z)\quad \text{for all } x, z\in X, \tag{1.1}$$

where X is an interval and $F'(X)$ is an interval extension of the derivative of f. This formula has important applications to several problems:

(i) the enclosure of the range of f over X,
(ii) the enclosure or iterative construction of zeros of f in X,
(iii) global optimization.

In case (i) we find

$$\bar f(X):=\{f(x)\mid x\in X\}\subseteq f(z)+F'(X)\cdot(X-z)\quad \text{if } z\in X. \tag{1.2}$$

For narrow intervals X, and natural conditions on $F'(X)$, this formula overestimates the range $\bar f(X)$ only by a term proportional to $(\mathrm{rad}\,X)^2$, cf. Caprani and Madsen [3], and since $\bar f(X)$ usually has a radius of order rad X, (1.2) is very accurate.

In case (ii) the error $\delta:=z-x^*$ of an approximation $z\in X$ to a zero $x^*\in X$ of f is by (1.1) contained in the solution of the linear interval "equation"

$$F'(X)\delta=f(z), \tag{1.3}$$

cf. Moore [10], Alefeld and Herzberger [2], Krawczyk [7]. Finally, in case (iii), knowledge of a good enclosure of the range of f over small interval regions allows us to avoid convergence of minimization algorithms to a nonglobal minimum, cf. Hansen [6].

It has been observed repeatedly (see e.g. Moore [10], Hansen [4], Ratschek [13], Krawczyk and Nickel [9]) that (1.1) can be replaced by various more specific relations which yield smaller intervals in place of $F'(X)$. Here we are concerned with the relation

$$f(x)-f(z)=f[x, z](x-z) \tag{1.4}$$

with resulting sharper bounds for the applications. For example, (1.2) and (1.3) become

$$\bar f(X)\subseteq f(z)+F[X, z](X-z)\quad \text{if } z\in X, \tag{1.2a}$$

$$F[X, z]\delta=f(z); \tag{1.3a}$$

moreover, under natural conditions, the so-called "centered form" (1.2a) still overestimates $\bar f(X)$ only by $O((\mathrm{rad}\,X)^2)$ (see Hansen [5], Krawczyk and Nickel [9]). While

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at least in the one-dimensional case, the "slope" $f[x, z]$ is, as a function of x and z, uniquely defined by (1.4) and continuity, there are many ways to express $f[x, z]$ explicitly. Now different expressions for $f[x, z]$ lead generally to different results when evaluated in interval arithmetic with X an interval. Moreover, different expressions may vary considerably in their evaluation cost. In particular, the centered forms of Ratschek [13], defined for rational functions given as quotients of polynomials, the evaluation cost is proportional to the square of the degree n of $f(x)$. Recently, Alefeld [1] has given an algorithm for polynomials with evaluation cost $O(n)$ only, which is of the same order as the cost for the evaluation of the derivative $F'(X)$.

The present paper proposes a new method for the calculation of the slopes of functions f defined by arbitrary rational expressions, whose cost is proportional to the number of operations involved in f. The method, which imitates analytic differentiation, is recursive in nature and reduces to Alefeld's method in case that the arithmetic expression is Horner's scheme for the evaluation of a polynomial. Compared with Ratschek's centered forms (and similar methods) the present method has two advantages: it has lower complexity, and it is more flexible since it does not require the sometimes costly transformation of a rational expression to the normal form as quotient of two polynomials.

The paper is organized as follows. We write real (complex) numbers and vectors with lower case letters, intervals and interval vectors with capital letters. Section 2 defines rational expressions, their interval extensions and slopes. Section 3 gives a new and elementary proof of the quadratic approximation property. The new method is compared with Ratschek's method and the mean value form in § 4; there are also some numerical examples. Finally, § 5 treats an extension of the method to the complex case and compares the results with a centered form studied by Petcović [12].

2. Rational expressions and associated slopes. We define (real) *rational expressions in the variables* $x_1, \cdots, x_n$ (or simply: an *expression*) as the elements of the smallest set $\mathfrak{R}$ satisfying the following properties

(i) $c\in\mathfrak{R}$ for all $c\in\mathbb{R}$,
(ii) $x_1, \cdots, x_n\in\mathfrak{R}$,
(iii) $g\in\mathfrak{R}\Rightarrow(-g)\in\mathfrak{R}$,
(iv) $g, h\in\mathfrak{R}\Rightarrow(g+h), (g-h), (g*h), (g/h)\in\mathfrak{R}$.

For simplicity of notation we adopt the ALGOL 60 priority rules for arithmetic expressions to save brackets. We say that an expression p is a *subexpression* of f if either $f=-g$ and p is g or a subexpression of g, or if $f=g\circ h$ with $\circ\in\{+,-,*,/\}$ and p is g, or h, or a subexpression of g or h. The *value* $f(x)$ of an expression f at the vector $x:=(x_1, \cdots, x_n)^T$ is obtained by interpreting the operations as operations between real numbers. $f(x)$ is defined for all $x\in\mathbb{R}^n$ for which no subexpression occurring in a denominator has the value zero; we then say that *f can be evaluated at x*. In a similar way, the value $F(X)$ of f at the interval vector $X:=(X_1, \cdots, X_n)^T$ is obtained by substituting X_i for x_i and interpreting the operations in interval arithmetic; $F(X)$ can be evaluated for all X for which the value of no subexpression occurring in a denominator contains zero. The resulting *interval extension* F of the function defined by the expression f has the following property:

If F can be evaluated at X, then it can be evaluated at $x\in X$ and all $Z\subseteq X$, and the relations

$$x\in X\Rightarrow f(x)\in F(X), \tag{2.1}$$

$$Z\subseteq X\Rightarrow F(Z)\subseteq F(X), \tag{2.2}$$

hold.

For example, the expression (in the single variable x)

$$f = -x*(x+1)/(x-1), \tag{2.3}$$

which in full form reads

$$f = ((-x)*((x+1)/(x-1))),$$

has the subexpressions

$$1, x, -x, x+1, x-1, (x+1)/(x-1);$$

f can be evaluated for all real $x \neq 1$, and the corresponding interval extension F can be evaluated for all intervals $X \not\ni 1$.

Suppose now that f is a rational expression in the single variable x, and that the corresponding function is defined for $x \in D$. Then the *slope* (divided difference) of f at $x, z \in D$, defined as

$$f[x,z] := \begin{cases} (f(x)-f(z))/(x-z) & \text{if } x \neq z, \\ f'(x) & \text{if } x = z, \end{cases}$$

is the unique continuous solution of the equation

$$f(x) - f(z) = f[x,z](x-z) \quad \text{for } x, z \in D. \tag{2.4}$$

The following theorem shows that the slope can be built up recursively from the slopes of the subexpressions of f.

THEOREM 1. *Suppose that f, g, h are rational expressions in a single variable x, which can be evaluated at some $x, z \in \mathbb{R}$. Then*

$$f = c \in \mathbb{R} \Rightarrow f[x,z] = 0, \tag{2.5}$$

$$f = x \quad \Rightarrow f[x,z] = 1, \tag{2.6}$$

$$f = -g \quad \Rightarrow f[x,z] = -g[x,z], \tag{2.7}$$

$$f = g \pm h \Rightarrow f[x,z] = g[x,z] \pm h[x,z], \tag{2.8}$$

$$f = g*h \quad \Rightarrow f[x,z] = g[x,z]*h(x) + g(z)*h[x,z], \tag{2.9}$$

$$f = g/h \quad \Rightarrow f[x,z] = (g[x,z] - h[x,z]*f(z))/h(x). \tag{2.10}$$

Proof. This is obvious for (2.5) to (2.8). If $f = g*h$, then

$$\begin{aligned} f(x) - f(z) &= g(x)h(x) - g(z)h(z) \\ &= (g(x)-g(z))h(x) + g(z)(h(x)-h(z)) \\ &= g[x,z](x-z)h(x) + g(z)h[x,z](x-z) = f[x,z](x-z), \end{aligned}$$

with $f[x,z]$ given by (2.9). If $f = g/h$, then

$$\begin{aligned} f(x) - f(z) &= (g(x) - h(x)f(z))/h(x) \\ &= (g(x) - g(z) - (h(x)-h(z))f(z))/h(x) \\ &= (g[x,z](x-z) - h[x,z](x-z)f(z))/h(x) = f[x,z](x-z), \end{aligned}$$

with $f[x,z]$ given by (2.10). □

It is obvious from Theorem 1 that the slope of a rational expression in x is given by a rational expression in x and z. Therefore, these formulae can be used to obtain an interval extension $F[X,Z]$ for the slope, an *interval slope*. It is easy to see that $F[X,Z]$ can be evaluated at intervals X, Z iff F can be evaluated at X and Z.

We remark that for $x = z$ the formulae (2.5)-(2.10) simply reduce to analytic differentiation formulae for rational expressions, namely (cf. Rall [14])

$$\begin{aligned} & f = c \in \mathbb{R} \Rightarrow f'(x) = 0, \\ & f = x \quad \Rightarrow f'(x) = 1, \\ & f = -g \quad \Rightarrow f'(x) = -g'(x), \\ & f = g \pm h \Rightarrow f'(x) = g'(x) \pm h'(x), \\ & f = g*h \quad \Rightarrow f' = g'(x)*h(x) + g(x)*h'(x), \\ & f = g/h \quad \Rightarrow f'(x) = (g'(x) - h'(x)*f(x))/h(x). \end{aligned} \tag{2.11}$$

Hence the particular interval extension $F'(X)$ for f' derived from (2.11) is related to the interval slope $F[X,Z]$ by the relation

$$F'(X) = F[X,X], \tag{2.12}$$

so that by inclusion isotonicity,

$$F[X,z] \subseteq F'(X) \quad \text{for } z \in X. \tag{2.13}$$

Therefore, for the applications mentioned in the introduction, the interval slopes defined by (2.5)-(2.10) represent a genuine improvement over interval extensions of the derivative, if the latter is computed by (2.11).

We now consider the maximal number of operations required to compute $F(X)$, its derivative, its slope, and the two associated centered forms. The cost for forming auxiliary quantities, like $f(z)$ and $F(X)$ for the slope is also included since their subexpressions are used in the multiplication and division formulae. But here, and for operations with constants or multiplication or division by x, some optimization is possible which results in decreased cost (cf. Table 1 with the examples given in § 4); therefore here worst case results are given. The maximal number of operations needed for the computation of the expression listed in the first column of the table below is then $\alpha + \alpha^+ N^+ + \alpha^* N^*$, where N^+ denotes the number of additions and subtractions, and N^* denotes the number of multiplications and divisions involved in the formation of the rational expression f. As an example, we explain $\alpha^+ = 3$ for $F[X,z]$: Counted is one addition each for $f(z) = g(z) \pm h(z)$, $F(X) = G(X) \pm H(X)$, $F[X,z] = G[X,z] \pm H[X,z]$ (the first two values are needed only if f occurs as a subexpression of a product or quotient).

TABLE 1

	α	α^+	α^*
$F(X)$	0	1	1
$F'(X)$	0	2	4
$F[X,z]$	0	3	5
$f(z)+F'(X)(X-z)$	3	3	5
$f(z)+F[X,z](X-z)$	3	3	5

One sees that the improved centered form can be evaluated at the same cost as the mean value form, and that the total number of operations is less than five times the number of operations in f. We note that a slightly different count is obtained if, in the multiplication and division formulae, $H(X)$ is replaced by the centered form

$h(z)+H'(X)(X-z)$, resp. $h(z)+H[X,z](X-z)$; this saves the recursive calculation and storage of $H(X)$, and implies the values $\alpha=0$, $\alpha^*=2$, $\alpha^*=6$ for the centered forms.

For rational expressions in several variables defined for $x \in D$, we call any continuous functions $f[\cdot,\cdot]: D \times D \to \mathbb{R}^n$ satisfying

$$f(x)-f(z)=f[x,z](x-z) \quad \text{for } x, z \in D \tag{2.4}$$

a *slope* for f; multiplication is now interpreted as formation of the inner product. Unlike in the one-dimensional case, the slope is no longer determined by (2.4). But the proof of Theorem 1 generalizes immediately to the new situation and shows that (2.7)–(2.10) *define* a slope for f if slopes for g and h are already available. Together with the obvious formulae (2.5) and

$$f=x_i \Rightarrow f[x,z]=e^{(i)}, \tag{2.6a}$$

where $e_j^{(i)}=\delta_{ij}$, this again defines a rational expression for the slope.

Remark. Hansen [4] proposed a heuristic procedure for the construction of a slope $F[X,z]$ which is smaller than $F'(X)$. It can be shown that the radius of the resulting slope is always at least as large as that of the slope defined by Theorem 1.

3. Overestimation. In order to discuss the amount of overestimation involved in the computation of $F(X,z)$ and the centered form, we need some preparation. For an interval $A=[\underline{a},\bar{a}]$, we use the midpoint $\operatorname{mid}(A):=\frac{1}{2}(\bar{a}+\underline{a})$ and the radius $\operatorname{rad}(A):=\frac{1}{2}(\bar{a}-\underline{a})$, and define

$$|A|:=\max\{|a| \mid a \in A\}, \qquad \langle A\rangle:=\inf\{|a| \mid a\in A\}.$$

For interval vectors, we understand midpoint, absolute value and inequalities componentwise. Properties of midpoint, radius, and absolute value can be found e.g., in Alefeld and Herzberger [2]; beyond that we need the following formulae:

LEMMA 1. *For intervals A, B, C, the following relations hold*:

$$0 \notin A \Rightarrow \underline{a}\bar{a}=|A|\langle A\rangle \geqq \langle A\rangle^2, \tag{3.1}$$

$$A \subseteq B \Rightarrow \langle A\rangle \geqq \langle B\rangle, \tag{3.2}$$

$$\begin{aligned} A=B\cdot C \Rightarrow \operatorname{rad}(A) &\leqq \operatorname{rad}(B)|\operatorname{mid}(C)|+|B|\operatorname{rad}(C)\\ &\leqq \operatorname{rad}(B)|C|+|B|\operatorname{rad}(C), \end{aligned} \tag{3.3}$$

$$C \not\ni 0,\ A=B/C \Rightarrow \operatorname{rad}(A) \leqq (\operatorname{rad}(B)+\operatorname{rad}(C)|A|)/\langle C\rangle. \tag{3.4}$$

Proof. (3.1) and (3.2) are obvious, and (3.3) is formula (B 13) of Neumaier [11]. For the proof of (3.4) we note that $C^{-1}=[\bar{c}^{-1},\underline{c}^{-1}]$ so that $|C^{-1}|=\operatorname{Max}\{|\bar{c}^{-1}|,|\underline{c}^{-1}|\}=(\operatorname{Min}\{|\underline{c}|,|\bar{c}|\})^{-1}=\langle C\rangle^{-1}$ and $\operatorname{rad}(C^{-1})=\frac{1}{2}(\underline{c}^{-1}-\bar{c}^{-1})=\frac{1}{2}(\bar{c}-\underline{c})(\bar{c}\underline{c})^{-1}\leqq \operatorname{rad}(C)\langle C\rangle^{-2}$ by (3.1). Therefore (3.3) implies $\operatorname{rad}(A)=\operatorname{rad}(B/C)=\operatorname{rad}(B\cdot C^{-1})\leqq \operatorname{rad}(B)\cdot|C^{-1}|+|B|\operatorname{rad}(C^{-1})\leqq \operatorname{rad}(B)\langle C\rangle^{-1}+|B|\operatorname{rad}(C)\langle C\rangle^{-2}=(\operatorname{rad}(B)+\operatorname{rad}(C)|A|)/\langle C\rangle$ since $|A|=|B\cdot C^{-1}|=|B|\cdot|C^{-1}|=|B|\langle C\rangle^{-1}$. □

Crucial for our analysis of the centered form is the bound for the range $\bar{f}(X)$ of f over X given in the next lemma.

LEMMA 2. *Let f be a rational expression in n variables which can be evaluated at the interval vector X. Then, with $\check{x}:=\operatorname{mid}(X)$,*

$$\operatorname{rad}(\bar{f}(X)) \geqq p\cdot \operatorname{rad}(X), \tag{3.5}$$

where

$$p:=p(X):=\inf\{|f[\xi,\check{x}]| \mid \xi\in X\}. \tag{3.6}$$

Proof. We define two vectors x^*, $x_* \in X$ such that

$$\tfrac{1}{2}(f(x^*)-f(x_*)) \geqq p\cdot\operatorname{rad}(X); \tag{3.7}$$

clearly, (3.5) follows from (3.7). If the ith component of $f[\xi,z]$ is always positive we put

$$x_i^*:=\bar{x}_i, \qquad x_{*i}:=\underline{x}_i,$$

if it is always negative we put

$$x_i^*:=\underline{x}_i, \qquad x_{*i}:=\bar{x}_i,$$

and if it can become zero we put

$$x_i^*:=x_{*i}:=\check{x}_i.$$

Then x^*, $x_*\in X$, and by construction,

$$p\cdot\operatorname{rad}(X)\leqq f[x^*,\check{x}](x^*-\check{x})=f(x^*)-f(\check{x}),$$

$$p\cdot\operatorname{rad}(X)\leqq f[x_*,\check{x}](\check{x}-x_*)=f(\check{x})-f(x_*).$$

Addition of these formulae and division by two gives the required relation (3.7) and hence (3.5). □

Lemma 2 now allows the derivation of a neat bound for the radius of the centered form.

THEOREM 2. *Let f be a rational expression in n variables which can be evaluated at the interval vector X, and suppose that $z\in X$. Then the range $\bar{f}(X)$ and the centered form*

$$F_z(X):=f(z)+F[X,z](X-z) \tag{3.8}$$

are related by the inequality

$$0\leqq \operatorname{rad}(F_z(X))-\operatorname{rad}(\bar{f}(X))\leqq 3\operatorname{rad}(F[X,X])\operatorname{rad}(X). \tag{3.9}$$

Proof. Define $r:=\operatorname{rad}(F(X,X))$. For arbitrary $\xi\in X$ we have $f[\xi,\check{x}]\in F[X,X]$ and therefore $|f[\xi,\check{x}]|\geqq|F[X,X]|-2r$. Since this is independent of ξ, the vector p defined by (3.6) satisfies $p\geqq|F[X,X]|=2r$, or

$$|F[X,X]|\leqq p+2r. \tag{3.10}$$

Therefore componentwise application of (3.7) gives

$$\begin{aligned} \operatorname{rad}(F_z(X))=\operatorname{rad}(F[X,z](X-z))&\leqq \operatorname{rad}(F[X,X](X-z))\\ &\leqq r|\check{x}-z|+|F[X,X]|\operatorname{rad}(X-z),\\ \operatorname{rad}(F_z(X))&\leqq r\cdot\operatorname{rad}(X)+(p+2r)\cdot\operatorname{rad}(X) \quad \text{by (3.10)},\\ &=p\cdot\operatorname{rad}(X)+3r\cdot\operatorname{rad}(X)\\ &\leqq \operatorname{rad}(\bar{f}(X))+3r\cdot\operatorname{rad}(X) \quad \text{by (3.5)}. \end{aligned}$$

This implies the right-hand inequality of (3.9); the left-hand inequality follows from $\bar{f}(X)\subseteq F_z(X)$. □

We now proceed to showing that the radius of the centered form $F_z(X)$ overestimates that of the range $\bar{f}(X)$ only by $O(\|\operatorname{rad}(X)\|_2^2)$. By (3.9) it suffices to show that $\operatorname{rad}(F[X,X])$ is $O(\|\operatorname{rad}(X)\|_2)$. Since we defined $F[X,Z]$ as an interval extension of the rational expression $f[x,z]$, this follows immediately from the following result about the radius of arbitrary rational expressions (cf. Moore [10]).

THEOREM 3. *Let f be a rational expression in n variables which can be evaluated at X_0. Then there is a constant γ (depending on f and X_0 but not on X) such that*

$$X \subseteq X_0 \Rightarrow \operatorname{rad}(F(X)) \leqq \gamma \|\operatorname{rad}(X)\|_2. \tag{3.11}$$

Proof. Fix $X \subseteq X_0$ and put $\varepsilon := \|\operatorname{rad}(X)\|_2$. We proceed inductively and assume that the theorem holds for all subexpressions of f, i.e. for each subexpression g of f there is a constant γ_g such that the interval extension G of g satisfies

$$\operatorname{rad}(G(X)) \leqq \gamma_g \varepsilon. \tag{3.12}$$

(3.11) is obvious if f has no subexpressions ($\gamma = 0$ for constants, and $\gamma = 1$ for variables). If f has subexpressions, then one of the following cases applies.

Case 1. $f = -g$. Obviously (3.11) holds with $\gamma = \gamma_g$.

Case 2. $f = g \pm h$. Then

$$\operatorname{rad}(F(X)) = \operatorname{rad}(G(X) \pm H(X)) = \operatorname{rad}(G(X)) + \operatorname{rad}(H(X)) \leqq \gamma_g \varepsilon + \gamma_h \varepsilon$$

so that (3.11) holds with $\gamma = \gamma_g + \gamma_h$.

Case 3. $f = g * h$. Then by Lemma 1,

$$\operatorname{rad}(F(X)) = \operatorname{rad}(G(X) * H(X)) \leqq \operatorname{rad}(G(X))|H(X)| + |G(X)|\operatorname{rad}(H(X))$$
$$\leqq \gamma_g \varepsilon |H(X_0)| + |G(X_0)| \gamma_h \varepsilon$$

so that (3.11) holds with $\gamma = \gamma_g |H(X_0)| + |G(X_0)| \gamma_h$.

Case 4. $f = g/h$. Then by Lemma 1,

$$\begin{aligned}\operatorname{rad}(F(X)) &= \operatorname{rad}(G(X)/H(X))\\ &\leqq (\operatorname{rad}(G(X)) + \operatorname{rad}(H(X))|F(X)|)/\langle H(X)\rangle\\ &\leqq (\gamma_g \varepsilon + \gamma_h \varepsilon |F(X_0)|)/\langle H(X_0)\rangle,\end{aligned}$$

and (3.11) holds again with $\gamma = (\gamma_g + \gamma_h |F(X_0)|)/\langle H(X_0)\rangle$. This completes the induction. □

COROLLARY (quadratic approximation). *Let f be a rational expression in n variables which can be evaluated at X_0. Then there is a constant γ' depending on f, X_0 such that*

$$X \subseteq X_0 \Rightarrow |\operatorname{rad}(F_z(X)) - \operatorname{rad}(\bar{f}(X))| \leqq \gamma' \|\operatorname{rad}(X)\|_2^2. \tag{3.13}$$

Proof. $\gamma' := 3\gamma_{f[x,z]}$ works by Theorem 2 and the Cauchy-Schwarz inequality. □

Remarks. 1. The restriction $X \subseteq X_0$ in Theorem 3 and the Corollary is essential. This is demonstrated by the following counterexamples: $f := x/x$ has $\bar{f}([\varepsilon, 3\varepsilon]) = F_{2\varepsilon}([\varepsilon, 3\varepsilon]) = 1$, $F([\varepsilon, 3\varepsilon]) = [\frac{1}{3}, 3]$, and $f := 1/x$ has $\bar{f}([\varepsilon, 3\varepsilon]) = F([\varepsilon, 3\varepsilon]) = [1/3\varepsilon, 1/\varepsilon]$, $F_{2\varepsilon}([\varepsilon, 3\varepsilon]) = [0, 1/\varepsilon]$. An interval X_0 containing $[\varepsilon, 3\varepsilon]$ for all sufficiently small ε must contain zero, so that the division by X_0 is impossible.

2. If no component of $f'(z)$ is zero, then $p > 0$ for all sufficiently narrow intervals containing z so that by Lemma 2,

$$\|\operatorname{rad}(\bar{f}(X))\|_2 \geqq \beta \|\operatorname{rad}(X)\|_2$$

for suitable $\beta > 0$. Hence in this case, the radii of $F(X)$ and $\bar{f}(X)$ have precisely the same order as $\operatorname{rad}(X)$. On the other hand, if $f'(z) = 0$ then $p = 0$ for all intervals containing z, so that by the proof of Theorem 2,

$$\operatorname{rad}(\bar{f}(X)) \leqq \operatorname{rad}(F_z(X)) \leqq 3 \operatorname{rad}(F[X, X]) \operatorname{rad}(X).$$

Hence in this case, the radius of $\bar{f}(X)$ generally is of a smaller order than $\operatorname{rad}(X)$.

But the radius of $F(X)$ generally still has the order $O(\operatorname{rad}(X))$; cf. the trivial example $f := x - x$.

3. The proof of Theorem 2 immediately carries over to a proof that for $z \in X$, the mean value form

$$F_m(X) := f(z) + F'(X)(X - z)$$

satisfies the inequalities

$$0 \leqq \operatorname{rad}(F_m(X)) - \operatorname{rad}(\bar{f}(X)) \leqq 3 \operatorname{rad}(F'(X)) \operatorname{rad}(X).$$

4. If $z = \operatorname{mid}(X)$ then a slight change in the argument shows that the upper bound in (3.9) can be replaced by the sharper $2 \operatorname{rad} F[X, z] \operatorname{rad}(X)$.

5. Compared with other proofs of the quadratic approximation property, it is interesting to note that the present proof uses neither Lipschitz conditions for slope or derivative (as in [3], [9]) nor is based on the deep existence theorem of Miranda (as in [9]).

4. A comparison of slopes and centered forms. In this section we illustrate the ideas developed so far. Among others, we compare our centered forms with those of Ratschek [13]. He defines the kth order centered form of $f(x) = p(x)/q(x)$, the quotient of two polynomials p and q, by

$$\hat{F}_k(X) := \sum_{\nu=0}^{k-1} \frac{f^{(\nu)}(z)}{\nu!}(X - z)^\nu + \frac{\sum_{\nu=k}^{\infty} (t_\nu^{(k)}(z)/\nu!)(X - z)^\nu}{\sum_{\nu=0}^{\infty} (q^{(\nu)}(z)/\nu!)(X - z)^\nu}, \tag{4.1}$$

where

$$t_\nu^{(k)} = p^{(\nu)}(z) - \sum_{l=0}^{k-1} \binom{\nu}{l} f^{(l)}(z) q^{(\nu-l)}(z);$$

the formally infinite sums are finite since p and q are polynomials. For polynomials $f(x)$, i.e. $q(x) \equiv 1$, the form (4.1) is independent of k and reduces to the centered form of Hansen [5],

$$\hat{F}(X) := \sum_{\nu=0}^{\infty} \frac{f^{(\nu)}(z)}{\nu!}(X - z)^\nu. \tag{4.2}$$

For rational functions it is difficult to organize the computation of (4.1) for large k; for polynomials, (4.2) can be easily computed by the full Horner scheme in $O(n^2)$ operations. But, since f only involves $N = O(n)$ operations, the method presented here is considerably faster except for very small values of n.

Example 1. $f := x - 10/(x + 2/x)$. f can be evaluated for all real $x \neq 0$ and all intervals X not containing zero. The three subexpressions are $f_1 := 2/x$, $f_2 := x + f_1$, $f_3 := 10/f_2$; then $f = x - f_3$. The recursions for $f(z)$, $F(X)$, and $F[X, z]$ are therefore

$$\begin{aligned}
&F_1(X) = 2/X, && f_1(z) = 2/z, && F_1[X, z] = -f_1(z)/X,\\
&F_2(X) = X + F_1(X), && f_2(z) = z + f_1(z), && F_2[X, z] = 1 + F_1[X, z],\\
&F_3(X) = 10/F_2(X), && f_3(z) = 10/f_2(z), && F_3[X, z] = -F_3[X, z]f_3(z)/F_2(X),\\
&F(X) = X - F_3(X), && f(z) = z - f_3(z), && F[X, z] = 1 - F_3[X; z].
\end{aligned}$$

Since $F_3(X)$, $F(X)$, $f(z)$ are not needed for the slope, the slope can be evaluated with 12 operations, and the centered form $F_z(X)$ with 16 operations (compared with $N = 4$ for f). As an example, we take $X = [1, 3]$, $z = 2$. Then (computing $F'(X)$ as $F[X, X]$)

$$f(z) = -\tfrac{4}{3}, \quad F[X, z] = [1, \tfrac{7}{3}], \quad F'(X) = [-\tfrac{13}{5}, \tfrac{19}{5}],$$
$$F(X) = [-5, 1], \quad F_z(X) = [-\tfrac{11}{3}, 1], \quad F_m(X) = [-\tfrac{77}{15}, \tfrac{37}{15}].$$

To compute the range we observe that f has in X a minimum at $x = 1.0306589$, leading to

$$\bar{f}(X) = [-2.3350242, 0.2727272].$$

Since $f(1) = -\frac{7}{3}$, $f(3) = \frac{3}{11}$, f has a zero $x^* \in X$; indeed $x^* = \sqrt{8} = 2.82842712 \cdots$.

We have already seen that f is not monotone in X; therefore, no matter how the interval extension $F'(X)$ of the derivative is formed, $0 \in F'(X)$ so that the "equation" $F'(X)\delta = f(z)$ cannot be solved. But $0 \notin F[X, z]$ and $F[X, z]\delta = f(z)$ has the "solution" $\delta = f(z)/F[X, z] = [-\frac{4}{3}, -\frac{4}{7}]$; therefore

$$x^* \in X' := X \cap (z - \delta) = [2.5714285, 3].$$

Two further steps of the same process with $z' = \mathrm{mid}\,(X')$, $z'' = \mathrm{mid}\,(X'')$ gives the enclosures

$$x^* \in X'' = [2.8266851, 2.8300253],$$

$$x^* \in X''' = [2.8284271, 2.8284272].$$

Example 2. $f := (x^3 - 8x)/(x^2 + 2)$, with derivative $f' = 1 + 10(x^2 - 2)/(x^2 + 2)^2$. This is the normalized rational expression for the same function as in Example 1. Using Horner's scheme for the evaluation of nominator and denominator, we find for $X = [1, 3]$, $z = 2$ (this time computing $F'(X)$ from the given expression for f'):

$$F(X) = [-21, 3], \quad F'(X) = [-\tfrac{1}{9}, \tfrac{79}{9}], \quad F_m(X) = [-\tfrac{91}{9}, \tfrac{67}{9}].$$

Ratschek's first order centered form is

$$\hat{F}_1(X) = [-19, \tfrac{49}{3}].$$

The evaluation of his higher order forms is very cumbersome; moreover, all of Ratschek's centered forms fail, e.g., for $1 \leqq \underline{x} \leqq \bar{x}/4$ due to division by an interval containing zero. Clearly, the expression in Example 1 is preferable.

In the case of polynomials we shall be more specific. The value of a polynomial

$$a_0 + a_1 x + \cdots + a_n x^n \tag{4.3}$$

may be computed recursively by the Horner scheme

$$\begin{aligned} &h_0 := a_n, \\ &h_i := h_{i-1} * x + a_{n-i} \qquad (i = 1, \cdots, n), \\ &h(x) := h_n, \end{aligned} \tag{4.4}$$

or the power scheme

$$\begin{aligned} &p_0 := 1,\ p_1 := a_0, \\ &p_{2i} := p_{2i-2} * x, \qquad p_{2i+1} := p_{2i-1} + a_i * p_{2i} \qquad (i = 1, \cdots, n), \\ &p(x) := p_{2n+1}. \end{aligned} \tag{4.5}$$

This yields two different rational expressions $h(x)$ and $p(x)$ for the polynomial (4.3). The formulae of Theorem 1 lead (with $h_i' := h_i[x, z]$, $p_i' := p_i[x, z]$) to expressions for the slope, namely

$$\begin{aligned} &h_0' := 0, \\ &h_i' := h_{i-1}' * x + h_{i-1}(z) \qquad (i = 1, \cdots, n), \\ &h[x, z] := h_n' \end{aligned} \tag{4.6}$$

for the Horner scheme, and

$$\begin{aligned} &p_0' := p_1' := 0, \\ &p_{2i}' := p_{2i-2}' * x + p_{2i-2}(z), \qquad p_{2i+1}' = p_{2i-1}' + a_i * p_{2i}' \qquad (i = 1, \cdots, n), \\ &p[x, z] := p_{2n+1}' \end{aligned} \tag{4.7}$$

for the power scheme. By inspection we see that the Horner scheme (4.4) involves $N = 2n$ operations, but the corresponding slope (4.6) can be defined in $4n$ operations and the centered form in $4n + 3$ operations. Similarly, the power scheme (4.5) involves $N = 3n$ operations, and the corresponding slope (4.7) needs $5n$ operations and the centered form $7n + 3$ operations.

The recursion (4.6) for the interval extension $H[X, z]$ is the scheme J_1 of Alefeld [1]. He proves several optimality properties of $H[X, z]$. Another optimality property

$$H[X, z] \subseteq P[X, z] \quad \text{if } z = \mathrm{mid}\,(X), \tag{4.8}$$

is shown in Krawczyk [8]. Therefore, for the usual choice $z = \mathrm{mid}\,(X)$, the Horner scheme is preferable to the power scheme with respect to both operation count and narrow inclusion. On the other hand, the following example shows that a general inclusion comparison with the Hansen-Ratschek centered form (4.2) is impossible.

Example 3. For the polynomial $x^3 - x^2 - 2x + 2$ and $X = [0, 2]$, $z = 1$, the Hansen-Ratschek form (4.2) gives $\hat{F}(X) = [-4, 4]$, and the centered form $F_z(X) = f(z) + H[X, z](X - z)$ derived from (4.6) gives the narrower interval $F_z(X) = [-2, 2]$. On the other hand, for the polynomial $f(x) = x^3 - 3x^2 + 3x - 1$ and the same X and z, we have $\hat{F}[0, 2] = [-1, 1]$ which is better than $F_z(X) = [-3, 3]$.

Therefore, the higher effort spent to compute $\hat{F}[0, 2]$ may or may not be honoured by a narrower inclusion interval. However, for a large class of polynomials, no improvement is possible: Indeed, whenever $z = \mathrm{mid}\,(X)$ and all $h_i(z)$, as defined by (4.4), agree in sign, then $F_z(X) = \hat{F}(X)$; see Krawczyk [8].

5. The complex case. It is no problem to extend the discussion to the complex case: *complex rational expressions* are defined as in the first paragraph of § 2, but with (i) replaced by

$$c \in \mathfrak{R} \quad \text{for all } c \in \mathbb{C}; \tag{i*}$$

a complex expression can be evaluated at each $x \in \mathbb{C}^n$ for which no subexpression occurring in a denominator has the value zero. In order to be able to calculate with sets of complex numbers, we use discs

$$\langle z, r \rangle := \{\tilde{z} \in \mathbb{C} \,|\, |z - \tilde{z}| \leqq r\}$$

as *complex intervals* and define a *complex interval arithmetic* on the set of complex intervals by the rules

$$X_1 \pm X_2 := \langle z_1 \pm z_2, r_1 + r_2 \rangle, \tag{5.1}$$

$$X_1 * X_2 := \langle z_1 z_2, |z_1| r_2 + r_1 |z_2| + r_1 r_2 \rangle, \tag{5.2}$$

$$X_1 / X_2 := \left\langle \frac{z_1 \bar{z}_2}{|z_2|^2 - r_2^2}, \frac{|z_1| r_2 + r_1 |z_2| + r_1 r_2}{|z_2|^2 - r_2^2} \right\rangle \quad \text{if } 0 \notin X_2, \tag{5.3}$$

where $X_i = \langle z_i, r_i \rangle (i = 1, 2)$ and $\bar{z}$ denote the complex conjugate of z. This definition is equivalent to that given by Alefeld and Herzberger [2]; in particular, the operations are inclusion isotonic, so that the interval extension of a complex expression also satisfies the rules

$$x \in X \Rightarrow f(x) \in F(X),$$
$$Z \subseteq X \Rightarrow F(Z) \subseteq F(X).$$

The definition of a slope carries over, and Theorem 1 holds without change. However, in the complex case, the arithmetic for the computation of the slope $F[X, z]$ simplifies if z is the center of X. With $X = \langle z, r\rangle$ and the abbreviations

$$F[X, z] := \langle z_f, r_f\rangle, \quad F(X) := \langle \zeta_f, \rho_f\rangle, \quad f(z) := \langle f, 0\rangle$$

(and similarly for g and h) we easily obtain from (5.1)-(5.3) and Theorem 1 the recursion

$$\begin{aligned}
&f = c \in \mathbb{C} \Rightarrow z_f = 0,\ r_f = 0,\\
&f = x \quad\ \Rightarrow z_f = 1,\ r_f = 0,\\
&f = g \pm h \Rightarrow z_f = z_g \pm z_h,\ r_f = r_g + r_h,\\
&f = g * h \ \Rightarrow z_f = z_g\zeta_h + g z_h,\\
&\qquad\qquad\quad r_f = |z_g|\rho_h + r_g|\zeta_h| + r_g\rho_h + |g| r_h,\\
&f = g/h \ \Rightarrow z_f = \frac{(z_g - z_h f)\bar{\zeta}_h}{|\zeta_h|^2 - \rho_h^2},\\
&\qquad\qquad\quad r_f = \frac{|z_g - z_h f|\rho_h + (r_g + r_h|f|)(|\zeta_h| + \rho_h)}{|\zeta_h|^2 - \rho_h^2},
\end{aligned} \tag{5.4}$$

and the centered form $F_z(X) := f(z) + F[X, z](X - z)$ becomes

$$F_z(X) = \langle f(z), (|z_f| + r_f) r\rangle. \tag{5.5}$$

It can be shown that Lemma 1, and hence the quadratic approximation property, remain valid for complex intervals.

We now compare this centered form (5.4) with previously known complex centered forms. The survey of Petcović [12] gives as best form that resulting from the inclusion

$$\bar{f}(X) := \{f(\tilde{z}) | \tilde{z} \in X\} \subseteq \left\langle f(z), \sum_{k=1}^{\infty} \left| \frac{f^{(k)}(z)}{k!} \right| r^k \right\rangle =: \hat{F}(X) \tag{5.6}$$

for $X = \langle z, r\rangle$. In the simple case $f(x) = 1/x$, we get for both (5.5) and (5.6) the result

$$F_z(X) = \hat{F}(X) = \left\langle \frac{1}{z}, \frac{r}{|z|(|z| - r)} \right\rangle \quad \text{if } 0 \notin X$$

(which is slightly weaker than $1/X$ computed directly by (5.3)). If f is a polynomial then Petcović's centered form (5.6) agrees with that of Hansen-Ratschek (interpreted with complex discs); on the other hand, the relations (5.4) for the complex Horner scheme for the polynomial $a_0 + a_1 x + \cdots + a_n x^n$ leads to the following algorithm

$$\begin{aligned}
&p_n := z_n := a_n, \quad r_n := 0, \quad |X| := |z| + r, \qquad (i = 1(1)n - 1),\\
&p_{n-i} := p_{n-i+1} * z + a_{n-i}\\
&z_{n-i} := z_{n-i+1} * z + p_{n-i}\\
&r_{n-i} := r_{n-i+1} * |X| + |z_{n-i+1}| r,\\
&p_0 := p_1 z + a_0,
\end{aligned}$$

where $X = \langle z, r\rangle$, $f(z) = \langle p_0, 0\rangle$, and

$$F[X, z] = \langle z_1, r_1\rangle, \quad F_z(X) = \langle p_0, (|z_1| + r_1) r\rangle. \tag{5.7}$$

Again, (5.7) is computed with $7n - 5$ interval operations much more efficiently than the Petcović form (5.6). For the two examples given by Petcović [12], the inclusion (5.6) is slightly better than (5.7): For the polynomial

$$\begin{aligned}
&(0.471 + 0.062i) + (0.468 - 0.794i)x\\
&\quad + (0.662 + 0.472i)x^2 + (-0.155 + 0.513i)x^3\\
&\quad + (0.185 + 0.622i)x^4 + (0.465 - 0.966i)x^5\\
&\quad + (-0.703 + 0.143i)x^6
\end{aligned}$$

and $X = \langle 0.174 + 0.252i, 0.415\rangle$ Petcović's form (5.6) gives

$$\hat{F}(X) = \langle 0.68943 + 0.06242i, 0.58361\rangle,$$

whereas the new form (5.5) gives only

$$F_z(X) = \langle 0.68943 + 0.06242i, 0.65400\rangle.$$

Similarly, the polynomial

$$\begin{aligned}
&(0.423 + 0.594i) + (-0.055 + 0.158i)x\\
&\quad + (-0.071 + 0.021i)x^2 + (-0.691 - 0.543i)x^3\\
&\quad + (0.046 - 0.974i)x^4 + (0.565 - 0.363i)x^5\\
&\quad + (0.311 + 0.125i)x^6 + (0.101 + 0.416i)x^7
\end{aligned}$$

with $X = \langle 0.185 - 0.289i, 0.354\rangle$ gives

$$\hat{F}(X) = \langle 0.50180 + 0.68304i, 0.74373\rangle,$$
$$F_z(X) = \langle 0.50180 + 0.68304i, 0.74932\rangle.$$

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