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Existence of solutions of piecewise differentiable systems of equations

By

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Interval arithmetic is an ideal tool for the verification of the existence of a solution x^* of a system of equations $F(x^*) = 0$; see e.g. Alefeld [1], Kahan [2], Moore [7], Neumaier [9], Nickel [10], Qi [11], Rump [12]. A number of tests based on Krawczyk type operators were discussed in the literature after Moore's paper [7] appeared; but implicitly, Moore's result is already in Krawczyk [3; Satz 1]. These tests are known to work under the assumption that F satisfies an interval Lipschitz condition. However, tests based on Newton type operators required up to now the assumption that F is continuously differentiable. Hence the latter were not applicable to systems of equations involving the absolute value or the positive part of numbers. The present paper removes this restrictive assumption from the test.

The notation used follows Neumaier [8], [9]: $\mathbb{IR}$, $\mathbb{IR}^n$, $\mathbb{IR}^{n\times n}$ denote the set of real intervals, n-dimensional interval vectors, and $n \times n$ interval matrices, and $\mathbb{I}D := \{x \in \mathbb{IR}^n \mid x \subseteq D\}$ for $D \subseteq \mathbb{R}^n$. The interval hull of a bounded subset $\Sigma \subseteq \mathbb{R}^{n\times n}$ is defined as $\square\Sigma := [\inf \Sigma, \sup \Sigma]$. An interval matrix $A \in \mathbb{IR}^{n\times n}$ is called regular if all $\tilde{A} \in A$ are nonsingular; in this case $A^H b$ is defined as the interval hull of the solution set of the system of linear interval equations $\tilde{A}\tilde{x} = \tilde{b}\,(\tilde{A} \in A, \tilde{b} \in b)$. In particular

$$\tilde{A}^{-1} b \in A^H b \qquad \text{for all } \tilde{A} \in A,\ \tilde{b} \in b.$$

Note that A^H is not an interval matrix but a sublinear mapping from $\mathbb{IR}^n$ to $\mathbb{IR}^n$; $A^H b$ is generally a tighter enclosure for the solution set than the matrix-vector product $A^{-1} b$ with an optimal enclosure $A^{-1} = \square\{\tilde{A}^{-1} \mid \tilde{A} \in A\}$. For example, if

$$A = \begin{pmatrix} 2 & [-1, 0] \\ [-1, 0] & 2 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

then

$$A^H b = \begin{pmatrix} [0, 1/2] \\ [-1, -3/4] \end{pmatrix}, \ A^{-1} = \begin{pmatrix} [1/2, 2/3] & [0, 1/3] \\ [0, 1/3] & [1/2, 2/3] \end{pmatrix}, \ A^{-1} b = \begin{pmatrix} [-1/6, 2/3] \\ [-4/3, -2/3] \end{pmatrix}.$$

We say that A^H is regular if $0 \in A^H b$, $\varrho(b) = 0$ implies $b = 0$. Sufficient conditions for this are given in Neumaier [9]; in particular, A^H is regular for all M-matrices A and for all A such that $\|I - CA\|_\infty < \frac{1}{2}$ for some $C \in \mathbb{R}^{n\times n}$ (use $A^H b \subseteq (CA)^H (Cb)$ and Proposition 4 of [9] for the latter).

In the sequel, we shall assume the following interval Lipschitz condition at a point $\tilde{x}^0$:

Hypothesis L. $F: D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function, $x \in \mathbb{I}D$, $\tilde{x}^0 \in x$, and $A \in \mathbb{I}\mathbb{R}^{n \times n}$ is such that for all $\tilde{x} \in x$ there is a matrix $\tilde{A} \in A$ satisfying

$$F(\tilde{x}) - F(\tilde{x}^0) = \tilde{A}(\tilde{x} - \tilde{x}^0). \tag{1}$$

Then Moore's existence test can be phrased in the following, slightly generalized form:

Theorem. *Under hypothesis* L, *if there is a matrix* $C \in \mathbb{R}^{n \times n}$ *such that*

$$K(x) := \tilde{x}^0 - CF(\tilde{x}^0) - (CA - I)(x - \tilde{x}^0) \subseteq x$$

then x *contains a zero of* F. *Moreover, if* (1) *can be satisfied for all* $\tilde{x}, \tilde{x}^0 \in x$ *by some* $\tilde{A} \in A$ *then* $K(x) \subseteq \operatorname{int} x$ *implies that* x *contains a unique zero of* x.

Proof. Essentially in Qi [11]. □

A disadvantage of $K(x)$ and other Krawczyk type operators is the fact that for x with large radius, $K(x)$ may have large radius even when $\tilde{x}^0$ is a very good approximation of a zero. This disadvantage is not present for Newton type operators. Here we shall prove the following analogue of Moore's theorem.

Theorem. *Under hypothesis* L *we have*:

(i) *If* A *is regular and* $x^* \in x$ *is a zero of* F *then*

$$x^* \in \tilde{x}^0 - A^H F(\tilde{x}^0). \tag{2}$$

(ii) *If* A *and* A^H *are regular then the condition*

$$\tilde{x}^0 - A^H F(\tilde{x}^0) \subseteq x \tag{3}$$

implies that x *contains a zero of* F.

(iii) *If* (1) *can be satisfied for all* $\tilde{x}, \tilde{x}^0 \in x$ *by some* $\tilde{A} \in A$ *then* F *has at most one zero in* x.

Note that if F is continuously differentiable in D then (1) holds with $A = F'(x)$ and

$$\tilde{A} = \int_0^1 F'(\tilde{x}^0 + t(\tilde{x}^0 - \tilde{x}))\, dt.$$

In this case the theorem is essentially due to Moore [6] for part (i), and Nickel [10] for part (ii); an elementary proof based on the inverse function theorem is given in Neumaier [9].

The proof of the theorem is based on the following fixpoint theorem of Kakutani; for a proof see e.g. Zangwill and Garcia [13; Theorem 21.2.1].

Fixpoint Theorem (Kakutani). *Let* $D \subseteq \mathbb{R}^n$, *and let* D_0 *be a nonempty, compact, and convex subset of* D. *Suppose that the set-valued map* $G: D \subseteq \mathbb{R}^n \to \mathfrak{P}(\mathbb{R}^n)$ *has the following properties*:

(K1) *G is upper hemicontinuous in D_0; i.e. for any two convergent sequences $\tilde{x}^\ell(\ell = 0, 1, 2, \ldots)$, $\tilde{y}^\ell(\ell = 0, 1, 2, \ldots)$ with limits $\tilde{x} = \lim_{\ell \to \infty} \tilde{x}^\ell$, $\tilde{y} = \lim_{\ell \to \infty} \tilde{y}^\ell$, the implication*

$$\tilde{x}^\ell \in D_0, \ \tilde{y}^\ell \in G(x^\ell) \quad \text{for } \ell = 0, 1, 2, \ldots \Rightarrow \tilde{y} \in G(\tilde{x})$$

is valid.

(K2) *For every $\tilde{x} \in D_0$, $G(\tilde{x})$ is a nonempty, convex subset of D_0.*

Then there is at least one $x^* \in D_0$ with $x^* \in G(x^*)$. □

For a bounded set Σ of real vectors or matrices we define the convex hull $\langle \Sigma \rangle$ by $\langle \Sigma \rangle := \left\{ \sum_{i=1}^{s} p_i \xi_i \,\middle|\, \xi_i \in \Sigma, p_i \geqq 0, s \geqq 1, \sum_{i=1}^{s} p_i = 1 \right\}$, and the closed convex hull $\operatorname{conv} \Sigma$ as the closure of $\langle \Sigma \rangle$.

Proof of the theorem. Suppose first that $x^* \in x$ is a zero of F. Then $F(x^*) = 0$, and (1) with $\tilde{x} = x^*$ implies $\tilde{x} = \tilde{x}^0 - \tilde{A}^{-1} F(\tilde{x}^0) \in \tilde{x}^0 - A^H F(\tilde{x}^0)$. Therefore (i) holds. To prove (ii) we assume (3) and put $\mathfrak{C}_0 := \{\tilde{A}^{-1} \mid \tilde{A} \in A\}$, $\mathfrak{C} := \operatorname{conv} \mathfrak{C}_0$. Since A is regular and bounded, $\mathfrak{C}_0$ is bounded and $\mathfrak{C}$ is nonempty, convex and compact. We claim

$$(4) \qquad \tilde{C} F(\tilde{x}) \in A^H F(\tilde{x}) \quad \text{for all } \tilde{C} \in \mathfrak{C}, \ \tilde{x} \in D.$$

Indeed, if $\tilde{C} \in \langle \mathfrak{C}_0 \rangle$ then $\tilde{C} = \sum_{i=1}^{s} p_i \tilde{A}_i^{-1}$ with $\tilde{A}_i \in A$, $p_i \geqq 0$, $\Sigma p_i = 1$, hence $\tilde{C} F(\tilde{x}) = \Sigma p_i \tilde{A}_i^{-1} F(\tilde{x}) \in \operatorname{conv} \{\tilde{A}_i^{-1} F(\tilde{x}) \mid i = 1, \ldots, s\} \subseteq A^H F(\tilde{x})$. And if $\tilde{C} \in \mathfrak{C}$ then $\tilde{C} = \lim_{i \to \infty} \tilde{C}_i$ for suitable $\tilde{C}_i \in \langle \mathfrak{C}_0 \rangle$ whence $\tilde{C} F(\tilde{x}) = \lim_{i \to \infty} \tilde{C}_i F(\tilde{x}) \in A^H F(\tilde{x})$.

Now we show that the set-valued map $G\colon D \to \mathfrak{P}(\mathbb{R}^n)$ defined by

$$(5) \qquad G(\tilde{x}) := \{\tilde{x}^0 - \tilde{C} F(\tilde{x}^0) \mid \tilde{C} \in \mathfrak{C}, \tilde{C}(F(\tilde{x}) - F(\tilde{x}^0)) = \tilde{x} - \tilde{x}^0\}$$

satisfies assumptions (K1) and (K2) with the nonempty, compact and convex interval $D_0 := x$. Indeed if $\tilde{x}^\ell (\ell = 0, 1, 2, \ldots)$ and $\tilde{y}^\ell (\ell = 0, 1, 2, \ldots)$ are convergent sequences with limits $\tilde{x} = \lim_{\ell \to \infty} \tilde{x}^\ell$, $\tilde{y} = \lim_{\ell \to \infty} \tilde{y}^\ell$, and if $\tilde{x}^\ell \in D_0$, $\tilde{y}^\ell \in G(\tilde{x}^\ell)$ for $\ell = 0, 1, 2, \ldots$ then

$$(6) \qquad \tilde{y}^\ell = \tilde{x}^0 - \tilde{C}_\ell F(\tilde{x}^0), \qquad \tilde{C}_\ell(F(\tilde{x}^\ell) - F(\tilde{x}^0)) = \tilde{x}^\ell - \tilde{x}^0$$

for suitable $\tilde{C}_\ell \in \mathfrak{C}$. Since $\mathfrak{C}$ is compact, the sequence $\tilde{C}_\ell (\ell = 0, 1, 2, \ldots)$ has an accumulation point $\tilde{C} \in \mathfrak{C}$, and by continuity,

$$\tilde{y} = \tilde{x}^0 - \tilde{C} F(\tilde{x}^0), \qquad \tilde{C}(F(\tilde{x}) - F(\tilde{x}^0)) = \tilde{x} - \tilde{x}^0.$$

Therefore $\tilde{y} \in G(\tilde{x})$. Hence G is upper hemicontinuous in D_0 and (K1) holds.

To prove convexity of $G(\tilde{x})$ we assume that $\tilde{y}^\ell \in G(\tilde{x})$ for $\ell = 1, 2$. Then (6) holds for suitable $\tilde{C}_\ell \in \mathfrak{C}$ $(\ell = 1, 2)$. Hence, for $0 \leqq t \leqq 1$, the matrix $\tilde{C} := t\tilde{C}_1 + (1 - t)\, \tilde{C}_2 \in \mathfrak{C}$ satisfies

$$t\, \tilde{y}^1 + (1 - t)\, \tilde{y}^2 = \tilde{x}^0 - \tilde{C} F(\tilde{x}^0), \qquad \tilde{C}(F(\tilde{x}) - F(\tilde{x}^0)) = \tilde{x} - \tilde{x}^0;$$

therefore $t\, \tilde{y}^1 + (1 - t)\, \tilde{y}^2 \in G(\tilde{x})$ and $G(\tilde{x})$ is convex. Now if $\tilde{x} \in x$ then (1) holds for suitable $\tilde{A} \in A$, whence $\tilde{x}^0 - A^{-1} F(\tilde{x}^0) \in G(\tilde{x})$. Therefore $G(\tilde{x})$ is nonempty. Finally, (3) and (4) imply $G(\tilde{x}) \subseteq \tilde{x}^0 - A^H F(\tilde{x}^0) \subseteq x$, and (K2) holds.

Kakutani's fixpoint theorem now implies the existence of $x^* \in x$ with $x^* \in G(x^*)$. Hence there is a matrix $C^* \in \mathfrak{C}$ with

$$x^* = \tilde{x}^0 - C^* F(\tilde{x}^0), \qquad C^*(F(x^*) - F(\tilde{x}^0)) = x^* - \tilde{x}^0.$$

This implies $C^* F(x^*) = 0$, and by (4), $0 \in A^H F(x^*)$. Since A^H is regular, $F(x^*) = 0$ whence F has a zero $x^* \in x$. $\square$

Remark. It is likely that the theorem remains valid if the assumption "A^H is regular" in (ii) is dropped. At least, this assumption is superfluous if F is continuously differentiable and $A = F'(x)$.

For practical applications one usually chooses $\tilde{x}^0$ as an approximation to a zero (cf. Neumaier [9]). If the radius of x is not too large then the condition $\|I - CA\| < \frac{1}{2}$ is usually satisfied so that A and A^H are regular. Since for $n > 2$, $A^H b$ is very difficult to compute one computes an enclosure for it, e.g. by precondioned Gauss elimination.

Finally, we comment on the construction of a Lipschitz matrix A. If F is defined by arithmetic expressions then (1) holds with $A = F[x, \tilde{x}^0]$ and $\tilde{A} = F[\tilde{x}, \tilde{x}^0]$, where $F[\cdot, \cdot]$ is a slope of F as defined recursively in Krawczyk and Neumaier [5]. By allowing interval values for the slope we may extend the list of allowable operations considerably: We observe that if slopes for φ and g are known then

$$f = \varphi(g) \Rightarrow f[\tilde{x}, \tilde{y}] := \varphi[g(\tilde{x}), g(\tilde{y})]\, g[\tilde{x}, \tilde{y}]$$

is a slope for f; indeed

$$\begin{aligned} f[\tilde{x}, \tilde{y}]\,(\tilde{x} - \tilde{y}) &= \varphi[g(\tilde{x}), g(\tilde{y})]\, g[\tilde{x}, \tilde{y}]\,(\tilde{x} - \tilde{y}) \\ &= \varphi[g(\tilde{x}), g(\tilde{y})]\,(g(\tilde{x}) - g(\tilde{y})) \\ &= \varphi(g(\tilde{x})) - \varphi(g(\tilde{y})) = f(\tilde{x}) - f(\tilde{y}). \end{aligned}$$

Now if φ is a real-valued differentiable function like exp, sin, arctan, ... then $\varphi[\xi_1, \xi_2] = \varphi'(\xi)$ for some $\xi \in x$ and we may define an interval slope by adding to the list of recursive statements in Theorem 1 of [5] the rule

$$f = \varphi(g) \Rightarrow f[x, \tilde{x}^0] := \varphi'(g(x))\, g[x, \tilde{x}^0] \qquad \text{for } \tilde{x}^0 \in x \in \mathbb{IR};$$

however, for the square root we have $\operatorname{sqrt}[\xi_1, \xi_2] = 1/(\operatorname{sqrt}(\xi_1) + \operatorname{sqrt}(\xi_2))$ and hence the sharper rule

$$f = \operatorname{sqrt}(g) \Rightarrow f[x, \tilde{x}^0] := g[x, \tilde{x}^0]/(f(x) + f(\tilde{x}^0)) \qquad \text{for } \tilde{x}^0 \in x \in \mathbb{IR}$$

(cf. Krawczyk [4]). If φ is only Lipschitz continuous we can proceed similarly. In particular we have the rules

$$f = \operatorname{abs}(g) \Rightarrow f[x, \tilde{x}^0] := \operatorname{sgn}(g(x))\, g[x, \tilde{x}] \qquad \text{for } \tilde{x}^0 \in x \in \mathbb{IR},$$

$$f = g^+ \Rightarrow f[x, \tilde{x}^0] := \operatorname{pos}(g(x))\, g[x, \tilde{x}] \qquad \text{for } \tilde{x}^0 \in x \in \mathbb{IR};$$

here

$$\operatorname{abs}(x) := \begin{cases} -x & \text{if } x < 0 \\ [0, |x|] & \text{if } x \ni 0, \\ x & \text{if } x > 0 \end{cases} \qquad \operatorname{sgn}(x) := \begin{cases} -1 & \text{if } x < 0 \\ [-1, 1] & \text{if } x \ni 0, \\ 1 & \text{if } x > 0 \end{cases}$$

$$x^+ := \begin{cases} 0 & \text{if } x < 0 \\ [0, \bar{x}] & \text{if } x \ni 0, \\ x & \text{if } x > 0 \end{cases} \qquad \operatorname{pos}(x) := \begin{cases} 0 & \text{if } x < 0 \\ [0, 1] & \text{if } x \ni 0. \\ 1 & \text{if } x > 0 \end{cases}$$

These rules extend without change to the case of several variables, i.e. to $\tilde{x}^0 \in x \in \mathbb{I}\mathbb{R}^n$. In this way, one can define recursively and componentwise slopes $F[\cdot, \cdot]$, for every n-dimensional function F expressed by arithmetic expressions (involving $+$, $-$, $*$, $/$, exp, sin, arctan, sqrt, abs and similar functions) such that (1) holds with suitable $\tilde{A} \in F[\tilde{x}, \tilde{x}^0] \subseteq F[x, \tilde{x}^0] =: A$.

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