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Some Relations between Roots, Holes, and Pillars

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These are some observations originating in work by Conway and Sloane (*Proc. R. Soc. London* **A381** (1982), 275–283) on holes of the Leech lattice and by Lemmens and Seidel (*J. Algebra* **24** (1973), 494–512) on equiangular lines. We observe close relations which may be relevant for applications to the classification of maximal twographs with smallest eigenvalue -5, and of integral lattices generated by norm 4 vectors.

1. Root classes

Let L be an integral even lattice of minimum norm 4 and dimension n. On the set of all norm 2 vectors of $\mathbb{R}^n$ we define an equivalence relation $r \equiv s$ iff $r - s \in L$. For reasons explained in a moment we call the equivalence classes R of $\equiv$ the *root classes* of L. We are interested in *rich root classes* R, i.e. root classes R such that $|R| \geqslant n + 1$ and R spans $\mathbb{R}^n$, in the hope that their structure is sufficient to reconstruct L. Let R be a root class. Clearly

$$r - s \in L \text{ for all } r, s \in R, \tag{1.1}$$

$$r \in R + L, \quad (r, r) = 2 \Rightarrow r \in R. \tag{1.2}$$

For $r, s \in R$, $r \neq s$ we have $(r, s)^2 \leqslant (r, r)(s, s) = 4$ and

$$4 \leqslant (r - s, r - s) = (r, r) - 2(r, s) + (s, s) = 4 - 2(r, s) \tag{1.3}$$

whence

$$(r, s) = 2 - \tfrac{1}{2}(r - s, r - s) \in \{0, -1, -2\}. \tag{1.4}$$

In particular, $(r, s) \leqslant 0$ for $r, s \in R$, $r \neq s$ so that R is a (reduced or extended) fundamental system of roots of equal length 2.

We denote by h_R the greatest common divisor of the set $\{\Sigma c_r | c_r \in \mathbb{Z}, \Sigma c_r r = 0\}$, where R is the index set. We then have

$$\sum c_r r = 0, c_r \in \mathbb{Z} \Rightarrow \sum c_r \equiv 0 \bmod h_R, \tag{1.5}$$

$$h_R R \subseteq L; \tag{1.6}$$

the last relation (1.6) follows since $(\Sigma c_r)s = \Sigma c_r(s - r) \in L$ for all $s \in R$. Note that h_R is zero if and only if the roots in R are linearly independent; in particular $h_R \neq 0$ if R is a rich root class.

We remark that in the above setting, if we drop the assumption of minimum norm 4 for a root class R and $r, s \in R$ we only have $(r, s) \in \{0, \pm 1, \pm 2\}$ in place of (1.4); therefore R is still a set of roots (consider the root system defined by the set of norm 2 vectors of the integral lattice generated by R), but R is no longer necessarily a fundamental system.

Example. Let $n = 24$, and let $L = \Lambda_{24}$ be the Leech lattice. If r is a deepest hole nearest to the origin (Conway and Sloane [3]) then $(r, r) = 2$ and the root class R containing r is a fundamental system of one of the types $\hat{A}_1^{24}$, $\hat{A}_2^{12}$, $\hat{A}_3^8, \ldots, \hat{D}_{16}\hat{E}_8$, $\hat{D}_{24}$, depending on the type of the hole. In each case the root class is rich; moreover, L can be reconstructed from R by one of the 23 constructions described in Conway and Sloane [4], cf. also Borcherds [1].

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2. Weyl Vectors

From the discussion of the 23 constructions in [4] (where a reference to Bourbaki [2] is given), we take the following properties of fundamental systems.

Let $\hat{R}$ be an irreducible extended fundamental system. The adjacency matrix of the Dynkin diagram of $\hat{R}$ has a positive integral Perron vector c (corresponding to the eigenvalue 2) whose smallest component is 1; clearly

$$\sum c_r r = 0. \tag{2.1}$$

The number $h = \Sigma c_r$ is called the *Coxeter number* of $\hat{R}$, and $n = |\hat{R}| - 1$ is the dimension of the lattice $L(\hat{R})$ spanned by $\hat{R}$. The number of roots $r_x \in \hat{R}$ with $c_r = 1$ (which we call the *extremal roots* of $\hat{R}$) equals the determinant of $L(\hat{R})$, the order of $L(\hat{R})^*/L(\hat{R})$, and the order of the symmetry group of the Dynkin diagram of $\hat{R}$. For an extremal root r_x of $\hat{R}$, the set $\hat{R}\backslash\{r_x\}$ is a reduced fundamental system of roots, and there is a unique vector $w_x \in \langle\hat{R}\rangle$, the *Weyl vector* of $\hat{R}\backslash\{r_x\}$, such that

$$(r, w_x) = \begin{cases} h-1, & \text{for } r = r_x, \\ -1, & \text{for } r \in \hat{R}\backslash\{r_x\}; \end{cases} \tag{2.2}$$

the Weyl vector can alternatively be described as half the sum of the positive roots of the root system defined by $\hat{R}\backslash\{r_x\}$. The norm of the Weyl vector is

$$(w_x, w_x) = \frac{n}{12} h(h+1). \tag{2.3}$$

One easily sees that if r_x and r_y are distinct extremal roots of $\hat{R}$ then

$$\frac{1}{h}(w_x - w_y) \in L(\hat{R})^*. \tag{2.4}$$

For example (with orthonormal vectors e_i), if $\hat{R} = \hat{A}_n = \{e_i - e_{i+1} | i \bmod n+1\}$ and $r_x = e_{n+1} - e_1$ we have

$$h = n+1, \qquad w_x = \frac{1}{2}\sum_{j=1}^{n+1}(n+2-2j)e_j \quad \text{of norm } \frac{n(n+1)(n+2)}{12}, \tag{2.5}$$

and if $\hat{R} = \hat{D}_n = \{e_i - e_{i+1} | i = 1, \ldots, n-1\} \cup \{-e_1 - e_2, e_{n-1} + e_n\}$ and $r_x = -e_1 - e_2$ we have

$$h = 2n-2, \; w_x = \sum_{j=1}^{n}(n-j)e_j \qquad \text{of norm } \frac{n(n-1)(2n-1)}{6}. \tag{2.6}$$

Note that Conway and Sloane [4] consider instead the scaled Weyl vectors $g_x = -(1/h)w_x$.

3. Generalizing the 23 Constructions

We are interested in the question whether L can be reconstructed from the sublattice

$$L_R = \{\sum m_r r | m_r \in \mathbb{Z}, \sum m_r = 0\} = \{\sum m_r r | \sum m_r \equiv 0 \bmod h_R\} \tag{3.1}$$

generated by the differences of roots of a rich root class R. Since R is assumed rich, $h_R \neq 0$, and the dimensions of L_R and L agree. To reconstruct L we have to find the remaining vectors $z \in L\backslash L_R$. Since L is integral, $z \in L_R^*$; in particular $(z, r - s) \in \mathbb{Z}$ for $r, s \in R$, and $(z, h_R r) \in \mathbb{Z}$. Hence there is an integer α with

$$(z, r) \equiv -\frac{\alpha}{h_R} \bmod 1, \qquad \text{for all } r \in R. \tag{3.2}$$

Therefore, for the projection z_i of z to the subspace spanned by an irreducible component R_i of R we have

$$\left(z_i - \frac{\alpha w_x}{h_R}, r\right) \in \mathbb{Z}, \qquad \text{for} \quad r \in R_i, \tag{3.3}$$

where x is an extremal node of R_i (if R_i is extended) or of $\hat{R}_i$ (if R_i is reduced) and w_x is the corresponding Weyl vector. This implies

$$z_i \in \alpha w_i + L(R_i)^*. \tag{3.4}$$

Note that if R_i is extended then, by construction of R_i, the number h_R divides the Coxeter number h_i of R_i. But since R is rich, at least one R_i is extended, so that there are only finitely many possibilities for the z_i and hence for $z = \Sigma\, z_i$. The requirement that z has integral norm further restricts the possibilities for the z_i. To find the (only finitely many) lattices admitting a given fundamental system R as a rich root class one then has to check when two vectors with given components z_i (according to (3.4)) have integral inner product. Knowing this it is easy to construct the *glue code* $L/L_R \subseteq L_R^*/L_R$. Although this will involve considerable work, it is a finite process for each R. Preliminary calculations of the author suggest the following.

Conjecture. (2.3) *and* (3.2)–(3.4) *imply that* $n = \Sigma\, n_i < 48$.

The truth of the conjecture would make it feasible to find all lattices with a rich root class.

In principle the above analysis can be extended to all root classes R with $h_R \neq 0$; however then only the projection of L to the subspace spanned by L_R can be determined and further analysis is needed to reconstruct L.

4. Equiangular lines with angle arccos 1/5

Let L be an even lattice, and let $L^{(m)} = \{x \in L \mid (x, x) = m\}$. Fix $e \in L^{(6)}$, and put

$$X := \left\{x - \frac{e}{2} \,\middle|\, x \in L^{(4)}, (x, e) = 3\right\}.$$

Then X is a set of vectors of norm 5/2. Since for $\boldsymbol{x}, \boldsymbol{y} \in X$, $(\boldsymbol{x} - \boldsymbol{y}, \boldsymbol{x} - \boldsymbol{y})$ and $(\boldsymbol{x} + \boldsymbol{y} - \boldsymbol{e}, \boldsymbol{x} + \boldsymbol{y} - \boldsymbol{e})$ are nonnegative even numbers, vectors in X have mutual inner products in $\{\pm 1/2, \pm 3/2, \pm 5/2\}$; the lines along these vectors therefore have mutual angles arccos 1/5 or arccos 3/5. Moreover if L contains no norm 2 vectors then the inner products $\pm 3/2$ cannot occur since $(\boldsymbol{x} - \boldsymbol{e}/2, \boldsymbol{y} - \boldsymbol{e}/2) = \pm 3/2$ implies $\boldsymbol{x} - \boldsymbol{y} \in L^{(2)}$ for the upper sign and $\boldsymbol{x} + \boldsymbol{y} - \boldsymbol{e} \in L^{(2)}$ for the lower sign; therefore, in this case X defines at set of equiangular lines with angles arccos 1/5. Such sets have been studied by Lemmens and Seidel [6] and are in close relation with twographs whose smallest eigenvalue is -5 (Taylor [10]). Conversely, if X is a set of vectors of norm 5/2 along lines at angles arccos 1/5 or arccos 3/5 then, with a vector $\boldsymbol{e} \perp X$ of norm 6, the vectors $\boldsymbol{e}/2 \pm \boldsymbol{x}$ ($\boldsymbol{x} \in X$) have norm 4 and integral inner product and hence span an even lattice L. However, even if X is equiangular with angles arccos 1/5, it is not clear whether L has no norm 2 vectors.

Now suppose that X contains a 6-clique, i.e. six vectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_6 \in X$ with mutual inner product $-1/2$. Put $\boldsymbol{u} = (\boldsymbol{x}_1 + \boldsymbol{x}_2 + \boldsymbol{x}_3)/3$ and $\boldsymbol{v} = (\boldsymbol{x}_4 + \boldsymbol{x}_5 + \boldsymbol{x}_6)/3$. Then $(\boldsymbol{u}, \boldsymbol{u}) = -(\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{v}, \boldsymbol{v}) = 1/2$; in particular $\boldsymbol{v} = -\boldsymbol{u}$. Using $(\boldsymbol{x}_i, \boldsymbol{u}) = 1/2$ for $i \leqslant 3$ and $(\boldsymbol{x}_i, \boldsymbol{u}) = -1/2$ for $i \geqslant 4$, it is easily verified that the vectors $\boldsymbol{r}_i = \boldsymbol{u} - \boldsymbol{x}_i$ ($i = 1, 2, 3$), $\boldsymbol{s}_i = \boldsymbol{u} + \boldsymbol{x}_{i+3}$ ($i = 1, 2, 3$), and $\boldsymbol{t}_1 = \boldsymbol{e}/2 + 4$, $\boldsymbol{t}_2 = \boldsymbol{u} - \boldsymbol{e}/2$, $\boldsymbol{t}_3 = -2\boldsymbol{u}$ form an extended root system of type $\hat{A}_2^3$. Moreover, one easily sees that the 9 roots $\boldsymbol{r}_i, \boldsymbol{s}_i, \boldsymbol{t}_i$ ($i = 1, 2, 3$) belong to the same root class R of L. The remaining members of R are the roots $\boldsymbol{u} \pm \boldsymbol{x}$, where $\boldsymbol{x}$

belongs to the set

$$P := \{x \in X | (x, x_i) = 1 \text{ for } i \leqslant 3, (x, x_i) = -1 \text{ for } i \geqslant 4\}.$$

In the terminology of Lemmens and Seidel [6], P is the *pillar* determined by the sign distribution $\varepsilon = (+++---)$ on the 6-clique X. This explains the occurrence of Dynkin diagrams in the pillar analysis of Lemmens and Seidel. As an example, the pillars of the 276 equiangular lines in $\mathbb{R}^{23}$ constructed by Taylor [10] from the Leech lattice consist of 9 triangles, giving the root class $R = \hat{A}_2^{12}$ and the Leech lattice $L = \Lambda_{24}$. This fact is also related to the uniqueness proof of the corresponding regular twograph by Goethals and Seidel [5].

5. Holes

It remains to consider the question how to find rich root classes of a lattice L. In view of the examples from the Leech lattice is it natural to consider holes.

Let L be an even lattice of dimension n and minimal norm 4, and let f be a hole of radius $R = \sqrt{m}$ nearest to the origin. Then $(f, f) = m$. We call the hole *spherical*, *affine*, or *hyperbolic*, depending on whether $m < 2, m = 2, m > 2$. For example, by Conway and Sloane [3], all holes of the Leech lattice are spherical (small holes) or affine (deepest holes). Since f is a hole, the set Σ of nearest lattice points has $s + 1 \geqslant n + 1$ points $v_i = f - f_i$ $(i = 0, \ldots, s)$, say; thus $f = f_0$ and

$$(f_i, f_i) = m. \tag{5.1}$$

Since f is a hole, f is a strictly convex combination of the v_i, $f = \Sigma\, c_i(f - f_i)$, $c_i > 0$, $\Sigma\, c_i = 1$, so that

$$\sum c_i f_i = 0 \qquad \text{with } c_i > 0, \quad \sum c_i = 1; \tag{5.2}$$

moreover the v_i and hence the f_i span $\mathbb{R}^n$. In particular, if $m = 2$ then $\{f_0, \ldots, f_s\}$ is a rich root class.

Now (5.1) implies

$$(f_i, f_j) = m - \tfrac{1}{2}(f_i - f_j, f_i - f_j); \tag{5.3}$$

therefore, for $i \neq j$, $(f_i, f_j) = m - p$ with an integer $p \geqslant 2$. This implies that the vectors $f_0, \ldots, f_s$ have a Gram matrix of the form

$$\mathbf{G} = 2\mathbf{I} - \mathbf{A} + (m-2)\mathbf{J}, \tag{5.4}$$

where $\mathbf{A} = (a_{ij})$ is an integral nonnegative matrix and $\mathbf{J}$ is the matrix all of whose entries are 1. This allows us to define a multigraph Γ with vertices f_i, where for $i \neq j$, f_i and f_j are joined by a_{ij} edges, and $(f_i, f_j) = m - 2 - a_{ij}$. Clearly $\mathbf{G}$ is positive semidefinite of rank n, and by (5.2), $c = (c_0, \ldots, c_s)^{\mathrm{T}}$ is a positive null vector of $\mathbf{G}$; i.e. with $j = (1, \ldots, 1)^{\mathrm{T}}$, we have

$$\mathbf{G}c = 0, \qquad c > 0, \qquad j^{\mathrm{T}}c = 1. \tag{5.5}$$

We now distinguish several cases.

(a) For a spherical hole we have $m < 2$. Then $2\mathbf{I} - \mathbf{A}$ is positive definite whence $\mathbf{A}$ has largest eigenvalue < 2. Therefore, Γ is a disjoint union of spherical Dynkin diagrams.

(b) For an affine hole we have $m = 2$. Then $2\mathbf{I} - \mathbf{A}$ is positive semidefinite, and (5.5) implies $\mathbf{A}c = 2c > \mathit{0}$. Therefore Γ is a disjoint union of affine Dynkin diagrams.

(c) For a hyperbolic hole we have $m > 2$. Then $\mathbf{A}c > 2c > \mathit{0}$ whence the largest eigenvalue $\lambda_1(\mathbf{A})$ of $\mathbf{A}$ is > 2. But since $\mathbf{G}$ is positive semidefinite and $\mathbf{J}$ has rank 1, $2\mathbf{I} - \mathbf{A} = \mathbf{G} - (m-2)\mathbf{J}$ has inertia $(-1)(+1)^{n-1}0^{s+1-n}$ or $(-1)(+1)^n 0^{s-n}$. Therefore, the second largest eigenvalue $\lambda_2(\mathbf{A})$ of $\mathbf{A}$ is $\leqslant 2$,

$$\lambda_1(\mathbf{A}) > 2 \geqslant \lambda_2(\mathbf{A}). \tag{5.6}$$

In particular, Γ is connected. Since $2\mathbf{I} - \mathbf{A}$ may be regarded as the Gram matrix of a set of norm 2 vectors in hyperbolic space $\mathbb{R}^{n,1}$ we are justified in calling a multigraph Γ whose adjacency matrix $\mathbf{A}$ satisfies (5.6) a *hyperbolic multigraph*; cf. e.g. Maxwell [7], Neumaier [8]. From (5.5) we find the further relations

$$(\mathbf{A} - 2\mathbf{I})c = (m - 2)j, \qquad c > 0, \tag{5.7}$$

$$m = 2 + 1/j^{\mathrm{T}}c. \tag{5.8}$$

The fact that (5.7) must have a positive solution restricts the possibilities for Γ a little; however, e.g. all regular hyperbolic multigraphs with v points and valency k might still occur with $m = 2 + (k - 2)/v$. In particular, many sporadic strongly regular graphs, like the Hoffman–Singleton graph ($v = 50$, $m = 21/10$), the Higman–Sims graph ($v = 100$, $m = 11/5$) and the McLaughlin graph ($v = 275$, $m = 12/5$) are candidates for Γ; cf. [5], [9].

References

1. R. E. Borcherds, The Leech lattice, *Proc. R. Soc. London* **A398** (1985) 365–376.
2. N. Bourbaki, *Groupes et algèbres de Lie*, Hermann, Paris, 1968, ch. IV, V, VI.
3. J. H. Conway and N. J. A. Sloane, The covering radius of the Leech lattice, *Proc. R. Soc. London* **A380** (1982) 261–290.
4. J. H. Conway and N. J. A. Sloane, Twenty-three constructions for the Leech lattice, *Proc. R. Soc. London* **A381** (1982) 275–283.
5. J. M. Goethals and J. J. Seidel, The regular two-graph on 276 vertices, *Discrete Math.* **12** (1975), 143–158.
6. P. W. H. Lemmens and J. J. Seidel, Equiangular lines, *J. Algebra* **24** (1973), 494–512.
7. G. Maxwell, Hyperbolic trees, *J. Algebra* **54** (1978) 46–49.
8. A. Neumaier, The second largest eigenvalue of a tree, *Linear Algebra Appl.* **46** (1982), 9–25.
9. A. Neumaier and J. J. Seidel, Discrete hyperbolic geometry, *Combinatorica* **3** (1983), 219–237.
10. D. E. Taylor, Regular 2-graphs, *Proc. London Math. Soc.* **35** (1977), 257–274.

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