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ABSTRACT

As a continuation of my paper "New techniques for the analysis of linear interval equations" [*Linear Algebra Appl.* 58:273–325 (1984)], the interrelation between interval Gauss elimination and interval iteration is investigated. Main results are a new existence theorem for interval Gauss elimination (in the guise of a perturbation theorem), a convergence and comparison theorem for a general family of interval iteration schemes, and a new method for the calculation of the hull of the solution set of linear interval equations with inverse positive coefficient matrix.

1. INTRODUCTION

This paper continues the study of direct and iterative solution algorithms for linear interval equations begun in Neumaier [10]. The techniques introduced there are used here to investigate the convergence of the interval iteration

$$y^{l+1} := A^G(x + Ey^l) \qquad (l = 0, 1, 2, \dots), \tag{1.1}$$

where $A^G z$ denotes the result of interval Gauss elimination applied to the coefficient matrix A and right-hand side z. Under suitable assumptions the iteration converges and the limit $y = \lim_{l \to \infty} y^l$ is an enclosure for the solution set of the linear interval equation

$$\tilde{B}\tilde{y} = \tilde{x} \qquad (\tilde{B} \in B, \quad \tilde{x} \in x), \tag{1.2}$$

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where $B = A - E$. The special case where $A = L$ is a lower triangular matrix has been treated in [10], thus abstracting common features of the interval versions of iteration schemes named after Richardson, Jacobi, and Gauss and Seidel. Another special case, where A is the midpoint of B, has been suggested (in the more general context of nonlinear equations) by Alefeld and Platzöder [2] in the hope that the iteration (1.1) is useful also in cases where interval Gauss elimination with the matrix B breaks down. However, as a consequence of a perturbation theorem for Gauss elimination (Theorem 4.6) we are able to show that the natural sufficient conditions for convergence of (1.1) already imply that Gauss elimination for $B = A - E$ does not break down.

The theory developed also allows an interesting application to inverse positive matrices. It is shown (Theorem 5.1) that if a system of linear interval equations (1.2) with inverse positive B is preconditioned by multiplication with $C = \tilde{B}^{-1}$ where $\tilde{B} \geqslant \bar{B}$ is still inverse positive—i.e. *not*, as usually recommended (Hansen [7], Neumaier [10]), with the midpoint inverse—then the hull of the solution set of (1.2) remains unchanged. Since under these assumptions CB is an M-matrix, the hull can be computed iteratively. This provides an alternative to the procedure suggested by Beeck [5]. If B itself is already an M-matrix, it is shown that the explicit inversion of $\tilde{A}$ can be avoided (Theorem 5.3).

The paper heavily relies on Neumaier [10], and the reader is assumed to be familiar with that paper. After a review of some terminology introduced in [10] and a number of technical results presented in Section 2, we generalize in Section 3 the convergence theory of [10] to the iteration

$$y^{l+1} := S(x + Ty^l) \qquad (l = 0, 1, 2, \ldots), \tag{1.3}$$

where S and T are sublinear maps. Apart from the iteration (1.1), this also covers a recent iteration procedure of Mayer [8] using incomplete triangular decompositions of an H-matrix. Section 4 then specializes to the iteration (1.1); a convergence theorem, the perturbation theorem mentioned, and a comparison theorem for different splittings $B = A - E = A_0 - E_0$ are derived. Finally, Section 5 treats the inverse positive case.

2. NOTATION AND TECHNICAL RESULTS

We assume the reader to be familiar with Neumaier [10]; we shall make repeated use of notation, concepts, and rules explained there. However, for the convenience of the reader, some basic terminology is reviewed in this section. Moreover, several technical results are proved which simplify the presentation of the material in later sections.

We denote by $\mathbb{R}$, $\mathbb{R}^n$, and $\mathbb{R}^{n\times n}$ the sets of real numbers, n-dimensional column vectors, and $n \times n$ matrices, and by $\mathbb{IR}$, $\mathbb{IR}^n$, and $\mathbb{IR}^{n\times n}$ the sets of real closed intervals, n-dimensional interval vectors, and $n \times n$ interval matrices, respectively. If $x = [\underline{x}, \bar{x}] \in \mathbb{IR}^n$, then we write $\operatorname{int} x := (\underline{x}, \bar{x})$ for the *interior*, $\operatorname{mid} x := \check{x} := (\bar{x} + \underline{x})/2$ for the *midpoint*, $\operatorname{rad} x := \rho(x) := (\bar{x} - \underline{x})/2$ for the *radius*, and $|x| := |\check{x}| + \rho(x) = \max\{|\tilde{x}| \mid \tilde{x} \in x\}$ for the *absolute value* of x; the *distance* of $x, y \in \mathbb{IR}^n$ is defined as the vector $q(x, y) := |\check{x} - \check{y}| + |\rho(x) - \rho(y)| = \sup(|\underline{x} - \underline{y}|, |\bar{x} - \bar{y}|)$. Similar definitions apply to scalar intervals and to interval matrices. The *Ostrowski operator* $\langle \cdot \rangle$ associates with an interval $a \in \mathbb{IR}$ the number $\langle a \rangle := \min\{|\tilde{a}| \mid \tilde{a} \in a\}$ [so that $\langle a \rangle = 0$ if $0 \in a$ and $\langle a \rangle = \min(|\underline{a}|, |\bar{a}|) = |\check{a}| - \rho(a)$ otherwise], and with a matrix $A \in \mathbb{IR}^{n\times n}$ the matrix $A' = \langle A \rangle$ with entries $A'_{ii} := \langle A_{ii} \rangle$, $A'_{ik} := -|A_{ik}|$ for $i \neq k$. The spectral radius of a real matrix $A \in \mathbb{R}^{n\times n}$ is denoted by $\sigma(A)$.

A map $S: \mathbb{IR}^n \to \mathbb{IR}^n$ is called *sublinear* if the axioms

(S1) $x \subseteq y \Rightarrow Sx \subseteq Sy$ (inclusion isotonicity),
(S2) $\alpha \in \mathbb{R} \Rightarrow S(\alpha x) = \alpha(Sx)$ (homogeneity),
(S3) $S(x \pm y) \subseteq Sx \pm Sy$ (subadditivity)

are valid for all $x, y \in \mathbb{IR}^n$. The prime example of a sublinear map is multiplication by an interval matrix $A \in \mathbb{IR}^{n\times n}$, i.e. the map A^M which maps x to Ax. The absolute value of a sublinear map S is the unique nonnegative matrix $|S| \in \mathbb{R}^{n\times n}$ satisfying $Sx = |S|x$ for $x = f^{(i)} := [-e^{(i)}, e^{(i)}]$ ($i = 1, \ldots, n$), where $e^{(i)}$ is the ith column of the identity matrix. A sublinear map is called *normal* if

(S4) $\rho(Sx) \geqslant |S|\rho(x)$

holds for all $x \in \mathbb{IR}^n$; the other condition (S5) required in [10] for normal maps is in fact already a consequence of (S1)–(S3):

LEMMA 2.1. *Let $S: \mathbb{IR}^n \to \mathbb{IR}^n$ be a sublinear map. Then*

$$q(Sx, Sy) \leqslant |S| q(x, y) \qquad \textit{for all} \quad x, y \in \mathbb{IR}^n. \tag{2.1}$$

Moreover, if $B \in \mathbb{R}^{n\times n}$ is a nonnegative matrix such that $q(Sx, Sy) \leqslant Bq(x, y)$ for all $x, y \in \mathbb{IR}^n$, then $|S| \leqslant B$.

Proof. (2.1) was shown in Proposition 3 of Neumaier [11]. To prove the second part we put $x = f^{(i)}$. The assumption then implies $|S|x = |Sx| = q(Sx,0) \leqslant Bq(x,0) = B|x|$; hence the ith column of $|S|$ is majorized by the ith column of B. Since i is arbitrary, $|S| \leqslant B$. ■

Later we shall need several properties of the distance not mentioned in [10].

LEMMA 2.2. *For all $A, B \in \mathbb{IR}^{n\times n}$ we have*

$$B \subseteq A + [-q(B,A), q(B,A)]. \tag{2.2}$$

In particular,

$$|B| \leqslant |A| + q(B,A), \qquad \langle B\rangle \geqslant \langle A\rangle - q(B,A). \tag{2.3}$$

Proof. Put $A' := A + [-q(B,A), q(B,A)]$. Then

$$|\check{B} - \check{A}'| = |\check{B} - \check{A}| = q(B,A) - |\rho(B) - \rho(A)|$$

$$\leqslant q(B,A) + \rho(A) - \rho(B) = \rho(A') - \rho(B),$$

which implies $B \subseteq A'$. Hence (2.2) holds, and (2.3) follows from properties of the absolute value and the Ostrowski operator. ■

LEMMA 2.3. *Let $x, y, y', z \in \mathbb{IR}^n$. Then*

$$q(x, x+z) = |z|, \tag{2.4}$$

$$x \subseteq y \quad \Rightarrow \quad y = x + z \quad \textit{for some } z \in \mathbb{IR}^n, \tag{2.5}$$

$$x \subseteq y \subseteq y' \quad \Rightarrow \quad q(x,y) \leqslant q(x,y'). \tag{2.6}$$

Proof. Put $y := x + z$. Then $\check{y} = \check{x} + \check{z}$, $\rho(y) = \rho(x) + \rho(z)$, whence $q(x,y) = |\check{x} - \check{y}| + (\rho(x) - \rho(y)) = |\check{z}| + \rho(z) = |z|$. This implies (2.4). Now if $x \subseteq y$ then $r := \rho(y) - \rho(x) \geqslant 0$, and $z := \check{y} - \check{x} + [-r, r]$ satisfies $y = x + z$ and $q(x,y) = |z|$. In particular, (2.5) holds. Similarly, if $x \subseteq y'$ then there is $z' \in \mathbb{IR}^n$ with $y = x + z'$ and $q(x,y') = |z'|$. If now $y \subseteq y'$ then $z \subseteq z'$, whence $q(x,y) = |z| \leqslant |z'| = q(x,y')$. Hence (2.6) holds. ■

LEMMA 2.4. *The operator $\beta: \mathbb{IR} \times \mathbb{IR} \to \mathbb{R}$ defined by*

$$\beta(x,y) := |x| + q(x,y) \tag{2.7}$$

has the following properties:

(i) *If $x_l, y_l \in \mathbb{IR}$, $x_l \subseteq y_l$ for $l = 1, \dots, i$, then*

$$\beta(x_1 \cdots x_i, y_1 \cdots y_i) \leqslant \beta(x_1, y_1) \cdots \beta(x_i, y_i). \tag{2.8}$$

(ii) *If $x, y \in \mathbb{IR}$, $x \subseteq y$, and $q(x,y) < \langle x\rangle$, then*

$$\beta(x^{-1}, y^{-1}) \leqslant (\langle x\rangle - q(x,y))^{-1}. \tag{2.9}$$

Proof. (i): By Lemma 2.3 we can find $z_l \in \mathbb{IR}$ such that $y_l = x_l + z_l$, $q(x_l, y_l) = |z_l|$, and hence $\beta(x_l, y_l) = |x_l| + |z_l|$. Since $x_1x_2 \subseteq y_1y_2 = (x_1 + z_1)(x_2 + z_2) \subseteq x_1x_2 + x_1z_2 + z_1x_2 + z_1z_2$, we have

$$\begin{aligned}
\beta(x_1x_2, y_1y_2) &= |x_1x_2| + q(x_1x_2, y_1y_2) \\
&\leqslant |x_1x_2| + q(x_1x_2, x_1x_2 + x_1z_2 + z_1x_2 + z_1z_2) && \text{by (2.6)} \\
&= |x_1x_2| + |x_1z_2 + z_1x_2 + z_1z_2| && \text{by (2.4)} \\
&\leqslant |x_1||x_2| + |x_1||z_2| + |z_1||x_2| + |z_1||z_2| \\
&= (|x_1| + |z_1|)(|x_2| + |z_2|) = \beta(x_1, y_1)\beta(x_2, y_2).
\end{aligned}$$

Now (2.8) follows by induction.

(ii): Put $r := q(x,y)$ and $y' := x + [-r, r]$. By assumption, $0 \notin y'$, whence $\langle y'\rangle = \langle x\rangle - r > 0$. Since $x \subseteq y \subseteq y'$, $x^{-1} \subseteq y^{-1} \subseteq y'^{-1}$, Lemma 2.3 implies

$$\begin{aligned}
q(x^{-1}, y^{-1}) &\leqslant q(x^{-1} y'^{-1}) \\
&= \max\left(|\underline{y}'^{-1} - \underline{x}^{-1}|, |\bar{y}'^{-1} - \bar{x}^{-1}|\right) \\
&= \max\left(\frac{r}{|\underline{x}(\underline{x} - r)|}, \frac{r}{|\bar{x}(\bar{x} - r)|}\right) \leqslant \frac{r}{\langle x\rangle(\langle x\rangle - r)}.
\end{aligned}$$

Therefore

$$\beta(x^{-1}, y^{-1}) = |x^{-1}| + q(x^{-1}, y^{-1})$$
$$\leqslant \frac{1}{\langle x\rangle} + \frac{r}{\langle x\rangle(\langle x\rangle - r)} = \frac{1}{\langle x\rangle - r}. \qquad \blacksquare$$

We shall also need the following results on nonnegative matrices.

LEMMA 2.5. *Let $M, M', \Delta, \Delta' \in \mathbb{R}^{n\times n}$ be nonnegative matrices such that*

$$M' \leqslant (I - M\Delta)^{-1} M, \tag{2.10}$$

and suppose that the spectral radius of $M(\Delta + \Delta')$ is less than one. Then $\sigma(M'\Delta') < 1$ and

$$0 \leqslant (I - M\Delta')^{-1} M' \leqslant (I - M(\Delta + \Delta'))^{-1} M. \tag{2.11}$$

Proof. By assumption there is a vector $u > 0$ such that $M(\Delta + \Delta')u < u$. Since $M\Delta u < u$, $M\Delta$ has spectral radius < 1, and $(I - M\Delta)^{-1} \geqslant 0$. Moreover $M\Delta' u < (I - M\Delta)u$, whence $M'\Delta' u < (I - M\Delta)^{-1} M\Delta' u < u$. Hence $\sigma(M'\Delta') < 1$, too, and $(I - M'\Delta')^{-1} \geqslant 0$. To show (2.11) we introduce the matrix

$$N := (I - M\Delta)^{-1}[I - M(\Delta + \Delta')] = I - (I - M\Delta)^{-1} M\Delta'. \tag{2.12}$$

Clearly $N \leqslant I$ and $Nu = (I - M\Delta)^{-1}(u - M(\Delta + \Delta')u) > 0$. Hence N is an M-matrix; in particular $N^{-1} \geqslant 0$. Now (2.10) and (2.12) imply $N \leqslant I - M'\Delta'$, hence $0 \leqslant (I - M'\Delta')^{-1} \leqslant N^{-1}$, and therefore $0 \leqslant (I - M'\Delta')^{-1} M' \leqslant N^{-1} M' \leqslant N^{-1}(I - M\Delta)^{-1} M = (I - M(\Delta + \Delta'))^{-1} M$. ■

LEMMA 2.6. *Let $A \in \mathbb{IR}^{n\times n}$ and $x \in \mathbb{IR}^n$.*

(i) *If $A \geqslant 0$ then $Ax = [A_1\underline{x}, A_2\bar{x}]$ for suitable $A_1, A_2 \in A$.*
(ii) *If $\underline{A} = 0$ then $Ax = [\bar{A}\inf(\underline{x}, 0), \bar{A}\sup(\bar{x}, 0)]$.*

Proof. (i): With A_1, A_2 defined by

$$(A_1)_{ik} := \underline{A}_{ik} \quad \text{if } \underline{x}_k \geqslant 0, \qquad (A_1)_{ik} := \bar{A}_{ik} \quad \text{otherwise},$$

$$(A_2)_{ik} := \bar{A}_{ik} \quad \text{if } \bar{x}_k \geqslant 0, \qquad (A_2)_{ik} := \underline{A}_{ik} \quad \text{otherwise},$$

we have

$$(Ax)_i = \sum_k [\underline{A}_{ik}, \bar{A}_{ik}][\underline{x}_k, \bar{x}_k]$$
$$= \sum_k [(A_1)_{ik}\underline{x}_k, (A_2)_{ik}\bar{x}_k] = [A_1\underline{x}, A_2\bar{x}]_i.$$

(ii): We have

$$(Ax)_i = \sum_k [0, \bar{A}_{ik}][\underline{x}_k, \bar{x}_k]$$
$$= \sum_k [\bar{A}_{ik}\inf(\underline{x}_k, 0), \bar{A}_{ik}\sup(\bar{x}_k, 0)]$$
$$= [\bar{A}\inf(\underline{x}, 0), \bar{A}\sup(\bar{x}, 0)]_i. \qquad \blacksquare$$

3. SUBLINEAR ITERATION

In this section we consider the fixpoint iteration

$$y^{l+1} := S(x + Ty^l)$$

where S, T are sublinear maps. This generalizes the special iteration

$$y^{l+1} = L^F(x + Ey^l)$$

considered in Section 7 of [10] for triangular splittings $A = L - E$ of an H-matrix $A \in \mathbb{IR}^{n\times n}$, and serves (as in [10]) as a useful general model for a number of iterative procedures for the solution of linear interval equations.

THEOREM 3.1. *Let $S, T: \mathbb{IR}^n \to \mathbb{IR}^n$ be sublinear maps satisfying $\sigma(|S||T|) < 1$. Then, for every $x \in \mathbb{IR}^n$, the following statements hold:*

(i) *The equation*

$$y = S(x + Ty) \tag{3.1}$$

has a unique solution $y \in \mathbb{IR}^n$.

(ii) *For all starting vectors* $y^0 \in \mathbb{IR}^n$, *the iteration*

$$y^{l+1} := \mathrm{S}(x + Ty^l) \qquad (l = 0, 1, \ldots) \tag{3.2}$$

converges to the solution y *of* (3.1), *and*

$$\| q(y^{l+1}, y) \| \leqslant \beta \| q(y^l, y) \| \tag{3.3}$$

for any monotone norm satisfying

$$\| |\mathrm{S}| |T| \| = \beta < 1. \tag{3.4}$$

(iii) *If* $y^1 \subseteq y^0$, *then for all* $i \geqslant 1$,

$$y \subseteq y^i \subseteq y^{i-1} \subseteq \cdots \subseteq y^0. \tag{3.5}$$

(iv) *If* $y^0 \subseteq y^1$, *then for all* $i \geqslant 1$,

$$y^0 \subseteq \cdots \subseteq y^{i-1} \subseteq y^i \subseteq y. \tag{3.6}$$

Proof. Since $\sigma(|\mathrm{S}||T|) < 1$, there exists a monotone norm such that (3.4) holds. For fixed $x \in \mathbb{IR}^n$, the map $\Psi : \mathbb{IR}^n \to \mathbb{IR}^n$ defined by

$$\Psi y := \mathrm{S}(x + Ty)$$

satisfies $q(\Psi y, \Psi z) = q(\mathrm{S}(x + Ty), \mathrm{S}(x + Tz))$

$$\leqslant |\mathrm{S}| q(x + Ty, x + Tz)$$

$$= |\mathrm{S}| q(Ty, Tz) \leqslant |\mathrm{S}||T| q(y, z)$$

by Lemma 2.1, whence by (3.4)

$$\| q(\Psi y, \Psi z) \| \leqslant \beta \| q(y, z) \|.$$

Since $\mathbb{IR}^n$ is a locally complete metric space with respect to the metric $d(y, z) := \|q(y, z)\|$, a generalization of the Banach fixpoint theorem by Schröder [14] shows that Ψ has a unique fixpoint $y \in \mathbb{IR}^n$, and for arbitrary $y^0 \in \mathbb{IR}^n$, the iteration $y^{l+1} = \Psi y^l$, i.e. (3.2), converges to y with speed determined by (3.3). This proves (3.1) and (ii).

For the proof of (iii) we first note that Ψ is inclusion isotone. Hence if $y^l \subseteq y^{l-1}$ then $y^{l+1} = \Psi y^l \subseteq \Psi y^{l-1} = y^l$. So if $y^1 \subseteq y^0$ then $y^l \subseteq y^k$ for all $l \geqslant k$, and for $l \to \infty$ we find $y \subseteq y^k$. This implies (iii). Statement (iv) follows in the same way by reversing the inclusion signs. ∎

Theorem 3.2. *Let* $\mathrm{S}, T : \mathbb{IR}^n \to \mathbb{IR}^n$ *be sublinear maps satisfying* $\sigma(|\mathrm{S}||T|) < 1$. *Then there is a unique map* $P_{\mathrm{S},T} : \mathbb{IR}^n \to \mathbb{IR}^n$ *such that*

$$P_{\mathrm{S},T} x = \mathrm{S}(x + TP_{\mathrm{S},T} x) \qquad \text{for all} \quad x \in \mathbb{IR}^n. \tag{3.7}$$

$P_{\mathrm{S},T}$ *is sublinear with absolute value*

$$|P_{\mathrm{S},T}| \leqslant (I - |\mathrm{S}||T|)^{-1} |\mathrm{S}|. \tag{3.8}$$

Moreover we have the implications

$$\mathrm{S}(x + Ty) \subseteq y \quad \Rightarrow \quad P_{\mathrm{S},T} x \subseteq y, \tag{3.9}$$

$$y \subseteq \mathrm{S}(x + Ty) \quad \Rightarrow \quad y \subseteq P_{\mathrm{S},T} x. \tag{3.10}$$

Proof. The uniqueness of $P_{\mathrm{S},T}$ and (3.9), (3.10) are immediate consequences of Theorem 3.1. We verify the sublinearity axioms for $P_{\mathrm{S},T}$. If $x \subseteq y$ then

$$\mathrm{S}(x + TP_{\mathrm{S},T} y) \subseteq \mathrm{S}(y + TP_{\mathrm{S},T} y) = P_{\mathrm{S},T} y,$$

whence $P_{\mathrm{S},T} x \subseteq P_{\mathrm{S},T} y$ by (3.9). Hence $P_{\mathrm{S},T}$ is inclusion isotone. Homogeneity is immediate. To show subadditivity we put $z := P_{\mathrm{S},T} x \pm P_{\mathrm{S},T} y$, so that

$$\mathrm{S}(x \pm y \pm Tz) \subseteq \mathrm{S}(x \pm y + TP_{\mathrm{S},T} x \pm TP_{\mathrm{S},T} y)$$

$$\subseteq \mathrm{S}(x + TP_{\mathrm{S},T} x) \pm \mathrm{S}(y + TP_{\mathrm{S},T} y)$$

$$= P_{\mathrm{S},T} x \pm P_{\mathrm{S},T} y = z.$$

Now (3.9) implies $P_{\mathrm{S},T}(x \pm y) \subseteq z = P_{\mathrm{S},T} x \pm P_{\mathrm{S},T} y$. Therefore $P_{\mathrm{S},T}$ is sublinear. Finally

$$q(P_{S,T}x, P_{S,T}y) = q(S(x + TP_{S,T}x), S(y + TP_{S,T}y))$$
$$\leq |S| q(x + TP_{S,T}x, y + TP_{S,T}y)$$
$$\leq |S| (q(x, y) + q(TP_{S,T}x, TP_{S,T}y))$$
$$\leq |S| q(x, y) + |S||T| q(P_{S,T}x, P_{S,T}y)$$

by Lemma 2.1, so that

$$(I - |S||T|) q(P_{S,T}x, P_{S,T}y) \leq |S| q(x, y).$$

Multiplication with the nonnegative matrix $(I - |S||T|)^{-1}$ and application of the second part of Lemma 2.1 now implies (3.8). ■

We shall refer to $P_{S,T}$ as the *fixpoint map* of the iteration (3.2). As an easily verifiable existence condition for $P_{S,T}$ we prove:

PROPOSITION 3.3. *Let S and T be normal sublinear maps, and let* $x, y \in \mathbb{IR}^n$.

(i) *If*

$$\rho(S(x + Ty)) < \rho(y) \tag{3.11}$$

then $\sigma(|S||T|) < 1$ *and* $P_{S,T}$ *exists.*

(ii) *If*

$$S(x + Ty) \subseteq \operatorname{int} y \tag{3.12}$$

then $\sigma(|S||T|) < 1$, $P_{S,T}$ *exists, and*

$$P_{S,T}x \subseteq \operatorname{int} y.$$

Proof. Since S, T are normal, (3.11) implies

$$\rho(y) > \rho(S(x + Ty)) \geq |S| \rho(x + Ty) = |S| (\rho(x) + \rho(Ty))$$
$$\geq |S| \rho(Ty) \geq |S||T| \rho(y) \geq 0.$$

Hence $u := \rho(y) > 0$ satisfies $|S||T|u < u$, so that $\sigma(|S||T|) < 1$, and $P_{S,T}$ exists by Theorem 3.2. Moreover, (3.12) clearly implies (3.11); hence by Theorem 3.2 we have $P_{S,T}x \subseteq y$ and $P_{S,T}x = S(x + TP_{S,T}x) \subseteq S(x + Ty) \subseteq \operatorname{int} y$. ■

PROPOSITION 3.4. *If S and T are normal sublinear maps satisfying* $\sigma(|S||T|) < 1$, *then* $P_{S,T}$ *is normal, too, and (3.8) holds with equality.*

Proof. Apply Proposition 3 of [10] with $R = P_{S,T}$. ■

REMARK. If S, T are normal sublinear maps but $\sigma(|S||T|) = \sigma \geq 1$, then the above results are no longer valid. Indeed, let u be a Perron vector for $|S||T|$, i.e. $0 \neq u \geq 0$, $|S||T|u = \sigma u$. If $\sigma > 1$, then the iteration (3.2) need not converge: If $x = 0$, $y^0 = [-u, u]$, then by rule (R8) of [10] we have $y^l = \sigma^l[-u, u]$ for all $l \geq 0$, and since $\sigma > 1$, no limit exists. And if $\sigma = 1$, then at least the uniqueness of (3.1) is lost: For $x = 0$, each $y = \alpha[-u, u]$ with $\alpha \geq 0$ is a solution.

As the special cases discussed in [10], the above results are useful for the study of iteration methods for the solution of linear interval equations. Let $\Sigma \subseteq \mathbb{R}^{n \times n}$ be a set of $n \times n$ matrices; Σ is called *regular* if all $\tilde{A} \in \Sigma$ are regular. In this case we say that a map $\Sigma^I : \mathbb{IR}^n \to \mathbb{IR}^n$ is an *inverse* of Σ if Σ^I is sublinear and

$$\tilde{A}^{-1}\tilde{x} \in \Sigma^I x \qquad \text{for all} \quad \tilde{A} \in \Sigma, \quad \tilde{x} \in x \in \mathbb{IR}^n. \tag{3.13}$$

If $\Sigma = A \in \mathbb{IR}^{n \times n}$, then these definitions reduce to those given in Neumaier [11].

THEOREM 3.5. *Let* $\Sigma^I := \mathbb{IR}^n \to \mathbb{IR}^n$ *be an inverse of the regular matrix set* $\Sigma \subseteq \mathbb{R}^{n \times n}$, *and let* $E \in \mathbb{IR}^{n \times n}$ *be such that*

$$\sigma(|\Sigma^I||E|) < 1. \tag{3.14}$$

Then the set

$$\Sigma - E := \{\tilde{A} - E \mid \tilde{A} \in \Sigma, \tilde{E} \in E\}$$

is regular, and the fixpoint map P of the iteration

$$y^{l+1} := \Sigma^I(x + Ey^l) \qquad (l = 0, 1, 2, \ldots) \tag{3.15}$$

is an inverse of $\Sigma - E$.

Proof. (3.14) implies the existence of $u > 0$ with $|\Sigma^I||E|u < u$. If $\tilde{A} \in \Sigma$, $\tilde{E} \in E$, then $|\tilde{A}^{-1}\tilde{E}|u \leqslant |\tilde{A}^{-1}||\tilde{E}|u \leqslant |\Sigma^I||E|u < u$, whence $\|\tilde{A}^{-1}\tilde{E}\|_u < 1$. Therefore $I - \tilde{A}^{-1}\tilde{E}$ and hence $\tilde{A} - \tilde{E} = \tilde{A}(I - \tilde{A}^{-1}\tilde{E})$ are regular, whence also $\Sigma - E$ is regular. Now let P denote the fixpoint map of (3.15). If $\tilde{x} \in x \in \mathbb{IR}^n$ and $\tilde{y} := (\tilde{A} - \tilde{E})^{-1}\tilde{x}$, then $\tilde{x} = \tilde{A}\tilde{y} - \tilde{E}\tilde{y}$, whence $\tilde{y} = \tilde{A}^{-1}(\tilde{x} + \tilde{E}\tilde{y}) \in \Sigma^I(\tilde{x} + E\tilde{y})$. Therefore (3.10) implies $\tilde{y} \in Px$. Hence $(\tilde{A} - \tilde{E})^{-1}\tilde{x} \in Px$ for all $\tilde{A} \in \Sigma$, $\tilde{E} \in E$, $\tilde{x} \in x$, and P is an inverse of $\Sigma - E$. ∎

In the special case where $\Sigma = L$ is a lower triangular interval matrix and $\Sigma^I = L^F$ is forward elimination, the preceding results just reduce to the discussion in [10] of fixpoint iteration with strong triangular splittings. Another case was recently considered by Mayer [8]. He shows how to construct certain incomplete factorizations of an interval H-matrix A, resulting in a unit lower triangular matrix $L \in \mathbb{IR}^{n\times n}$, a nonsingular upper triangular matrix $R \in \mathbb{IR}^{n\times n}$, and an error matrix $E \in \mathbb{IR}^{n\times n}$ such that every $\tilde{A} \in A$ has a decomposition $\tilde{A} = \tilde{L}\tilde{R} - \tilde{E}$ with suitable $\tilde{L} \in L$, $\tilde{R} \in R$, $\tilde{E} \in E$; his convergence theorem is thus the case $\Sigma = \{\tilde{L}\tilde{R} \mid \tilde{L} \in L,\ \tilde{R} \in R\}$, $\Sigma^I = R^F L^F$ of Theorems 3.5 and 3.1 above. In the next section we shall discuss another interesting case.

4. FIXPOINT ITERATION AND GAUSS ELIMINATION

If $A \in \mathbb{IR}^{n\times n}$ has a (nonsingular) triangular decomposition (L_A, R_A), then, as in [10], we denote by $A^G x$ the result of Gauss elimination applied to the coefficient matrix A and the right-hand side $x \in \mathbb{IR}^n$. By [10, Section 5], A^G is a normal sublinear map with absolute value

$$|A^G| = \langle R_A\rangle^{-1}\langle L_A\rangle^{-1}. \tag{4.1}$$

PROPOSITION 4.1. *Let $A, B \in \mathbb{IR}^{n\times n}$. If A^G exists and $B \subseteq A$, then B^G exists and*

$$B^G x \subseteq A^G x \qquad \textit{for all} \quad x \in \mathbb{IR}^n; \tag{4.2}$$

in particular,

$$|B^G| \leqslant |A^G|. \tag{4.3}$$

Proof. Since Gauss elimination is defined in terms of inclusion isotone arithmetic operations, B^G exists and (4.2) holds. By applying (4.2) to $x = f^{(i)}$ $(i = 1, \ldots, n)$ we arrive at (4.3). ∎

LEMMA 4.2. *If $A \in \mathbb{IR}^{n\times n}$ has a triangular decomposition (L_A, R_A), then there are $u, v \in \mathbb{R}^n$ such that*

$$0 < u, \qquad 0 \leqslant v \leqslant \langle L_A\rangle\langle R_A\rangle u, \tag{4.4}$$

and for any such u, v we have

$$|A^G|v \leqslant u. \tag{4.5}$$

Proof. Let $v > 0$, and put $u := |A^G|v$. Then $u > 0$, since no row of $|A^G|$ consists of zeros only, and $\langle L_A\rangle\langle R_A\rangle u = v$. Hence (4.4) holds. Conversely, (4.4) implies $|A^G|v \leqslant |A^G|\langle L_A\rangle\langle R_A\rangle u = u$. ∎

We now consider the special case of Theorem 4.3 applied to $\Sigma = A$ and $\Sigma^I = A^G$.

THEOREM 4.3. *Let $A, B \in \mathbb{IR}^{n\times n}$, and suppose that A has a triangular decomposition (L_A, R_A) and $B = A - E$. If*

$$\sigma(|A^G||E|) < 1, \tag{4.6}$$

equivalently if there is $u \in \mathbb{R}^n$ such that

$$0 < u, \qquad |E|u < \langle L_A\rangle\langle R_A\rangle u, \tag{4.7}$$

then B is regular, and the fixpoint map P of the iteration

$$y^{l+1} := A^G(x + Ey^l) \qquad (l = 0, 1, 2, \ldots) \tag{4.8}$$

is an inverse of B. Moreover, P is normal and

$$|P| = (I - |A^G||E|)^{-1}|A^G| = (\langle L_A\rangle\langle R_A\rangle - |E|)^{-1}. \tag{4.9}$$

Proof. We begin with showing the equivalence of (4.6) and (4.7). If (4.6) holds, then by continuity of the spectral radius there is a matrix $\Delta > 0$ such that $|A^G|(|E| + \Delta)$ still has spectral radius < 1. Hence there is a vector $w > 0$ such that $u := |A^G|(|E| + \Delta)w < w$ and therefore $|E|u \leqslant |E|w < (|E| + \Delta)w = \langle L_A\rangle\langle R_A\rangle u$. Conversely, if (4.7) holds, then $|A^G||E|u < |A^G|\langle L_A\rangle\langle R_A\rangle u = u$, so that $\sigma(|A^G||E|) < 1$. Now by Theorem 3.5, the condition (4.6) implies that P is an inverse of B, and since $S := A^G$ and $T := E^M$ are normal maps, Proposition 3.4 implies that

$$|P| = \left(I - |A^G||E^M|\right)^{-1}|A^G| = \left(I - |A^G||E|\right)^{-1}|A^G|$$

$$= \left(I - \langle R_A\rangle^{-1}\langle L_A\rangle^{-1}|E|\right)^{-1}\langle R_A\rangle^{-1}\langle L_A\rangle^{-1}$$

$$= \left(\langle L_A\rangle\langle R_A\rangle - |E|\right)^{-1}. \qquad ■$$

REMARK. If $A = L$ is lower triangular, then the iteration (4.8) reduces to the iteration $y^{l+1} := L^F(x + Ey^l)$ considered in Section 7 of [10]. Indeed, it is not difficult to show that if A is regular and lower triangular, then A^G exists and agrees with the forward substitution map A^F. This is trivial if A has a unit diagonal; in general the triangular decomposition is given by $R_A := \operatorname{Diag}(A_{11}, \dots, A_{nn})$,

$$(L_A)_{ik} := \begin{cases} A_{ik}A_{kk}^{-1} & \text{for} \quad i > k, \\ 1 & \text{for} \quad i = k, \\ 0 & \text{for} \quad i < k \end{cases}$$

(note that $a/b = ab^{-1} = b^{-1}a$ for $a, b \in \mathbb{IR}$, $0 \notin b$). Now suppose that $y := A^F x$ and $z := L_A^F x$ satisfy

$$y_k = z_k / A_{kk} = A_{kk}^{-1} z_k \qquad \text{for} \quad k < i;$$

this certainly holds for $i = 1$. Then

$$y_i = \frac{x_i - \sum_{k<i} A_{ik} y_k}{A_{ii}}$$

$$= \frac{x_i - \sum_{k<i} A_{ik} A_{kk}^{-1} z_k}{A_{ii}}$$

$$= \frac{x_i - \sum_{k<i} (L_A)_{ik} z_k}{A_{ii}} = \frac{z_i}{A_{ii}},$$

since multiplication of intervals is associative. By induction we find $y_i = z_i / A_{ii}$ for $i = 1, \dots, n$, whence

$$y = R_A^F z = R_A^F L_A^F x = A^G x.$$

COROLLARY 4.4. *Let $A, B \in \mathbb{IR}^{n\times n}$, and suppose that A^G exists and $B = A - E$. If there are $x, y \in \mathbb{IR}^n$ such that*

$$\rho\left(A^G(x + Ey)\right) < \rho(y)$$

then $\sigma(|A^G||E|) < 1$ and the iteration (4.8) converges.

Proof. Apply Proposition 3.3(i). ■

In order to prove a comparison theorem for different splittings $B = A - E = A_0 - E_0$ we need a perturbation theorem for Gauss elimination which is also of independent interest. By continuity of Gauss elimination as a function of the matrix coefficients it is clear that the existence of A^G implies the existence of B^G for $B \in \mathbb{IR}^{n\times n}$ sufficiently close to A. We shall prove that, in this sense, B is sufficiently close to A if (4.6) or (4.7) holds. The proof is based on the following result.

PROPOSITION 4.5. *Let (L, R) be a triangular decomposition of $A \in \mathbb{IR}^{n\times n}$, and let $A' \supseteq A$ be such that for suitable $u \geqslant 0$, $v > 0$ we have*

$$q(A, A')u \leqslant \langle L\rangle\langle R\rangle u - v.$$

Then A' has a triangular decomposition (L', R'), and

$$\langle L'\rangle\langle R'\rangle u \geqslant v.$$

Proof. This is trivial for $n = 1$; hence we proceed by induction on n and assume the statement to be true for some dimension n. Let $A_0 \in \mathbb{IR}^{(n+1)\times(n+1)}$ have the triangular decomposition (L_0, R_0). Then

$$A_0 = \begin{pmatrix} \alpha & a^T \\ b & B \end{pmatrix}, \qquad L_0 = \begin{pmatrix} 1 & 0 \\ b\alpha^{-1} & L \end{pmatrix}, \qquad R_0 = \begin{pmatrix} \alpha & a^T \\ 0 & R \end{pmatrix},$$

where (L, R) is the triangular decomposition of the Schur complement

$$A := B - b\alpha^{-1}a^T.$$

Let

$$A_0' = \begin{pmatrix} \alpha' & a'^T \\ b' & B' \end{pmatrix}, \qquad u_0 = \begin{pmatrix} \mu \\ u \end{pmatrix} \geqslant 0, \qquad v_0 = \begin{pmatrix} \nu \\ v \end{pmatrix} > 0,$$

$$q(A_0, A_0') = \begin{pmatrix} \varepsilon & e^T \\ f & E \end{pmatrix}$$

be such that $A_0' \supseteq A_0$ and

$$q(A_0, A_0')u_0 \leqslant \langle L_0\rangle\langle R_0\rangle u_0 - v_0.$$

Then

$$\varepsilon\mu + e^T u \leqslant \langle\alpha\rangle\mu - |a|^T u - \nu, \tag{4.10}$$

$$f\mu + Eu \leqslant \langle L\rangle\langle R\rangle u - |b|\langle\alpha\rangle^{-1}(\langle\alpha\rangle\mu - |a|^T u) - v; \tag{4.11}$$

in particular $\mu(\langle\alpha\rangle - \varepsilon) > 0$ and hence

$$\varepsilon < \langle\alpha\rangle.$$

We compute a bound for the distance between A and the Schur complement

$$A' := B' - b'\alpha'^{-1}a'^T$$

of A_0'. Since $A_0' \supseteq A_0$, Lemma 2.4(i) implies

$$\begin{aligned} q(A, A') &= q(B, B') + q(b\alpha^{-1}a^T, b'\alpha'^{-1}a'^T) \\ &= q(B, B') - |b|\langle\alpha\rangle^{-1}|a|^T + \beta(b\alpha^{-1}a^T, b'\alpha'^{-1}a'^T) \\ &\leqslant q(B, B') - |b|\langle\alpha\rangle^{-1}|a|^T + \beta(b, b')\beta(\alpha^{-1}, \alpha'^{-1})\beta(a, a')^T; \end{aligned}$$

here β is understood componentwise (note that each component of $b\alpha^{-1}a^T$ is a product). Hence by Lemma 2.4(ii),

$$q(A, A') \leqslant E - |b|\langle\alpha\rangle^{-1}|a|^T + (|b| + f)(\langle\alpha\rangle - \varepsilon)^{-1}(|a| + e)^T$$

and by (4.11) and (4.10) therefore

$$\begin{aligned} q(A, A')u &\leqslant Eu - |b|\langle\alpha\rangle^{-1}|a|^T u + (|b| + f)(\langle\alpha\rangle - \varepsilon)^{-1}(|a| + e)^T u \\ &\leqslant \langle L\rangle\langle R\rangle u - v - (|b| + f)\mu + (|b| + f)(\langle\alpha\rangle - \varepsilon)^{-1}(|a|^T u + e^T u) \\ &\leqslant \langle L\rangle\langle R\rangle u - v - (|b| + f)(\langle\alpha\rangle - \varepsilon)^{-1}\nu. \end{aligned}$$

By our induction hypothesis, A' has a triangular decomposition (L', R') with

$$\langle L'\rangle\langle R'\rangle u \geqslant v + (|b| + f)(\langle\alpha\rangle - \varepsilon)^{-1}\nu.$$

Therefore, with

$$L_0' := \begin{pmatrix} 1 & 0 \\ b'\alpha'^{-1} & L \end{pmatrix}, \qquad R_0' := \begin{pmatrix} \alpha & a'^T \\ 0 & R' \end{pmatrix},$$

(L_0', R_0') is a triangular decomposition of A_0', and

$$\begin{aligned} \langle L_0'\rangle\langle R_0'\rangle u_0 &\geqslant \begin{pmatrix} (\langle\alpha\rangle - \varepsilon)\mu - (|a| + e)^T u \\ \langle L'\rangle\langle R'\rangle u - (|b| + f)(\langle\alpha\rangle - \varepsilon)^{-1}\nu \end{pmatrix} \\ &\geqslant \begin{pmatrix} \nu \\ v \end{pmatrix} = v_0, \end{aligned}$$

which completes the induction. ■

THEOREM 4.6. *Let $A, B \in \mathbb{IR}^{n\times n}$, and suppose that A^G exists. If*

$$\sigma(|A^G|q(B, A)) < 1, \tag{4.12}$$

or equivalently, if there is $u \in \mathbb{R}^n$ such that

$$0 < u,\ q(B, A)u < \langle L_A\rangle\langle R_A\rangle u, \tag{4.13}$$

where (L_A, R_A) is the triangular decomposition of A, then B^G exists and

$$|B^G| \leqslant (I - |A^G|q(B, A))^{-1}|A^G| = (\langle L_A\rangle\langle R_A\rangle - q(B, A))^{-1}. \tag{4.14}$$

Proof. By the argument given in the proof of Theorem 4.3, the conditions (4.12) and (4.13) are equivalent. Hence suppose that (4.12) holds. Let us put

$$\Delta := q(B, A), \qquad A' := A + [-\Delta, \Delta].$$

Then

$$A \subseteq A', \qquad q(A, A') = \Delta.$$

By (4.12), $I - |A^G|\Delta$ has a nonnegative inverse, whence for arbitrary $v > 0$ we have

$$u := (I - |A^G|\Delta)^{-1}|A^G|v \geqslant 0,$$

$$|A^G|\Delta u = u - |A^G|v,$$

$$\Delta u = \langle L_A\rangle\langle R_A\rangle|A^G|\Delta u = \langle L_A\rangle\langle R_A\rangle u - v.$$

Hence, by Proposition 4.5, A' has a triangular decomposition (L', R'), and $\langle L'\rangle\langle R'\rangle u \geqslant v$; in particular, A'^G exists. But by Lemma 2.2 we have $B \subseteq A'$, whence by Proposition 4.1, B^G exists and $|B^G| \leqslant |A'^G|$. Hence

$$|B^G|v \leqslant |A'^G|\langle L'\rangle\langle R'\rangle u = u = (I - |A^G|\Delta)^{-1}|A^G|v.$$

Since $v > 0$ was arbitrary, this implies

$$|B^G| \leqslant (I - |A^G|\Delta)^{-1}|A^G| = (\langle L_A\rangle\langle R_A\rangle - \Delta)^{-1}.$$

Now (4.14) follows, since $\Delta = q(B, A)$. ∎

REMARK. Since $q(B, \check{B}) = \rho(B)$, the theorem implies that B^G exists whenever $|\check{B}^G|\rho(B)$ has spectral radius less than one. This sufficient criterion should be compared with the well-known sufficient criterion $\sigma(|\check{B}^{-1}|\rho(B)) < 1$ for the regularity of $B \in \mathbb{IR}^{n\times n}$. But note that B^G may exists even when $\sigma(|\check{B}^G|\rho(B)) \geqslant 1$; an example is provided by

$$B = \begin{pmatrix} [1,3] & 4 \\ 4 & 4 \end{pmatrix}.$$

COROLLARY 4.7. *Let $A, B \in \mathbb{IR}^{n\times n}$ and suppose that A^G exists. If*

$$\operatorname{rad}(A^G(q(B, A)z)) < \rho(z) \tag{4.15}$$

for some $z \in \mathbb{IR}^n$, then B^G exists.

Proof. Put $E := [-q(B, A), q(B, A)]$, $x := 0$, $y := z$, and apply Proposition 3.3(i) with $S = A^G$, $T = E^M$. We get $\sigma(|A^G|q(B, A)) = \sigma(|S||T|) < 1$, so that Theorem 4.5 applies. ∎

COROLLARY 4.8. *Under the assumptions of Theorem 4.3, B^G exists and $|B^G| \leqslant |P|$.*

Proof. By Lemma 2.3, applied componentwise, $B = A - E$ implies $q(A, B) = |E|$. Therefore (4.6) and (4.12) are equivalent, and (4.9) and (4.14) show that $|B^G| \leqslant |P|$. ∎

REMARKS.

(1) The remark after Proposition 3.5 now implies that if B^G does not exist, then either the iteration (4.8) may not converge, or the uniqueness of the fixpoint may be lost.

(2) The iteration (4.8) with $A = \check{B}$, $E = [-\rho(B), \rho(B)]$ occurs implicitly in the more general context of nonlinear equations in papers by Alefeld and Platzöder [2] and Schrempp [13]. One hope was that, based on the fact that a nonsingular triangular decomposition of B exists almost certainly, a broader class of equations could be solved with (4.8) than with standard interval Gauss elimination. However, Corollary 4.8 shows that this hope is not justified. It implies that in fact the domain of applicability of Gauss elimination is at least as large as that of any iteration of the form (4.8). Moreover, for symmetric right-hand sides ($\check{x} = 0$), where $B^G x = |B^G|x$, $Px = |P|x$ (since B^G and P are normal), no iteration of the form (4.8) can give a sharper result than Gauss elimination. Thus, in general, Gauss elimination has a remarkable superiority over iterative methods.

(3) However, if B is an M-matrix and A is chosen suitably, then we may have equality in Corollary 4.8, and iteration may give more accurate results. Indeed, it is even possible to get by iteration the hull of the solution set; see Theorem 5.3 below.

(4) Note also that an inequality very similar to (4.15) plays a key role in the convergence proof of Alefeld and Platzöder [2].

COROLLARY 4.9 (Alefeld [1], Neumaier [10]). *If $B \in \mathbb{IR}^{n\times n}$ is an H-matrix, then B^G exists and $|B^G| \leqslant \langle B\rangle^{-1}$.*

Proof. Let $B = D - E$ be the Jacobi splitting of B. If B is an H-matrix, then D is a regular diagonal matrix and $|D^G||E| = \langle D\rangle^{-1}|E|$ has spectral radius <1. Hence Theorem 4.6 applies with $A = D$, $q(B, A) = |E|$. Therefore B^G exists and

$$|B^G| \leqslant (\langle L_A\rangle\langle R_A\rangle - q(B,A))^{-1} = (\langle D\rangle - |E|)^{-1} = \langle B\rangle^{-1}. \qquad ■$$

Finally we present the promised comparison theorem; note that $q(B, A) = |E|$ if $B = A - E$.

THEOREM 4.10. *Let $A, B \in \mathbb{IR}^{n\times n}$, and suppose that A^G exists and*

$$\sigma(|A^G|q(B,A)) < 1.$$

If A_0 is "between" B and A, i.e., if

$$q(B, A_0) + q(A_0, A) = q(B, A),$$

then A_0^G exists and

$$[I - |A_0^G|q(B, A_0)]^{-1}|A_0^G| \leqslant [I - |A^G|q(B,A)]^{-1}|A^G|.$$

Proof. The assumption implies that $|A^G|q(A_0, A) \leqslant |A^G|q(B, A)$. Hence $\sigma(|A^G|q(A_0, A)) < 1$, and A_0^G exists by Theorem 4.6. Now apply Lemma 2.5 with $M := |A^G|$, $M' := |A_0^G|$, $\Delta := q(A_0, A)$ and $\Delta' := q(B, A_0)$. ■

COROLLARY 4.11. *Suppose that $B \in \mathbb{IR}^{n\times n}$ is an H-matrix, and let $B = A - E$ be a direct splitting of B. Then A^G exists and*

$$|B^G| \leqslant (I - |A^G||E|)^{-1}|A^G| \leqslant \langle B\rangle^{-1}.$$

Proof. If $u > 0$ satisfies $\langle B\rangle u > 0$, then $\langle B\rangle = \langle A\rangle - |E|$ implies $\langle A\rangle u > 0$. Hence A is an H-matrix, A^G exists, and the left-hand inequality follows from Theorem 4.5. The other part follows from Theorem 4.10 by comparing $B = A - E$ with the Jacobi splitting; cf. the proof of Corollary 4.9. ■

5. THE INVERSE POSITIVE CASE

Let $A \in \mathbb{IR}^{n\times n}$ be regular. Then, as discussed in [10], the hull inverse A^H is the sublinear map which maps $x \in \mathbb{IR}^n$ to the hull

$$A^H x := \square\{\tilde{A}^{-1}\tilde{x} \mid \tilde{A} \in A,\ \tilde{x} \in x\}$$

of the solution set of the system of linear interval equations

$$\tilde{A}\tilde{y} = \tilde{x} \qquad (\tilde{A} \in A,\quad \tilde{x} \in x).$$

Here we show that in case

$$A^{-1} := \square\{\tilde{A}^{-1} \mid \tilde{A} \in A\}$$

is nonnegative, the hull inverse can be obtained as a fixpoint map. This gives a new method for the computation of the hull of the solution set of linear interval equations with inverse positive coefficient matrix; cf. Beeck [5] for previous approaches. We also show that preconditioning of such equations with certain nonnegative matrices does not change the hull of the solution set. Finally, the special situation of M-matrices is considered.

THEOREM 5.1. *Let $A \in \mathbb{IR}^{n\times n}$ be such that $\underline{A}^{-1} \geqslant 0$, and suppose that $\tilde{A} \in \mathbb{R}^{n\times n}$ satisfies*

$$\tilde{A} \geqslant \overline{A}, \qquad C := \tilde{A}^{-1} \geqslant 0. \tag{5.1}$$

Then:

(i) *A is regular, $A^{-1} \geqslant 0$, and the hull inverse A^H is the fixpoint map of the iterations*

$$y^{l+1} := Cx + (I - CA)y^l \qquad (l = 0,1,2,\ldots) \tag{5.2}$$

and

$$y^{l+1} := \tilde{A}^{-1}[x + (\tilde{A} - A)y^l] \qquad (l = 0,1,2,\ldots). \tag{5.3}$$

(ii) *CA is an M-matrix, and for all $x \in \mathbb{IR}^n$,*

$$A^H x = (CA)^H(Cx) = (CA)^F(Cx). \tag{5.4}$$

Proof. Since $A \subseteq [\underline{A}, \tilde{A}]$, Lemma 12 of [10] implies $A^{-1} \geqslant 0$. To show the existence of the fixpoint maps of (5.2) and (5.3) we put

$$\Delta := I - CA = \tilde{A}^{-1}\tilde{A} - \tilde{A}^{-1}A = \tilde{A}^{-1}(\tilde{A} - A) \geqslant 0$$

by rule (B20) of [10]. Pick $v > 0$. Then $u := \underline{A}^{-1}v > 0$, since no row of $\underline{A}^{-1}$ consists of zeros only. Hence $|\Delta|u = \tilde{A}^{-1}(\tilde{A} - \underline{A})u = u - \tilde{A}^{-1}\underline{A}u = u - \tilde{A}^{-1}v < 0$, so that $\sigma(|\Delta|) < 1$. Since $CA = I - \Delta \leqslant I$ and $\langle CA \rangle u \geqslant u - |\Delta|u > 0$, CA is an M-matrix; in particular CA and hence A are regular. Now $CA = I - \Delta$ is a strong splitting of CA; hence by Theorem 8 of [10], the fixpoint map P of (5.2) exists, and by Proposition 12 of [10],

$$(CA)^F(C\mathbf{x}) \subseteq P\mathbf{x} \qquad \text{for all} \quad \mathbf{x} \in \mathbb{IR}^n. \tag{5.5}$$

Moreover, since $|\tilde{A}^{-1}||\tilde{A} - A| = \tilde{A}^{-1}(\tilde{A} - \underline{A}) = |\Delta|$, Theorem 3.5 applies with $\Sigma := \{\tilde{A}\}$, $\Sigma^I := (\tilde{A}^{-1})^M$, $E := \tilde{A} - A$ and shows that the fixpoint map P' of (5.3) exists.

Now suppose that $\mathbf{x} \in \mathbb{IR}^n$, and put $y := P'\mathbf{x}$. By rules (B20) and (B22) of [10],

$$y = \tilde{A}^{-1}[\mathbf{x} + (\tilde{A} - A)y] = \tilde{A}^{-1}\mathbf{x} + \tilde{A}^{-1}[(\tilde{A} - A)y]$$
$$\supseteq \tilde{A}^{-1}\mathbf{x} + [\tilde{A}^{-1}(\tilde{A} - A)]y = C\mathbf{x} + \Delta y = C\mathbf{x} + (I - CA)y,$$

whence by Theorem 8(iii) of [10],

$$P\mathbf{x} \subseteq y = P'\mathbf{x}. \tag{5.6}$$

By definition of y we have $y = \tilde{A}^{-1}(\mathbf{x} + Ey)$, whence $E = \tilde{A} - A \geqslant 0$. By Lemma 2.6 above

$$y = \tilde{A}^{-1}[\underline{x} + E_1\underline{y}, \bar{x} + E_2\bar{y}] = [\tilde{A}^{-1}(\underline{x} + E_1\underline{y}), \tilde{A}^{-1}(\bar{x} + E_2\bar{y})]$$

for suitable $E_1, E_2 \in E$ (note that $\tilde{A}^{-1} \geqslant 0$ is thin). Now $\tilde{A}_1 := \tilde{A} - E_1$ and $\tilde{A}_2 := \tilde{A} - E_2$ belong to $\tilde{A} - E = A$, hence are invertible, and

$$\tilde{A}_1\underline{y} = \tilde{A}\underline{y} - E_1\underline{y} = \underline{x}, \qquad \tilde{A}_2\bar{y} = \tilde{A}\bar{y} - E_2\bar{y} = \bar{x}.$$

Therefore $\underline{y} = \tilde{A}_1^{-1}\underline{x}$, $\bar{y} = \tilde{A}_2^{-1}\bar{x}$ belong to $A^H\mathbf{x}$, whence

$$P'\mathbf{x} = y \subseteq A^H\mathbf{x}. \tag{5.7}$$

Since $A^H\mathbf{x} \subseteq (CA)^H(C\mathbf{x}) \subseteq (CA)^F(C\mathbf{x})$, we see from (5.5), (5.6), and (5.7) that

$$A^H\mathbf{x} = (CA)^H(C\mathbf{x}) = (CA)^F(C\mathbf{x}) = P\mathbf{x} = P'\mathbf{x}.$$

Since $\mathbf{x}$ was arbitrary, $A^H = P = P'$, and (5.4) holds. ∎

REMARKS.

(1) If $A \in \mathbb{IR}^{n \times n}$ is regular and $A^{-1} \geqslant 0$, then the assumptions of Theorem 5.1 are certainly satisfied with $\tilde{A} = \bar{A}$. However, for practical reasons (to simplify the inversion of $\tilde{A}$) a matrix $\tilde{A} \geqslant \bar{A}$ with more zero entries than $\bar{A}$ may be used as long as $\tilde{A}^{-1} \geqslant 0$.

(2) The iteration (5.3) is not suited to practical calculation, since an exact inverse $\tilde{A}^{-1}$, or an enclosure of it, is needed. In the presence of rounding errors one gets, however, only an approximation $C \approx \tilde{A}^{-1}$. The iteration (5.2) still works in this situation: if z is an enclosure of the fixpoint of (5.2), or an enclosure of $(CA)^F(C\mathbf{x})$ obtained by preconditioned Gauss-Seidel iteration, then we still have $A^H\mathbf{x} \subseteq (CA)^F(C\mathbf{x}) \subseteq z$, and since equality would hold if $C = \tilde{A}^{-1}$ and z is the fixpoint of (5.2) or $(CA)^F(C\mathbf{x})$, the overestimation in $A^H\mathbf{x} \subseteq z$ is entirely due to rounding errors in the computation of C and the iteration for the computation of z, and therefore usually small.

(3) It is remarkable that in the situation discussed, preconditioning with $C = \tilde{A}^{-1}$ preserves the hull of the solution set. In contrast to this, the most recommended procedure of preconditioning with the midpoint inverse (Hansen [7], Neumaier [10]) may expand the hull slightly. However, the amount of overestimation remains small; cf. Neumaier [12].

(4) The explicit inversion of $\tilde{A}$ is very time and storage consuming if $\tilde{A}$ is a high-dimensional sparse matrix. Hence it is interesting that in the case of M-matrices, the inversion of $\tilde{A}$ can be avoided and replaced by a triangular factorization of $\tilde{A}$. This is based on the following observation.

PROPOSITION 5.2. *Let $A \in \mathbb{IR}^{n \times n}$ be an M-matrix. Then*

$$A^G\mathbf{x} \subseteq A^{-1}\mathbf{x} \qquad \textit{for all} \quad \mathbf{x} \in \mathbb{IR}^n. \tag{5.8}$$

Moreover, if $\rho(A) = 0$ then

$$A^G\mathbf{x} = A^{-1}\mathbf{x} \qquad \textit{for all} \quad \mathbf{x} \in \mathbb{IR}^n. \tag{5.9}$$

Proof. By results of Barth and Nuding [3] and Beeck [4], $A^Gz = A^Hz$ if $0 \leqslant z$, $0 \geqslant z$, or $0 \in z$; in particular, this holds if z has at most one nonzero component. Now $x = \Sigma_i x_i e^{(i)}$ and $A^G(x_i e^{(i)}) = A^H(x_i e^{(i)}) \subseteq A^{-1}(x_i e^{(i)})$ imply $A^G x \subseteq \Sigma_i A^G(x_i e^{(i)}) \subseteq \Sigma_i A^{-1}(x_i e^{(i)}) = A^{-1}x$, and (5.8) follows.

Now suppose that $\rho(A) = 0$. If $A = LR$ is the triangular decomposition of A, then L and R are themselves M-matrices (Fiedler and Pták [6]). Hence $|A^G| = \langle R \rangle^{-1} \langle L \rangle^{-1} = R^{-1}L^{-1} = A^{-1}$. But by Example 1 in Section 3 of [10], $(A^{-1})^M$ is linear and has the same absolute value and kernel, whence by Theorem 2 of [10], $A^G = (A^{-1})^M$. This implies (5.9). ■

THEOREM 5.3. *Let $A \in \mathbb{IR}^{n\times n}$ be an M-matrix, and suppose that $\bar{A} \leqslant \tilde{A} = \langle \tilde{A} \rangle$. Then $\tilde{A}$ is an M-matrix, and the hull inverse is the fixpoint map of the iteration*

$$y^{l+1} := \tilde{A}^G[x + (\tilde{A} - A)y^l] \qquad (l = 0, 1, 2, \ldots). \tag{5.10}$$

Proof. Let $u > 0$ be such that $Au > 0$. Then $\tilde{A}u \geqslant \bar{A}u \geqslant Au > 0$, so that $\tilde{A}$ is an M-matrix. In particular $\tilde{A}^{-1} \geqslant 0$ and Theorem 5.1 applies. The assertion now follows, since by Proposition 5.2, (5.2) and (5.8) generate the same sequence. ■

REMARK. For a similar result see Mayer [9].

We close with an example showing that (5.9) need not be valid if A is not thin. Consider the diagonally dominant M-matrix

$$A = \begin{pmatrix} 1 & [-\frac{1}{2}, 0] \\ [-\frac{1}{2}, 0] & 1 \end{pmatrix}$$

whose triangle decomposition (L, R) and inverse are given by

$$L = \begin{pmatrix} 1 & 0 \\ [-\frac{1}{2}, 0] & 1 \end{pmatrix}, \qquad R = \begin{pmatrix} 1 & [-\frac{1}{2}, 0] \\ 0 & [\frac{3}{4}, 1] \end{pmatrix},$$

$$A^{-1} = \begin{pmatrix} [1, \frac{4}{3}] & [0, \frac{2}{3}] \\ [0, \frac{2}{3}] & [1, \frac{4}{3}] \end{pmatrix}.$$

From this one easily finds for

$$x = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

that

$$A^H x = \begin{pmatrix} [0, 1] \\ [-2, -\frac{3}{2}] \end{pmatrix}, \qquad A^G x = \begin{pmatrix} [-\frac{1}{3}, 1] \\ [-\frac{8}{3}, -\frac{3}{2}] \end{pmatrix}, \qquad A^{-1}x = \begin{pmatrix} [-\frac{1}{3}, \frac{4}{3}] \\ [-\frac{8}{3}, -\frac{4}{3}] \end{pmatrix},$$

so that

$$A^H x \subsetneqq A^G x \subsetneqq A^{-1}x.$$

REFERENCES

1 G. Alefeld, Über die Durchführbarkeit des Gaußschen Algorithmus bei Gleichungen mit Intervallen als Koeffizienten, *Comput. Suppl.* 1:15–19 (1977).

2 G. Alefeld and J. Platzöder, A quadratically convergent Krawczyk-like algorithm, *SIAM J. Numer. Anal.* 20:210–219 (1983).

3 W. Barth and E. Nuding, Optimale Lösung von Intervallgleichungssystemen, *Computing* 12:117–125 (1974).

4 H. Beeck, Zur scharfen Aussenabschätzung der Lösungsmenge bei linearen Intervallgleichungssystemen, *Z. Angew. Math. Mech.* 54:T208–T209 (1974).

5 H. Beeck, Zur Problematik der Hüllenbestimmung von Intervallgleichungssystemen, in *Interval Mathematics* (K. Nickel, Ed.), Springer Lecture Notes Comput. Sci., 29, pp. 150–159, 1975.

6 M. Fiedler and V. Pták, On matrices with non-positive off-diagonal elements and positive principal minors, *Czechoslovak Math. J.* 12:382–400 (1962).

7 E. Hansen, Interval arithmetic in matrix computations, *SIAM J. Numer. Anal.* 2:308–320 (1965).

8 G. Mayer, Enclosing the solution set of linear systems with inaccurate data by iterative methods based on incomplete *LU*-decompositions, *Computing*, 35:189–206 (1985).

9 G. Mayer, Comparison theorems for an iterative method based on strong splittings, *SIAM J. Numer. Anal.*, to appear.

10 A. Neumaier, New techniques for the analysis of linear interval equations, *Linear Algebra Appl.* 58:273–325 (1984).

11 A. Neumaier, Interval iteration for zeros of systems of equations, *BIT* 25:256–273 (1985).

12 A. Neumaier, Overestimation in linear interval equations, *SIAM J. Numer. Anal.*, to appear.

13 G. Schrempp, Intervall-Iterationsverfahren für nichtlineare Gleichungssysteme, Freiburger Intervall-Berichte 84/6, 1984.

14 J. Schröder, Das Iterationsverfahren bei allgemeinem Abstandsbegriff, *Math. Z.* 66:111–116 (1956).

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