Computing 40, 85–90 (1988)

Computing

A Note on Moore's Interval Test for Zeros of Nonlinear Systems

Shen Zuhe, Nanjing, and **A. Neumaier**, Freiburg i. Br.

Received November 11, 1986; revised September 7, 1987

Abstract — Zusammenfassung

A Note on Moore's Interval Test for Zeros of Nonlinear Systems. We study relations between Moore's interval test and Miranda's theorem. As an application we combine the (real) Newton iteration with a computational test for Miranda's hypothesis by Moore and Kioustelidis to find an approximate solution to the system $f(x)=0$ with specified error bounds.

AMS Subject Classifications: 65H10, 65G10.

Key words: Interval arithmetic, existence test, Newton iteration.

Eine Bemerkung zu Moores Intervalltest für Nullstellen nichtlinearer Gleichungssysteme. Es werden Beziehungen zwischen Moores Test und dem Zwischenwertsatz von Miranda hergeleitet. Als Anwendung wird ein konstruktiver Test von Moore und Kioustelidis mit dem reellen Newton-Verfahren kombiniert, um Nullstellennäherungen mit vorgegebener Genauigkeit zu konstruieren.

Let $f: D\subset\mathbb{R}^n\to\mathbb{R}^n$ be a continuously differentiable function and let F' be an inclusion monotone interval extension of the derivative f' on the interval vector $X\subset D$, i.e. a mapping which assigns to each interval vector $X_0\subset X$ an interval matrix $F'(X_0)\subset\mathbb{R}^{n\times n}$ such that

$$F'(X_0)=f'(x) \quad \text{if} \quad X_0=[x,x] \quad (x\in\mathbb{R}^n)$$
$$X_1\subset X_2\subset X \Rightarrow F'(X_1)\subset F'(X_2).$$

For the investigation of solutions of the equation

$$f(x)=0 \tag{1}$$

we consider the Krawczyk operator

$$K(X)=\check{x}-Yf(\check{x})+(I-YF'(X))(X-\check{x}),$$

where Y is a suitable $n\times n$ interval matrix, usually an approximation of the inverse of a matrix in $F'(X)$, and $\check{x}$ is the mid-point of X. Moore [2], [3] introduced simple computational tests for the existence of a solution to (1). We state here a sharpened version due to L. Qi.

On the other hand, a well-known theorem by Bolzano states that whenever a continuous real function changes sign in some interval it has a zero in this interval. An n-dimensional generalization of this theorem is due to C. Miranda. We state an equivalent but slightly more general result:

Theorem 1 (*Qi* [5]): *If*

$$K(X) \subset \mathrm{Int}(X) \tag{2}$$

then X contains a unique solution x^* of (1). *x^* is a regular zero of* (1) *and the ordinary iterative method*

$$x^{(k+1)} = x^{(k)} - Yf(x^{(k)}) \tag{3}$$

converges to the unique solution x^* from any starting point $x^{(0)}$ in X. □

Here $\mathrm{Int}(X)$ denotes the interior of X and a zero x^* of (1) is called regular if $f'(x^*)$ is nonsingular.

Theorem 2 (*Miranda* [1]): *Let*

$$(X)_i^+ = \{x \in X, x_i = \check{x}_i + d_i\}, \ (X)_i^- = \{x \in X, x_i = \check{x}_i - d_i\}$$

be the n pairs of parallel, opposite faces of the interval vector $X = [\check{x} - d, \check{x} + d]$, $d = (d_1, d_2, \ldots, d_n)^T$. If there is a permutation $(r_1, r_2, \ldots, r_n)$ of $(1, 2, \ldots, n)$ such that

$$f_i(x) f_i(y) \leq 0 \ \textit{for all} \ x \in (X)_{r_i}^+, \ y \in (X)_{r_i}^- \ (i = 1, 2, \ldots, n) \tag{4}$$

then f has at least one solution x^* in X. □

The purpose of this note is to show that Miranda's theorem can be combined with the iteration (3) to determine an approximate solution to the system (1) with specified accuracy.

Miranda's theorem can be violated for interval vectors X in arbitrarily small neighbourhoods of a regular zero $x^* \in X$. Take $n = 2$ and consider

$$f(x) = \begin{pmatrix} x_1 + x_2 \\ x_1 - x_2 \end{pmatrix}, \quad X = \begin{pmatrix} [-\varepsilon, \varepsilon] \\ [-2\varepsilon, 2\varepsilon] \end{pmatrix}.$$

Then X contains the regular zero $x^* = 0$ as an interior point, but (4) cannot be satisfied for a permutation (r_1, r_2) of $(1, 2)$. The example shows that to use Miranda's theorem as a local existence test we must precondition the equation in a suitable way. As for the Krawczyk operator we consider

$$g(x) = Yf(x), \tag{5}$$

where Y is a nonsingular $n \times n$-matrix. Then $g(x)$ has the same zeros as $f(x)$. Moreover, if $\check{x}$ is an approximation of a regular zero x of f, then the choice $Y = f'(\check{x})^{-1}$ yields

$$\begin{aligned} g(x) &= Y\big(f(x^*) + f'(x^*)(x - x^*) + o(\varepsilon)\big) \\ &= \big(f'(\check{x})^{-1} f'(x^*)\big)(x - x^*) + o(\varepsilon) \\ &= \big(1 + o(1)\big)(x - x^*) + o(\varepsilon) \\ &= x - x^* + o(\varepsilon) \end{aligned}$$

for all $x \in \mathbb{R}^n$ with $\| x - x^* \| \leq \varepsilon$, ε small. This implies that (4) holds (with g in place of f) for any interval vector $X = [\check{x} - d, \check{x} + d]$ with $|\check{x} - x^*| \leq \frac{1}{2} d$, $\| d \|$ sufficiently small. To verify (4) numerically, Moore and Kioustelidis [4] gave the following test which we shall use for bounding the error of an approximate solution to the system (1).

Theorem 3: *Let X, $(X)_i^+$ and $(X)_i^-$ be defined as in Theorem 2. Let $g(x) = Yf(x)$. Then system (1) has at least one solution in $X = [\check{x} - d, \check{x} + d]$ if the following conditions are satisfied for all $i = 1, 2, \ldots, n$:*

$$g_i(\check{x} + d_i e_i)\, g_i(\check{x} - d_i e_i) \leq 0, \tag{6}$$

$$\sum_{j \neq i} |[YF'((X)_i^+)]_{ij}|\, d_j \leq |g_i(\check{x} + d_i e_i)|, \tag{7}$$

$$\sum_{j \neq i} |[YF'((X)_i^-)]_{ij}|\, d_j \leq |g_i(\check{x} - d_i e_i)|. \quad \square \tag{8}$$

We would like to know that the conditions of Theorem 3 can be satisfied locally. But first we show that Theorem 3 is stronger than the existence statement of Theorem 1.

Theorem 4: *If $K(X) \subset X$ then* (6)–(8) *hold.*

Proof: First we note that $K(X) \subset X$ is equivalent with

$$-g(\check{x}) + (I - YF'(X))[-d, d] \subset [-d, d]$$

and therefore with

$$|g(\check{x})| + |I - YF'(X)|\, d \leq d, \tag{9}$$

or, in components,

$$|g_i(\check{x})| + |[I - YF'(X)]_{ii}|\, d_i + \sum_{j \neq i} |[YF'(X)]_{ij}|\, d_j \leq d_i.$$

But by the mean value theorem,

$$\begin{aligned} g_i(\check{x} \pm d_i e_i) &\in g_i(\check{x}) \pm [YF'(X)]_{ii}\, d_i \\ &= \pm d_i + g_i(\check{x}) \mp [I - YF'(X)]_{ii}\, d_i, \end{aligned}$$

whence

$$\begin{aligned} \pm g_i(\check{x} \pm d_i e_i) &\geq d_i - |g_i(\check{x})| - |[I - YF'(X)]_{ii}|\, d_i \\ &\geq \sum_{j \neq i} |[YF'(X)]_{ij}|\, d_j \geq 0, \end{aligned}$$

which implies (6)–(8). $\square$

A slight variation of the proof allows an application to the iteration (3).

Theorem 5: *Suppose that $(d_1, d_2, \ldots, d_n)^T$ is a positive vector (the error bound required in advance). If $K(X_0) \subset \mathrm{Int}(X_0)$, where $X_0 = [x - \gamma d, x + \gamma d]$ for some $\gamma \geq 1$, then the conditions* (6), (7), (8) *are satisfied for all $i = 1, 2, \ldots, n$ with $X^{(k)} = [x^{(k)} - d, x^{(k)} + d]$ in place of X_0 for all iterates $x^{(k)}$ of* (3) *with sufficiently large k and any starting point $x^{(0)}$ in X_0, and, for such k, the i-th component $x_i^{(k)}$ of the approximate solution $x^{(k)}$ of* (1) *is accurate to within $\pm d_i$ for each $i = 1, 2, \ldots, n$.*

Proof: As before, $K(X_0) \subset \operatorname{Int} X_0$ implies

$$|g(\check{x})| + |I - YF'(X_0)|\gamma d < \gamma d. \tag{9 a}$$

Therefore $|I - YF'(X)|d < d$ which shows that (9) holds if $\check{x}$ is sufficiently close to the zero x^*. But by Theorem 1, $\lim x^{(k)} = x^*$; hence we can continue as in the last proof with $\check{x} = x^{(k)}$, provided that k is sufficiently large. □

Example: As an illustration we consider the two-dimensional nonlinear system

$$f(x) = \begin{pmatrix} x_1^2 + x_2^2 - 1 \\ x_1 - x_2^2 \end{pmatrix} = 0, \tag{10}$$

taken from [4]. The matrix of partial derivatives of the system (8) is

$$f'(x) = \begin{pmatrix} 2x_1 & 2x_2 \\ 1 & -2x_2 \end{pmatrix}.$$

The natural interval extension of $f'(x)$ is

$$F'(X) = \begin{pmatrix} 2X_1 & 2X_2 \\ 1 & -2X_2 \end{pmatrix},$$

where X_1, X_2 are intervals. Starting with the interval $X^{(0)}$ given by

$$X^{(0)} = \begin{pmatrix} [0.5, 0.8] \\ [0.6, 0.9] \end{pmatrix}, \quad m(X^{(0)}) = \begin{pmatrix} 0.65 \\ 0.75 \end{pmatrix}$$

and an approximate inverse Y of $f'(m(X^{(0)}))$:

$$Y = \begin{pmatrix} 0.434783 & 0.434783 \\ 0.289855 & -0.376812 \end{pmatrix},$$

we have

$$K(X^{(0)}) = \begin{pmatrix} [0.559793, \ 0.677173] \\ [0.744275, \ 0.830363] \end{pmatrix}$$

and clearly $K(X^{(0)}) \subset \operatorname{Int}(X^{(0)})$. Therefore, following Theorems 1, 3 and 5, we can use the point Newton iteration and the computational test (6)–(8) for the accuracy of the approximate solution. For example, let us suppose that we want to find an index ℓ such that the approximate solution $x^{(\ell)}$ is accurate to within $\pm 10^{-6}$ for each $i = 1, \dots, n$. Now, starting with the initial

$$m(X^{(0)}) = \begin{pmatrix} 0.65 \\ 0.75 \end{pmatrix}$$

and the nonsingular matrix Y in (9), the first iteration gives

$$x^{(1)} = m(X^{(0)}) - Yf(m(X^{(0)})) = \begin{pmatrix} 0.618478 \\ 0.787318 \end{pmatrix}.$$

If $x^{(1)}$ would satisfy our accuracy requirements, then

$$X^{(1)} = \begin{pmatrix} [0{,}618477, \ 0.618479] \\ [0.787317, \ 0.787319] \end{pmatrix},$$

would contain a zero. With

$$(X^{(1)})_1^+ = \begin{pmatrix} 0.618479 \\ [0.787317, 0.787319] \end{pmatrix}, \quad (X^{(1)})_1^- = \begin{pmatrix} 0.618477 \\ [0.787317, 0.787319] \end{pmatrix},$$

$$(X^{(1)})_2^+ = \begin{pmatrix} [0.618477, 0.618479] \\ 0.787319 \end{pmatrix}, \quad (X^{(1)})_2^- = \begin{pmatrix} [0.618477, 0.618479] \\ 0.787317 \end{pmatrix},$$

we are in a position to check conditions (6), (7) and (8). We find that

$$g_1(x^{(1)} + d_1 e_1) = 0.3152275 \cdot 10^{-1}, \quad g_1(x^{(1)} - d_1 e_1) = 0.3152070 \cdot 10^{-1},$$

$$g_2(x^{(1)} + d_2 e_2) = -0.3731781 \cdot 10^{-1}, \quad g_2(x^{(1)} - d_2 e_2) = -0.3731988 \cdot 10^{-1},$$

and

$$[YF'((X^{(1)})_1^+)]_{12}\, d_2 = [-0.1788139 \cdot 10^{-11}, 0.1788139 \cdot 10^{-11}],$$

$$[YF'((X^{(1)})_1^-)]_{12}\, d_2 = [-0.1788139 \cdot 10^{-11}, 0.1788139 \cdot 10^{-11}],$$

$$[YF'((X^{(1)})_2^+)]_{21}\, d_1 = [-0.5960464 \cdot 10^{-12}, 0.5960464 \cdot 10^{-12}],$$

$$[YF'((X^{(1)})_2^-)]_{21}\, d_1 = [-0.5960464 \cdot 10^{-12}, 0.5960464 \cdot 10^{-12}].$$

While conditions (7) and (8) are satisfied at this step, condition (6) is not. Hence $x^{(1)}$ was not accurate enough to satisfy our test. The following table shows some data of the successive iteration. It is clear that conditions (6), (7) and (8) are satisfied in the third step. The approximate solution

$$x^{(3)} = \begin{pmatrix} 0.6180340 \\ 0.7861514 \end{pmatrix}$$

is accurate to within $\pm 10^{-6}$ as required and

$$g(x^{(3)}) = \begin{pmatrix} 0.0000001 \\ 0.0000009 \end{pmatrix}, \quad f(x^{(3)}) = \begin{pmatrix} 0.0000015 \\ 0.0000014 \end{pmatrix}.$$

Table

k	1	2	3
$g_1(x^{(k)} + d_1 e_1)$	$.31 \cdot 10^{-1}$	$.44 \cdot 10^{-3}$	$.11 \cdot 10^{-5}$
$g_1(x^{(k)} - d_1 e_1)$	$.31 \cdot 10^{-1}$	$.44 \cdot 10^{-3}$	$-.90 \cdot 10^{-6}$
$g_2(x^{(k)} + d_2 e_2)$	$-.37 \cdot 10^{-1}$	$.11 \cdot 10^{-2}$	$.19 \cdot 10^{-5}$
$g_2(x^{(k)} - d_2 e_2)$	$-.37 \cdot 10^{-1}$	$.11 \cdot 10^{-2}$	$-.10 \cdot 10^{-6}$
$\lvert [YF'((X^{(k)})_1^+)]_{12}\, d_2 \rvert$	$.17 \cdot 10^{-11}$	$.18 \cdot 10^{-11}$	$.17 \cdot 10^{-11}$
$\lvert [YF'((X^{(k)})_1^-)]_{12}\, d_2 \rvert$	$.17 \cdot 10^{-11}$	$.18 \cdot 10^{-11}$	$.17 \cdot 10^{-11}$
$\lvert [YF'((X^{(k)})_2^+)]_{21}\, d_1 \rvert$	$.59 \cdot 10^{-12}$	$.56 \cdot 10^{-12}$	$.59 \cdot 10^{-12}$
$\lvert [YF'((X^{(k)})_2^-)]_{21}\, d_1 \rvert$	$.59 \cdot 10^{-12}$	$.56 \cdot 10^{-12}$	$.59 \cdot 10^{-12}$
$f(x^{(k)})_1$	.0023863	.0000015	.0000000
$f(x^{(k)})_2$	.0013927	.0000014	−.0000000
$(x^{(k)})_1$	.6184783	.6180341	.6180340
$(x^{(k)})_2$	.7873188	.7861523	.7861514

References

[1] Miranda, C.: Un 'Osservatione su un teorema di Brower. Bolletino Unione Math. Ital. Ser. II, *3*, 5–7 (1940).

[2] Moore, R. E.: A test for existence of solutions to nonlinear systems. SIAM J. Numer. Anal. *14*, 611–615 (1977).

[3] Moore, R. E.: A computational test for convergence of iterative methods for nonlinear systems. SIAM J. Numer. Anal. *15*, 1194–1196 (1978).

[4] Moore, R. E., Kioustelidis, J. B.: A simple test for accuracy of approximate solutions to nonlinear (or linear) systems. SIAM J. Numer. Anal. *17*, 521–529 (1980).

[5] Qi, L.: A generalization of the Krawczyk-Moore algorithm. In: Interval Mathematics 1980 (Nickel, K., ed.), pp. 481–488. New York: Academic Press 1980.

Prof. Shen Zuhe
Mathematics Department
Nanjing University
Nanjing
People's Republic of China

Prof. Dr. Arnold Neumaier
Institut für Angewandte Mathematik
Universität Freiburg i. Br.
Hermann-Herder-Strasse 10
D-7800 Freiburg i. Br.
Federal Republic of Germany

Printed in Austria
Druck Paul Gerin, A 1021 Wien