Proceedings A 91 (3), September 26, 1988

Discrete measures for spherical designs, eutactic stars and lattices

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Communicated by Prof. J.H. van Lint at the meeting of November 30, 1987

1. INTRODUCTION

Designs essentially deal with the approximation of certain simple sets Σ of vectors by a nice subcollection. This refers to spherical designs (approximating all unit vectors), orthogonal arrays (approximating all (0, 1)-vectors), $t-(v,k,\lambda)$-designs (approximating all (0,1)-vectors of fixed weight), and in general to cubature formulae for the approximation of integrals over Σ. The strength t indicates the degree for which the approximation is exact. In the present paper we propose a general setting for some of these notions and their generalizations in terms of measures for the Euclidean vector space $V=\mathbb{R}^d$. We define $d\xi$ to be a measure of strength t whenever

$$\int_V f_k(x)d\xi(x) = \int_V \|x\|^k d\xi(x) \cdot \int_S f_k(x)d\sigma(x)$$

holds for all homogeneous polynomials f_k of degree k in d variables, for $k=1,2,\dots,t$. Thus we compare and confront the Borel measure $d\sigma$ for the unit sphere S with general measures $d\xi$ for V, whose support may be finite or infinite, of variable or of constant weight, and sometimes subject to additional restrictions. This covers cubature formulae for the unit sphere, spherical t-designs and eutactic stars and integral lattices in $\mathbb{R}^d$. In particular, the restrictive unit length condition is removed from the theory of spherical designs. We will be interested in the combinatorial aspects of such discrete approximations of the Borel measure of the unit sphere. There are applications to the design of statistical experiments and to discrete quadrature, which will be considered in a future publication.

In Section 2 various equivalent definitions are given for the notion of a measure of strength t. These definitions are phrased in terms of monomials and polynomials in d variables. They are collected in Theorem 2.6. Section 3 specializes to the case of finite support (X, w), where $X \subset \mathbb{R}^d$ is a finite set and $w: X \to \mathbb{R}^+$ is a weight function. In Theorem 3.2 it is proved that (X, w) of strength $2e$ implies

$$|X| \geq \binom{d+e}{e},$$

provided X is distributed over at least $1+[\frac{1}{2}e]$ concentric spheres. The proof uses the difference in harmonic analysis for solid and for spherical polynomials of degree $\leq e$. In Section 4 the condition of strength 2 in the case of finite support is related to the geometric notion of eutactic stars.

The final Section 5 deals with lattices, in particular, with even, integral, unimodular, extremal lattices. We recall certain facts from modular forms, in relation to the theta-series of the lattice. In the case of extremal lattices, Theorem 5.2 gives a bound for the Poincaré series of the spaces of the harmonic theta-series of various degrees. This relates to Venkov's result on the strength of the layers of the lattice, and poses the problem of the strength of the lattice itself.

The paper contains several open problems and directions for future research.

2. MEASURES OF STRENGTH t

Let V denote a real vector space of finite dimension d, provided with a positive definite inner product (x, y) and norm $(x, x) = \|x\|^2$. Let $e_1, e_2, \ldots, e_d$ be an orthonormal basis. Let $S := \{x \in V : (x, x) = 1\}$ denote the unit sphere provided with the standard Borel measure $d\sigma$, normalized such that the total measure equals 1. We wish to approximate $d\sigma$ by other measures $d\xi$ on V, subject to certain conditions. In particular we shall be interested in measures $d\xi$ whose support is finite or denumerable.

First some notation will be introduced, cf. [21], [10]. For the vector x with components $(x_1, x_2, \ldots, x_d)$, let $\underset{\sim}{x}^k$ denote the symmetric tensor $\otimes^k x$ whose components are the monomials $x_1^{k_1} x_2^{k_2} \cdots x_d^{k_d}$, $\sum_{i=1}^d k_i = k$. For a fixed k the linear combinations of these monomials constitute $\mathrm{Hom}_k(V)$, the linear space of the homogeneous polynomials in d variables of degree k, which has dimension

$$\binom{d+k-1}{d-1}.$$

Two further linear spaces of polynomials will be used. $\mathrm{Hom}_k(V)$ contains as a subspace $\mathrm{Harm}_k(V)$, the harmonic polynomials of degree k, the kernel of the action on $\mathrm{Hom}_k(V)$ of the Laplace operator

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_d^2}.$$

$\mathrm{Hom}_k(V)$ is contained as a subspace in $\mathrm{Pol}_k(V)$, the polynomials of degree $\leq k$. The linear extension of the trace inner product

$$(\underline{x}^k, \underline{y}^k) = (x, y)^k$$

serves a positive definite inner product for $\mathrm{Hom}_k(V)$. Let us calculate the inner product with itself of the tensor

$$D_k := \int_S \underline{x}^k d\sigma(x).$$

2.1. LEMMA

If k is odd, then $D_k = 0$. If k is even, then

$$(D_k, D_k) = \frac{1 \cdot 3 \cdot \cdots \cdot (k-1)}{d(d+2) \cdots (d+k-2)}.$$

PROOF

$$(D_k, D_k) = \int_S \int_S (x, y)^k d\sigma(x) d\sigma(y) =$$

$$= \int_S (x, y)^k d\sigma(x) \cdot \int_S d\sigma(y) = \int_S x_i^k d\sigma(x),$$

by putting $y = e_i$. This equals zero if k is odd, and the expression above if k is even.

□

We now pose the problem of finding measures $d\xi$ for V such that the tensor

$$C_k := \int_V \underline{x}^k d\xi(x)$$

only differs from D_k by a constant factor, depending on k.

DEFINITION

A *measure* $d\xi$ has *strength* t if there exists constants μ_k such that

$$C_k = \mu_k D_k, \text{ for } k = 0, 1, \ldots, t.$$

For odd k, this says that $C_k = 0$. For even k we show that the constants μ_k are just the moments

$$\mu_k(d\xi) := \int_V \|x\|^k d\xi(x).$$

2.2. LEMMA

$$(C_k, D_k) = \mu_k(d\xi)(D_k, D_k).$$

PROOF. Writing $x=\|x\|u$, with $u\in S$, we find from Lemma 2.1 that

$$(\underline{x}^k, D_k)=\|x\|^k(\underline{u}^k, D_k)=\|x\|^k \int_S (u,x)^k d\sigma(x)=$$

$$=\|x\|^k \int_S x_1^k d\sigma(x)=\|x\|^k(D_k, D_k).$$

Therefore,

$$(C_k, D_k)=(\int_V \underline{x}^k d\xi(x), D_k)=\int_V (\underline{x}^k, D_k)d\xi(x)=\int_V \|x\|^k(D_k, D_k)d\xi(x),$$

which proves the lemma. □

2.3. COROLLARY

For a measure $d\xi$ of strength t and even $k\le t$,

$$\mu_k=\int_V \|x\|^k d\xi(x).$$

PROOF. Substitute $C_k=\mu_k D_k$ in Lemma 2.2. □

The proportionality of the tensors C_k and D_k, which occurs in the definition of strength, is the extremal case with respect to the following generalized Sidelnikov inequality, cf. [19], [10].

2.4. THEOREM

$$\int_V\int_V (x,y)^k d\xi(x)d\xi(y)\ge\mu_k^2(D_k, D_k),$$

with equality iff $C_k=\mu_k D_k$.

PROOF

$$0\le(C_k-\mu_k D_k, C_k-\mu_k D_k)=(C_k, C_k)-2\mu_k(C_k, D_k)+\mu_k^2(D_k, D_k).$$

This implies the theorem, since for any k we have

$$(C_k, C_k)=\int_V\int_V (x,y)^k d\xi(x)d\xi(y), \quad (C_k, D_k)=\mu_k(D_k, D_k).$$ □

2.5. REMARK

It is interesting to compare and confront the formula in Theorem 2.4 with

$$\int_V\int_V Q_k((x,y))d\xi(x)d\xi(y)\ge 0$$

(both the inequalities and the equalities). Here $Q_k(z)$ denotes the k-th Gegenbauer polynomial in one variable z, cf. [9], [10]. For the finite constant weight case it is explained in Remark 3.3 of [10], cf. also [17], that the present formula is actually stronger than the formula in Theorem 2.4.

Summarizing, we have the following equivalent conditions for measures of strength t.

2.6. THEOREM

For a measure $d\xi$ on V and a positive integer t, the following conditions are equivalent:

(i) $d\xi$ has strength t.

(ii) For every positive integer $k \le t$,

$$\int_V \underline{x}^k d\xi(x) = \int_V \|x\|^k d\xi(x) \int_S \underline{x}^k d\sigma(x).$$

(iii) For every positive integer $k \le t$,

$$\int_V \int_V (x, y)^k d\xi(x) d\xi(y) = \Big(\int_V \|x\|^k d\xi(x)\Big)^2 (D_k, D_k).$$

(iv) For every homogeneous polynomial f_k of degree k $(0 < k \le t)$,

$$\int_V f_k(x) d\xi(x) = \int_V \|x\|^k d\xi(x) \int_S f_k(x) d\sigma(x).$$

(v) For every harmonic polynomial h_k of degree k $(0 < k \le t)$ and every integer l $(0 \le 2l \le t - k)$,

$$\int_V (x, x)^l h_k(x) d\xi(x) = 0.$$

PROOF. The equivalence of (i) and (ii), (iii), (iv) follows from Corollary 2.3, Theorem 2.4, and the definition of $\mathrm{Hom}_k(V)$, respectively. For (v), we iterate the decomposition

$$f_k(x) = h_k(x) + (x, x) f_{k-2}(x)$$

under the action of the Laplace operator Δ, and observe that the integral over S of any harmonic polynomial of degree ≥ 1 vanishes. □

2.7. REMARK

Trivial examples of measures of strength t (for arbitrary t) are *rotatable* measures, i.e. measures $d\xi$ which are uniform on each sphere with center 0, so that

$$d\xi(x) = d\varrho(\|x\|) \cdot d\sigma(u), \text{ for } x = \|x\| u,\ u \in S,$$

cf. [6], [13], [14]. This is the reason why the study of optimal rotable designs by Kiefer [14] yields such elegant results. In a separate paper we shall apply the present theory to extend Kiefer's results to the non-rotable case.

3. FINITE SUPPORT

We consider measures $d\xi$ of strength t which have finite support X, so that integrals turn into weighted sums. Let (X, w) denote the finite set X, of cardinality n, with positive weights w_x, $x \in X$. Then

$$\int_V f(x) d\xi(x) = \sum_{x \in X} w_x f(x),$$

and the conditions for strength t of Theorem 2.6 can be specialized to this situation.

For finite *spherical* support $X \subset S$ condition (iv) of Theorem 2.6 amounts to

$$\int_S f(x)d\sigma(x) = (\sum_{x\in X} w_x)^{-1}(\sum_{x\in X} w_x f(x)),$$

for $f \in \mathrm{Pol}_t(S)$, the linear space of all polynomials of degree $\leq t$ restricted to S. This is a *cubature formula* of strength t for the sphere S. We refer to [11] for the definition and for the construction of such cubature formulae, in particular from orbits of finite subgroups of the orthogonal group.

In the case of equal weights $w_x = 1/n$ the condition (iv) reads

$$\frac{1}{n}\sum_{x\in X} f(x) = \int_S f(x)d\sigma(x), \quad f \in \mathrm{Pol}_t(S),$$

and X is a *spherical t-design*. This notion was introduced and developed in [9], [10], [1], [2].

We turn to the general case of a finite weighted set (X, w) of strength t in $V = \mathbb{R}^d$, and put $e = [\frac{1}{2}t]$. In order to obtain lower bounds for $|X|$ we investigate the linear space $\mathrm{Pol}_e(V)$ of all polynomials of degree $\leq e$ in d variables. This space is spanned by the monomials of degree $\leq e$, hence its dimension equals

$$1 + d + \binom{d+1}{2} + \binom{d+2}{3} + \cdots + \binom{d+e-1}{e} = \binom{d+e}{e}.$$

In $\mathrm{Pol}_e(V)$ there are two types of inner product for polynomials f and g, viz.

$$\langle f, g\rangle := (f(\partial_x)g)(0); \; (f, g) := \int_S f(x)g(x)d\sigma(x).$$

Both inner products are symmetric. $\langle f, g\rangle$ is positive definite on $\mathrm{Pol}_e(V)$, whereas (f, g) is positive semidefinite on $\mathrm{Pol}_e(V)$ and definite on $\mathrm{Pol}_e(S)$. Between these inner products the following relation holds, cf. [7] Theorem 3.8 and [5]:

$$\langle f, g\rangle = (f, g)d(d+2)\cdots(d+2k-2), \text{ for } f \in \mathrm{Harm}_k(V),\ g \in \mathrm{Hom}_k(V).$$

With respect to $\langle\ ,\ \rangle$ the following decompositions are orthogonal, cf. [21] IV.2.1 and [7] 3.2:

$$\mathrm{Pol}_e(V) = \sum_{m=0}^{e} \mathrm{Hom}_m(V); \quad \mathrm{Hom}_m(V) = \sum_{2l=0}^{m} \mathrm{Harm}_{m-2l}(V)(x, x)^l.$$

Restriction to the unit sphere S in V yields the decomposition

$$\mathrm{Pol}_e(S) = \mathrm{Hom}_e(S) + \mathrm{Hom}_{e-1}(S) = \sum_{k=0}^{e} \mathrm{Harm}_k(S),$$

which is orthogonal with respect to (;). For future reference we fix an arbitrary (,)-orthonormal basis $\mathscr{H} = \cup \mathscr{H}_k$ for $\mathrm{Pol}_e(S)$ following this decomposition.

We recall [21] that the spherical harmonics $\mathrm{Harm}_k(S)$ and the solid harmonics $\mathrm{Harm}_k(V)$ constitute isomorphic spaces, but that $\mathrm{Pol}_e(S)$ and $\mathrm{Pol}_e(V)$ are not isomorphic. Indeed, any $h \in \mathrm{Harm}_k(S)$ corresponds to the polynomials

$$h(x), (x,x)h(x), \ldots, (x,x)^i h(x), \ldots, (x,x)^{[\frac{1}{2}(e-k)]} h(x) \in \mathrm{Pol}_e(V).$$

We call this set of $1+[\frac{1}{2}(e-k)]$ polynomials of degrees $k, k+2, \ldots, k+2i, \ldots, e$ or $e-1$, respectively, the *fiber* $\mathscr{F}_h$ *corresponding to* $h \in \mathrm{Harm}_k(V)$. Clearly $(\ ,\ )$ is constant on any fiber, which expresses the degeneracy of $(\ ,\ )$ on $\mathrm{Pol}_e(V)$. We shall use the *special basis*

$$\mathscr{F} := \bigcup_{h \in \mathscr{H}} \mathscr{F}_h$$

for $\mathrm{Pol}_e(V)$. It is the union of the fibers which correspond to the elements of the $(\ ,\ )$-orthonormal basis $\mathscr{H}$ for $\mathrm{Pol}_e(S)$.

In order to describe finite weighted sets (X, w) of strength $t=2e$, we introduce the following matrices whose rows are indexed by the $x \in X$. F denotes the matrix whose columns are indexed by the $f \in \mathscr{F}$, with entries $F_{x,f} = f(x)$. For $h \in \mathscr{H}_k$, F_h denotes the matrix whose columns are indexed by integers i $(0 \le 2i \le e-k)$, with entries $(F_k)_{x,i} = \|x\|^{k+2i}$, and W is the diagonal matrix with diagonal entries $W_{x,x} = w_x$. We denote transposition by a dash.

We first prove the following Kronecker decomposition.

3.1. LEMMA

If (X, w) has strength $t=2e$, then

$$F'WF = \bigoplus_{h \in \mathscr{H}} F'_h W F_h.$$

PROOF. Since (X, w) has strength t, Theorem 2.6 (iv) yields

$$\sum_{x \in X} w_x f(x) g(x) = \sum_{x \in X} w_x \|x\|^{a+b} \cdot \int_S f(x) g(x) d\sigma(x),$$

for $f \in \mathrm{Hom}_a(V)$, $g \in \mathrm{Hom}_b(V)$, $a+b \le t$. We apply this formula for the pairs of the polynomials of the special basis $\mathscr{F}$. If f and g belong to distinct fibers, then we get zero. If f and g belong to the same fiber $\mathscr{F}_h$, with $h \in \mathrm{Harm}_k(S)$, then they have degrees $k+2i$ and $k+2j$, say, and

$$\sum_{x \in X} w_x f(x) g(x) = \sum_{x \in X} w_x \|x\|^{2k+2i+2j}.$$

This equals the (i, j)-entry of the matrix $F'_h W F_h$, and the lemma is proved. □

3.2. THEOREM

If the finite weighted set (X, w) of strength $t=2e$ in $\mathbb{R}^d$ is distributed over at least $1+[\frac{1}{2}e]$ concentric spheres, then

$$|X| \ge \binom{d+e}{e}.$$

4. EUTACTIC STARS

We investigate the condition for strength 2, in the case of finite support (X, w), $X \subset \mathbb{R}^d$, $|X| = n$, $w : X \to \mathbb{R}^+$. Theorem 2.6 (iii) translates into the following arithmetical conditions:

$$\text{for } k=1: \sum_{x, y \in X} w_x w_y (x, y) = 0,$$

$$\text{for } k=2: \sum_{x, y \in X} w_x w_y (x, y)^2 = \Big(\sum_{x \in X} w_x (x, x) \Big)^2 \cdot \frac{1}{d}.$$

The condition for $k=1$ is equivalent to $\sum_{x \in X} w_x x = 0$, and (X, w) is *balanced*, that is, the center of mass is in the origin. Also the condition for $k=2$ has a geometric interpretation, in terms of the set

$$Y : = \{ x \sqrt{w_x} : x \in X \}.$$

Let $H = [(x, y)\sqrt{w_x w_y} : x, y \in X]$ denote the Gram matrix of Y, and let H have the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_d$, and 0 of multiplicity $n-d$. Then the condition for $k=2$ reads:

$$\text{trace } H^2 = (\text{trace } H)^2 / d,$$

$$\lambda_1^2 + \cdots + \lambda_d^2 = (\lambda_1 + \cdots + \lambda_d)^2 / d,$$

whence $\lambda_1 = \cdots = \lambda_d = : \lambda$, say, and $\lambda^{-1} H$ is an idempotent matrix:

$$H^2 = \lambda H, \quad \lambda = d^{-1} \sum_{x \in X} w_x \| x \|^2.$$

Following [17] the geometric significance of this condition is that the set $\lambda^{-\frac{1}{2}} Y$ is a *eutactic star* in $\mathbb{R}^d$, that is, the spanning orthogonal projection onto $\mathbb{R}^d$ of an orthonormal frame in any $\mathbb{R}^n$ which contains $\mathbb{R}^d$ as a subspace.

In the following theorems, the first of which just has been proved, the finite set X is supposed to span $\mathbb{R}^d$, and $\lambda : = d^{-1} \sum_{x \in X} w_x \| x \|^2$, as above.

4.1. THEOREM

The finite weighted set (X, w) in $\mathbb{R}^d$ has strength 2 iff (X, w) is balanced, and $\lambda^{-\frac{1}{2}} \{ x \sqrt{w_x} : x \in X \}$ is a eutactic star.

4.2. THEOREM

The finite weighted set (X, w) in $\mathbb{R}^d$ has strength 2 iff

$$GWJ = 0 \text{ and } GWG = \lambda G,$$

where $G = [(x, y) : x, y \in X]$, $W = \text{diag}\,(w_x : x \in X)$.

PROOF. $GWJ = 0$ is equivalent to $\sum_{x \in X} w_x x = 0$, and $GWG = \lambda G$ to $\lambda^{-1} H$ idempotent, where $H : = W^{\frac{1}{2}} G W^{\frac{1}{2}}$. Now use Theorem 4.1. $\square$

4.3. THEOREM

The finite weighted set (X, w) in $\mathbb{R}^d$ has strength 2 iff

$$\sum_{x\in X} w_x x = 0 \text{ and } \sum_{x\in X} w_x xx' = \hat{\lambda} I$$

for some number $\hat{\lambda}$; in this case $\hat{\lambda} = \lambda$.

PROOF. Theorem 2.6 (iv), for $k=2$, translates into

$$\sum_{x\in X} w_x x_i x_j = \Big(\sum_{x\in X} w_x (x, x)\Big) \int_S x_i x_j d\sigma(x).$$

Since for $i \neq j$ we have

$$\int_S x_i x_j d\sigma(x) = 0, \quad \int_S x_i^2 d\sigma(x) = \frac{1}{d} \int_S (x, x) d\sigma(x) = \frac{1}{d},$$

this reads as follows in terms of the column vector x:

$$\sum_{x\in X} w_x xx' = I\Big(\sum_{x\in X} w_x (x, x)/d\Big).$$

This implies the theorem, since $\sum w_x xx' = \hat{\lambda} I$ implies $\hat{\lambda} = \lambda$ (take traces). □

Important examples of eutactic stars come from extreme lattices. Recall that a lattice Λ is *extreme* if the quotient $n(\Lambda)^d/\det \Lambda$, where $n(\Lambda)$ is the minimum norm and $\det \Lambda$ is the discriminant of Λ, is locally maximal with respect to small changes of Λ, cf. [8]. By Voronoi's theorem a lattice is extreme iff it is perfect and eutactic. Following [8] a lattice is *perfect* iff the set X of the minimum norm vectors has the property that the rank 1 matrices xx^T, $x \in X$, span the space of symmetric matrices (equivalently, the tensors x^2, $x \in X$ span Pol_2 (V)). The lattice is *eutactic* iff the identity matrix is a positive linear combination of the rank 1 matrices xx^T, $x \in X$. Since X is antipodal, we can combine this with Remark 3.5 and Theorem 4.3, and obtain

4.4. THEOREM

A lattice is eutactic iff, with suitable weights, the minimum norm vectors in a eutactic lattice form a set of vectors of strength 3. In particular, the conclusion holds for extreme lattices. □

5. LATTICES

We now consider the case of a lattice Λ in $\mathbb{R}^d$. Let $R := \{(x, x) : x \in \Lambda\}$ denote the set of norms in Λ. A measure $d\xi$ is called *(Λ, t)-homogeneous* if support is Λ (hence discrete), and for each $r \in R$ the points in the layer $\Lambda_r := \{x \in \Lambda : (x, x) = r\}$ have the same weight $\gamma(r)$, where $\sum_{r\in R} r^l \gamma(r) < \infty$ for all integers l with $0 \leq 2l \leq t$. Introducing

$$f(\Lambda_r) := \sum_{x\in \Lambda_r} f(x),$$

for any polynomial f and layer Λ_r, we have

$$\int_V f(x)d\xi(x) = \sum_{x\in\Lambda} f(x)\gamma((x,x)) = \sum_{r\in R} \gamma(r)f(\Lambda_r).$$

DEFINITION

The *strength of a lattice* Λ is the maximum number t such that there is a (Λ, t)-homogeneous measure $d\xi$ of strength t.

The condition (v) of Theorem 2.6 for strength t reads

$$\sum_{r\in R} r^l\gamma(r)h_j(\Lambda_r)=0,\ h_j\in \mathrm{Harm}_j\,(V),\ 0<j\leq t-2l.$$

Since a homogeneous measure is antipodal, $d\xi(-x)=d\xi(x)$, this condition is automatically satisfied for all odd j; therefore, the strength of any lattice is an odd integer $t\geq 1$. Clearly, if all layers are spherical t-designs, then all $h_j(\Lambda_r)=0$ and every (Λ, t)-homogeneous measure has strength t. This raises two questions. Which lattices have the property that all layers are spherical t-designs? In addition, how far can the strength of the lattice go beyond this t by a suitable choice of the weights $\gamma(r)$?

Many extreme lattices have strength $t\geq 3$ (cf. Theorem 4.4), and it would be interesting to know more about the relationship between extreme lattices and strength. While this seems to be an intricate problem, recent work by Venkov [22] implies that lattices with a different extremal property, viz. the extremal unimodular lattices, have large strength. To discuss this, we need certain notions and results from the theory of modular forms (for $PSL(2,\mathbb{Z})$), to be used for the theta-series of the lattice. We follow the notation of [15], and refer for further details to [15], [16], [18], [20].

Examples of modular forms of weight k are the Eisenstein series E_k, defined by

$$E_k:=\frac{1}{2\zeta(k)}\sum_{(m,n)\neq(0,0)}\frac{1}{(mz+n)^k}=1-\frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n,$$

where $q=\exp\,(2\pi iz)$, $z\in\mathbb{C}$, the B_k are the Bernoulli numbers, and

$$2\zeta(k)=\sum_{n\neq 0}\frac{1}{n^k},\ \sigma_k(n)=\sum_{d|n} d^k.$$

A modular form of weight 12 which is a cusp form (i.e. where the Fourier expansion into powers of q has no constant term) is given by

$$\Delta:=\frac{1}{1728}(E_4^3-E_6^2)=q\prod_{n=1}^{\infty}(1-q^n)^{24}.$$

The linear space M_k of the modular forms of weight k, and the subspace M_k^0 of the cusp forms, are related by

$$M_k=M_k^0\oplus[E_k],\quad \Delta\cdot M_{k-12}=M_k^0$$

for even $k \geq 2$, and ≥ 12, respectively. The graded algebra of all M_k is generated by the algebraically independent forms E_4 and E_6:

$$M = \bigcup_{k=0}^{\infty} M_k = [E_4, E_6].$$

Hence, the *Poincaré series* of this algebra, i.e. the power series in λ whose coefficients are the dimensions $\dim M_k$, reads

$$\frac{1}{(1-\lambda^4)(1-\lambda^6)} = 1 + \lambda^4 + \lambda^6 + \lambda^8 + \lambda^{10} + 2\lambda^{12} + \lambda^{14} + 2\lambda^{16} + \cdots.$$

We return to lattices Λ in $\mathbb{R}^d$, which from now on are supposed to be even, integral, unimodular, and extremal, that is

$$(x, x) \in 2\mathbb{Z},\ (x, y) \in \mathbb{Z},\ \text{for all } x, y \in \Lambda,$$

$$\Lambda = \Lambda^{\#} := \{x \in \mathbb{R}^d : V_{y \in \Lambda}((x, y) \in \mathbb{Z})\},$$

$$\min_{0 \neq x \in \Lambda} (x, x) = 2\left[\frac{d}{24}\right] + 2 = 2\delta + 2.$$

Since $d \equiv 0 \pmod 8$ we found it convenient to put

$$d = 24(\delta + 1) - 8\varepsilon, \text{ with } \varepsilon \in \{3, 2, 1\}.$$

Hence, the layers Λ_r are empty for r odd and for $r = 2, 4, \ldots, 2\delta$.

The harmonic theta-series for Λ are defined by

$$\theta(z; h, \Lambda) := \sum_{r \in R} h(\Lambda_r) q^r,\ h \in \operatorname{Harm}_j(V),\ q = \exp(2\pi i z).$$

A classical result by E. Hecke and B. Schoeneberg, cf. [12], is the following.

5.1. THEOREM

$\theta(z; h, \Lambda)$, $h \in \operatorname{Harm}_j(V)$, is a modular form of weight $\frac{1}{2}d + j$, and a cusp form if $j > 0$.

As a consequence of this theorem, the set

$$\Theta_j := \{\theta(z; h, \Lambda) : h \in \operatorname{Harm}_j(V)\}$$

is a subspace of $M_{\frac{1}{2}d+j}$. For the Poincaré series of these subspaces we have:

5.2. THEOREM

$$\sum_{j \geq 0} \lambda^j \dim \Theta_j \leq 1 + \frac{\lambda^{4\varepsilon}}{(1-\lambda^4)(1-\lambda^6)}.$$

PROOF. (cf. Venkov [22]). For the extremal lattice Λ, $h \in \operatorname{Harm}_j(V)$, and $j > 0$, the theta-series reads

$$\theta(z; h, \Lambda) = h(\Lambda_{2\delta+2}) q^{2\delta+2} + h(\Lambda_{2\delta+4}) q^{2\delta+4} + \cdots,$$

and has weight $12(\delta+1)-4\varepsilon+j$. Iteration of the property $M_k^0=\Delta M_{k-12}$ implies that Θ_j is a subspace of $\Delta^{\delta+1}M_{j-4\varepsilon}$. A comparison of dimensions yields

$$\Theta_j=0 \text{ for } 0<j\leq 4\varepsilon-1;$$

$$\dim \Theta_j \leq \dim M_{j-4\varepsilon} \text{ for } j\geq 4\varepsilon.$$

Putting $j-4\varepsilon=i$ we obtain

$$\sum_{j\geq 4\varepsilon} \lambda^j \dim M_{j-4\varepsilon} = \sum_{i\geq 0} \lambda^{4\varepsilon+i} \dim M_i = \frac{\lambda^{4\varepsilon}}{(1-\lambda^4)(1-\lambda^6)},$$

which implies the assertion. □

5.3. COROLLARY (Venkov [22]).

For an extremal lattice in $\mathbb{R}^d$, every layer has strength $4\varepsilon-1$.

PROOF. For $o<j\leq 4\varepsilon-1$ we have $\Theta_j=0$, hence $h(\Lambda_r)=0$ for $h\in \mathrm{Harm}_j\,(V)$ and all r. □

5.4. COROLLARY

An extremal lattice has strength $\geq 4\varepsilon-1$.

PROOF. Use the criterion of Theorem 2.6 (v). □

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