The Graphs with Spectral Radius Between 2 and $\sqrt{2+\sqrt{5}}$

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Dedicated to Alan J. Hoffman on the occasion of his 65th birthday.

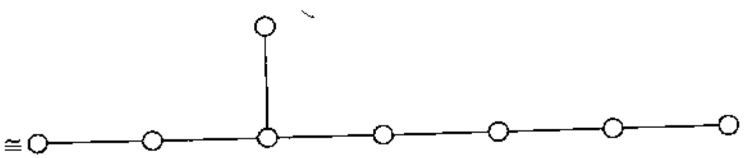
Submitted by Alexander Schrijver

ABSTRACT

We complete the determination of the graphs in the title, begun by Cvetković, Doob, and Gutman.

The spectral radius of a graph is the largest eigenvalue of its (0,1) adjacency matrix. Hoffman [3] shows that graphs G properly containing a circuit have largest eigenvalue $\lambda_{\max}(G) > \tau^{3/2} = \sqrt{2+\sqrt{5}} \approx 2.058171$ [where $\tau = (1+\sqrt{5})/2$] and that $\tau^{3/2}$ is a limit point of these numbers $\lambda_{\max}(G)$. Cvetković, Doob, and Gutman [1] classify the graphs G with $2 < \lambda_{\max}(G) \leqslant \tau^{3/2}$ and find that these are certain trees without vertices of degree at least 4, and with at most two vertices of degree 3.

More explicitly, let $T_{i,j,k}$ be the graph with $i + j + k + 1$ vertices consisting of three paths with i, j, and k edges, respectively, where these paths have one end vertex in common, e.g.,

$T_{1,2,4} \cong$  ,

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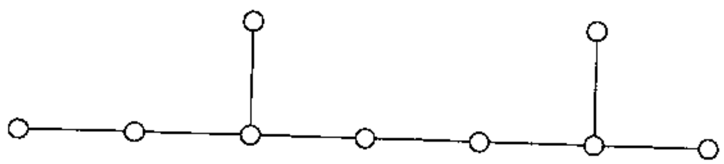
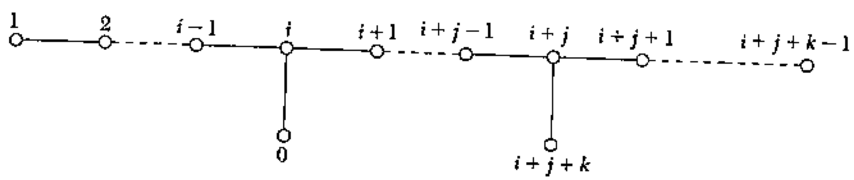
$T_{1,2,4}$ is the graph more commonly known as E_8. Also, let $H_{i,j,k}$ be the graph with $i+j+k+1$ vertices consisting of a path with $u=i+j+k-1$ vertices $x_1,\dots,x_u$ with two extra edges affixed at x_i and x_{u+1-k}, e.g.,

$H_{3,3,2} \cong$  .

The abovementioned authors show that if $2<\lambda_{\max}(G)\leqslant\tau^{3/2}$, then G is one of the graphs $T_{i,j,k}$ or $H_{i,j,k}$; furthermore, that the $T_{i,j,k}$ occurring are $T_{1,2,k}$ $(k\geqslant 6)$, $T_{1,3,k}$ $(k\geqslant 4)$, $T_{1,j,k}$ $(k\geqslant j\geqslant 4)$, $T_{2,2,k}$ $(k\geqslant 3)$, and $T_{2,3,3}$, and that each of these has largest eigenvalue less than $\tau^{3/2}$. Concerning the $H_{i,j,k}$, they did not succeed in determining the precise triples (i,j,k) occurring, but gave computer results for small (i,j,k). However, it is not difficult to determine for which i, j, k one has $\lambda_{\max}(H_{i,j,k})\leqslant\tau^{3/2}$. On the one hand, this can easily be read off from the explicit formulae for the characteristic polynomial of $H_{i,j,k}$ given in Goodman, de la Harpe, and Jones [2]; on the other hand, it follows immediately from the results in Neumaier [4]. Here, we shall follow the latter approach.

PROPOSITION. *$\lambda_{\max}(H_{i,j,k})\leqslant\tau^{3/2}$ if and only if $\tau^j\geqslant(\tau^i-2)(\tau^k-2)$, and equality does not occur.*

Proof. We apply Theorem 2.4 of [4]. We have to construct a partial λ-eigenvector for $\lambda=\tau^{3/2}=\mu+\mu^{-1}$, where $\mu=\tau^{1/2}$. Label the vertices of $H_{i,j,k}$ as follows:



and define a vector e with components

$$e_l=\frac{\mu^l-\mu^{-l}}{\alpha}\qquad(1\leqslant l\leqslant i),$$

$$e_0=e_i(\mu-\mu^{-1}),$$

$$e_{i+l}=\frac{\mu^{i-l}+\mu^{l-i}-2\mu^{-i-l}}{\alpha}\qquad(0\leqslant l\leqslant j),$$

$$e_{i+j+k}=\mu-\mu^{-1},$$

$$e_{i+j+l}=\frac{\mu^{k-l}-\mu^{l-k}}{\beta}\qquad(0\leqslant l<k),$$

where

$$\alpha=\mu^{i-j}+\mu^{j-i}-2\mu^{-i-j},\qquad\beta=\mu^k-\mu^{-k}.$$

Using the relation $\mu^2-\mu^{-2}=1$ (which holds because $\tau^2=\tau+1$), one easily checks that e is a positive partial λ-eigenvector with respect to the vertex $i+j$, and the exit value is

$$\begin{aligned}\epsilon&=\lambda-e_{i+j+k}-e_{i+j-1}-e_{i+j+1}\\&=2\mu^{-1}-\frac{\mu^{i-j+1}+\mu^{j-i-1}-2\mu^{-i-j+1}}{\mu^{i-j}+\mu^{j-i}-2\mu^{-i-j}}-\frac{\mu^{k-1}-\mu^{1-k}}{\mu^k-\mu^{-k}}.\end{aligned}$$

Distributing one of the μ^{-1} to each fraction gives

$$\epsilon=\frac{(\mu-\mu^{-1})(2-\mu^{2i})}{\mu^{2i}+\mu^{2j}-2}+\frac{\mu-\mu^{-1}}{\mu^{2k}-1},$$

and since $\mu^2=\tau$, this simplifies to

$$\epsilon=(\mu-\mu^{-1})\frac{\tau^j-(\tau^i-2)(\tau^k-2)}{(\tau^i+\tau^j-2)(\tau^k-1)}.$$

By Theorem 2.4 of [4], $\lambda_{\max}(H_{i,j,k}) \leqslant \lambda$ if and only if $\epsilon \geqslant 0$, and $\lambda_{\max}(H_{i,j,k}) = \lambda$ if and only if $\epsilon = 0$. But one easily checks that the latter cannot happen. ∎

Combining this with the results of Cvetković, Doob, and Gutman yields:

THEOREM. *Let G be a graph with $2 < \lambda_{\max}(G) \leqslant \tau^{3/2}$. Then G is one of the graphs $T_{i,j,k}$ (see above), or one of the graphs $H_{i,j,k}$, where $j \geqslant i + k$, or $i = 3$ and $j \geqslant k + 2$, or $i = 2$ and $j \geqslant k - 1$, or (i, j, k) is one of $(2,1,3)$, $(3,4,3)$, $(3,5,4)$, $(4,7,4)$, $(4,8,5)$. None of these graphs has $\lambda_{\max}(G) = \tau^{3/2}$.*

Now we can answer a question posed in [2]:

COROLLARY. *The set of spectral radii of graphs is not a closed subset of the real line.*

POSTSCRIPT

This result was obtained by both authors independently (fall 1986) after having received a copy of [2] from J. J. Seidel. Shortly afterwards we learned from him that Shearer [5] had shown that each real $\lambda \geqslant \tau^{3/2}$ is a limit point of spectral radii (from which our Corollary follows immediately, since spectral radii are algebraic integers) and that Godsil had remarked that $\sqrt{2+\sqrt{5}}$ cannot be a spectral radius, since otherwise all its conjugates would be eigenvalues, too, but $\sqrt{2-\sqrt{5}}$ is not real.

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