COMBINATORICA
Akadémiai Kiadó — Springer-Verlag

COMBINATORICA 9 (1) (1989) 59—67

DUALITY IN COHERENT CONFIGURATIONS

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Received July 1, 1986
Revised July 12, 1987

By introducing a duality operator in coherent algebras (i.e. adjacency algebras of coherent configurations) we give a new interpretation to Delsarte's duality theory for association schemes. In particular we show that nonnegative matrices and positive semidefinite matrices, (0, 1)-matrices and distance matrices, regular graphs and spherical 2-designs, distance regular graphs and Delsarte matrices are pairs of dual objects. Several "almost dual" properties which are not yet fully understood are also reported.

In his thesis, Delsarte [8] observed a formal duality in the theory of association schemes which becomes an actual duality in the case that an abelian group is present. In this note we shall develop certain aspects of this duality in the slightly more general context of coherent algebras. In particular, it turns out that graphs and distance matrices, introduced in Neumaier [12], are dual objects.

A *coherent algebra* is an algebra V of square complex $v\times v$-matrices (indexed by a set X with v elements) closed under componentwise (Schur) multiplication $\circ$ and under conjugate transposition $*$ such that the identity matrix I and the all-one matrix J belong to V. It is not difficult to show that the partition of $X\times X$ defined by the equivalence relation

$$(x, y) \equiv (x', y') \quad \text{iff} \quad A_{xy} = A_{x'y'} \quad \text{for all} \quad A\in V$$

turns X into a coherent configuration in the sense of Higman [10]; conversely, the adjacency algebra of a coherent configuration is a coherent algebra ([10], (2.8)). In particular, if V is commutative, X is an association scheme in the sense of Delsarte [8], and V is its Bose—Mesner algebra (cf. [5], where only the symmetric case is treated). See also Higman [21].

Many properties of coherent algebras and related configurations are formally dual and in some cases, this is due to the existence of a duality operator.

We call a coherent algebra V *selfdual* if there is a semi-linear *duality operator* τ on V such that

$$\begin{aligned}
A^{\tau\tau} &= vA && \text{for all } A\in V.\\
(\alpha A)^\tau &= \bar{\alpha}A^\tau && \text{for all } \alpha\in\mathbf{C} \text{ and } A\in V.\\
(A^*)^\tau &= (A^\tau)^* && \text{for all } A\in V.\\
(AB)^\tau &= A^\tau\circ B^\tau && \text{for all } A, B\in V.
\end{aligned}$$

AMS subject classification (1980): 05B20, 05C50

Note that then $A^\tau=(AI)^\tau=A^\tau\circ I^\tau$, so

$$I^\tau = J, \quad J^\tau = I^{\tau\tau} = vI,$$

$$(A\circ B)^\tau = \frac{1}{v^2}(A^{\tau\tau}\circ B^{\tau\tau})^\tau = \frac{1}{v^2}(A^\tau B^\tau)^{\tau\tau} = \frac{1}{v}A^\tau B^\tau;$$

in particular V is commutative.

Example 1. Let G be a finite abelian group of order v. A decomposition of G into a direct product $G=C_1\times\ldots\times C_t$ of cyclic groups C_i of order e_i with generators g_i determines a multiplicative inner product

$$\langle x,y\rangle = \prod \zeta_i^{x_i y_i}, \quad \text{if} \quad x=\prod g_i^{x_i}, \quad y=\prod g_i^{y_i},$$

(where ζ_i is a primitive e_i-th root of unity) with the following properties:

$$\langle x,y\rangle = \langle x,y\rangle, \quad \langle x,y\rangle^{-1} = \langle x^{-1},y\rangle,$$

$$\langle x,yz\rangle = \langle x,y\rangle\langle x,z\rangle,$$

$$\langle x,y\rangle = 1 \quad \text{for all} \quad y\in G \Leftrightarrow x=1.$$

$$\sum_{y\in G}\langle x,y\rangle = \begin{cases} v & \text{if} \quad x=1,\\ 0 & \text{otherwise,}\end{cases}$$

since

$$\sum_{y_1=0}^{e_1}\cdots\sum_{y_t=0}^{e_t}\zeta_i^{x_i y_i} = 0.$$

In particular, the map $x\to\langle x,\cdot\rangle$ is an isomorphism between G and its character group $\chi(G)$.

The semi-linear map τ obtained by extending the definition

$$x^\tau = \sum_{y\in G}\langle x,y\rangle y \quad \text{with} \quad \Big(\sum_{x\in G}a_x x\Big)^\tau = \sum_{x\in G}\bar{a}_x x^\tau$$

to the group algebra $\mathbf{C}G$ satisfies

$$x^{\tau\tau} = \Big[\sum_y\langle x,y\rangle y\Big]^\tau = \sum_y\overline{\langle x,y\rangle}y^\tau = \sum_y\langle x,y\rangle^{-1}y^\tau =$$

$$= \sum_y\sum_z\langle x^{-1},y\rangle\langle y,z\rangle z = \sum_z\sum_y\langle x^{-1}z,y\rangle z = vx,$$

$$(xy)^\tau = \sum_z\langle xy,z\rangle z = \sum_z\langle x,z\rangle\langle y,z\rangle z = \sum_z\langle x,z\rangle z\circ\sum_z\langle y,z\rangle z = x^\tau\circ y^\tau,$$

$$1^\tau = \sum_y\langle 1,y\rangle y = \sum_y y.$$

Let V be the centralizer algebra of G (cf. Wielandt [18]) in its regular action (=the set of all matrices that are invariant under G), $V=\{A=(a_{xy})_{x,y\in G}|a_{xz,yz}=a_{x,y}$ for all $x,y,z\in G\}$ is isomorphic to the group algebra of G via the isomorphism $a=\sum_x a_x x\to\Pi(a)=(a_{xy^{-1}})_{x,y\in G}$. V is a coherent algebra, and the map τ defined by $\Pi(a)^\tau=\Pi(a^\tau)$ is a duality operator on V.

Example 2. An association scheme is of Latin square type (negative Latin square type) if every (0, 1)-matrix with zero diagonal contained in the Bose—Mesner algebra V is the adjacency matrix of a pseudo Latin square graph (negative Latin square graph). Here a $\frac{\text{pseudo}}{\text{negative}}$ Latin square graph is a strongly regular graph with $\frac{\mu=s(s+1)}{\mu=r(r+1)}$ (cf. Cameron et al. [7]); in particular, imprimitive strongly regular graphs (i.e. disjoint unions of cliques and multipartite graphs) are degenerate pseudo Latin square graphs. E.g. the points of an affine plane (or more generally a net (Bruck [6])), together with a partition $(\Sigma_1, \ldots, \Sigma_s)$ of the line directions (points at infinity) defines an association scheme of Latin square type by calling x, y i-th associates if x and y are on a line with direction $\in\Sigma_i$. See also [19, 20].

Define for $A \in V$ the numbers k_A, d_A by

$$AJ = k_A J, \quad A\circ I = d_A I.$$

Then for any association scheme of Latin square type, the map τ with

$$\bar{A}^\tau = \pm nA + (n\mp 1)^{-1}(k_A \mp nd_A)(nI\mp J), \quad n = \sqrt{v}$$

(upper sign for Latin square type, lower sign for negative Latin square type) is a duality operator.

Proposition 1. *A commutative coherent algebra V has a unique basis $D_0, \ldots, D_s$ of (0, 1)-matrices such that $D_0=I$, $D_i^*\in\{D_0, \ldots, D_s\}$ for all i, $D_0+D_1+\ldots+D_s=J$.*

Proof. Every finite dimensional commutative algebra has a basis of idempotents. Let $D_0, \ldots, D_s$ be such a basis with respect to the $\circ$-multiplication. Then each D_i is a (0, 1)-matrix, and since $D_i\circ D_j=0$ for $i\neq j$, the linear combination of the D_i representing J must be $J=D_0+\ldots+D_s$. Similarly, D_i^* and I are represented by a sum of D_j $(j=0, \ldots, s)$ which is seen to contain only one term, and by a suitable permutation we may take $D_0=I$. ∎

From now on, let V be selfdual (and hence commutative).

Proposition 2. *Let $T=(\tau_{ij})_{i,j}$ be the matrix defined by*

$$D_i^\tau = \sum_{j=0}^{s} \tau_{ij} D_j.$$

Then D_i has row sums $k_i=\tau_{i0}$, and we have the relations

$$T\bar{T} = vI,$$

$$\tau_{ji}k_i = \bar{\tau}_{ij^*}k_{j^*}.$$

Proof. $(D_i^\tau)^\tau = \sum_{j=0}^{s} \bar{\tau}_{ij} D_j^\tau = \sum_{j=0}^{s}\sum_{l=0}^{s} \bar{\tau}_{ij}\tau_{jl}D_l$ but also $D_i^{\tau\tau}=vD_i$, so $\bar{T}T=vI$. Next we have

$$D_i^\tau \circ I = \sum_{j=0}^{s} \tau_{ij} D_j \circ I = \tau_{i0} D_0 = \tau_{i0} I,$$

hence $D_l J = \frac{1}{v}(D_l^\tau \circ J^\tau)^\tau = (D_l^\tau \circ I)^\tau = (\tau_{i0} I)^\tau = \tau_{i0} J$, giving $k_i = \tau_{i0}$. Finally

$$D_i^\tau D_j \circ I = \sum_{l=0}^{s} \tau_{il}(D_l D_j \circ I) = \sum_{l=0}^{s} \tau_{il}(\delta_{lj^*} k_{j^*} I) = \tau_{ij^*} k_{j^*} I$$

since $D_l D_j \circ I = \delta_{lj^*} k_{j^*}$, hence

$$\bar{\tau}_{ij^*} k_{j^*} J = (\tau_{ij^*} k_{j^*} I)^\tau = (D_i^\tau D_j \circ I)^\tau = \frac{1}{v}(D_i^\tau D_j)^\tau I^\tau =$$

$$= \frac{1}{v}(D_i^{\tau\tau} \circ D_j^\tau) J = (D_i \circ D_j^\tau) J = \tau_{ji} D_i J = \tau_{ij} k_i J,$$

since $D_i \circ D_j^\tau = D_i \circ \sum_l \tau_{jl} D_l = \sum_l \tau_{jl} D_i \circ D_l = \tau_{ji} D_i$; and we get $\bar{\tau}_{ij^*} k_{j^*} = \tau_{ji} k_i$. ∎

Proposition 3.

(i) *A has constant row sum $k \Leftrightarrow A^\tau$ has constant diagonal entries k.*

(ii) *A is a $(0, 1)$-matrix of valency $k \Leftrightarrow \frac{1}{v} A^\tau$ is idempotent of rank k.*

(iii) *A is nonnegative $\Leftrightarrow A^\tau$ is positive semidefinite.*

(iv) *The $\frac{1}{v} D_i^\tau$ are the irreducible idempotents of V.*

(v) *D_i has the eigenvalues $\bar{\tau}_{ij}$ with multiplicity k_j $(j=0, \dots, s)$.*

Proof.

(i) $AJ = kJ \Leftrightarrow A^\tau \circ I = kI$.

(ii) $A \circ A = A \Leftrightarrow \frac{1}{v} A^\tau \cdot \frac{1}{v} A^\tau = \frac{1}{v} A^\tau$, trace $\frac{1}{v} A^\tau = k$, so rank $\frac{1}{v} A^\tau = k$ (since the only eigenvalues are 0 and 1).

(iii) A is nonnegative $\Leftrightarrow A = \sum_{i=0}^{s} a_i D_i$, and all $a_i \geqq 0 \Leftrightarrow A^\tau = \sum_{i=0}^{s} a_i D_i^\tau$, and all $a_i \geqq 0 \Leftrightarrow A^\tau$ is positive semidefinite.

(iv) We have $\frac{1}{v} D_i^\tau \cdot \frac{1}{v} D_j^\tau = \frac{1}{v}(D_i \circ D_j)^\tau = \frac{1}{v} \delta_{ij} D_i^\tau$. If $\frac{1}{v} D_i^\tau = \frac{1}{v} E^\tau + \frac{1}{v} F^\tau$ is a decomposition of $\frac{1}{v} D_i^\tau$ into idempotents then $D_i = E + F$ is a decomposition into $(0, 1)$-matrices. But D_i is a basis element so this is impossible.

(v) $D_i^\tau = \sum_j \tau_{ij} D_j$ implies $D_i = \sum_j \bar{\tau}_{ij} \cdot \frac{1}{v} D_j^\tau$, and since $\frac{1}{v} D_j$ is idempotent, D_i has eigenvalues $\bar{\tau}_{ij}$ with multiplicity rank $\left(\frac{1}{v} D_j^\tau\right) = \text{trace}\left(\frac{1}{v} D_j\right) = k_j$. ∎

The *eigenmatrices* of a commutative coherent algebra (equivalently, of a symmetric association scheme) are the matrices $P = (P_i(j))_{i,j}$ and $Q = (Q_i(j))_{i,j}$ relating the basis of $(0, 1)$-matrices D_i and the basis of idempotents E_i as follows

$$D_i = \sum_{j=0}^{s} P_i(j) E_j; \quad vE_i = \sum_{j=0}^{s} Q_i(j) D_j.$$

Of course $PQ=vI$, and by the above results, we have $\bar{P}=Q=T$ for a self-dual algebra. Conversely, a commutative coherent configuration with $\bar{P}=Q$ is self-dual, since the semi-linear map τ with $D_i^\tau=vE_i$ is a duality operator.

If V_0 is a coherent subalgebra of a self-dual coherent algebra, then so is its dual V_0^τ; the valencies k_i', multiplicities f_i', and eigenmatrices P', Q' of V_0^τ are related to the valencies k_i, multiplicities f_i, and eigenmatrices P, Q of V_0 by

$$k_i'=f_i,\quad f_i'=k_i,\quad P'=\bar{Q},\quad Q'=\bar{P}.$$

In particular, we have

Theorem 1. *If a self-dual coherent algebra contains the adjacency matrix of a strongly regular graph Γ with eigenvalues k, r, s with multiplicites $1, f, v-1-f$, then it also contains the adjacency matrix of a strongly regular graph Γ with the dual parameters:*

$$(*)\qquad v'=v,\quad k'=f,\quad r'=\frac{v-k+s}{r-s},\quad s'=\frac{s-k}{r-s},\quad f'=k.$$

In particular, the hypothesis can only be satisfied if either Γ is a conference graph, or $r-s$ is an integer dividing v (then the known necessary conditions already imply that $(r-s)|(s-k)$).

Proof. The eigenmatrices of Γ are (cf. Lemma 2.2 of Cameron et al. [7])

$$P=\begin{pmatrix}1 & 1 & 1\\ k & r & s\\ l & -r-1 & -s-1\end{pmatrix},\quad Q=\begin{pmatrix}1 & 1 & 1\\ f & \frac{fr}{k} & -\frac{f(r+1)}{l}\\ g & \frac{gs}{k} & -\frac{g(s+1)}{l}\end{pmatrix},$$

where $l=v-1-k$, $g=v-1-f$, and a similar formula holds for the dual. Now it is clear that $v'=v$ and the relations $P'=Q$, $Q'=P$ give $k'=f, f'=k$, and $r'=\frac{fr}{k}$, $s'=\frac{-f(r+1)}{l}$. Now, using e.g. the formulas in Lemma 2.1 of Cameron et al. [7], we find

$$s'=-\frac{f(r+1)}{l}=-\frac{(k(s+1)+ls)(r+1)}{(r-s)l}=-\frac{l(k+rs)+ls(r+1)}{(r-s)l}=\frac{s-k}{r-s}$$

and

$$r'-s'=\frac{f}{kl}(rl+(r+1)k)=\frac{fg}{kl}(r-s)=\frac{v}{r-s}.\quad \blacksquare$$

Corollary 1 (Delsarte [8]) *If a stronly regular graph Γ has a regular abelian automorphism group then there is a dual strongly regular graph Γ' such that their parameters are related by A.* $\blacksquare$

Corollary 2 (Delsarte [8]). *The Bose-Mesner algebra V of a strongly regular graph Γ is self-dual iff $v=(r-s)^2$ iff Γ is a conference graph, a pseudo Latin square graph, or a negative Latin square graph.* $\blacksquare$

Next we relate the adjacency matrices of graphs in a self-dual coherent algebra with their dual, which are distance matrices (see Neumaier [12], [13], [14]). We shall also establish a relation with distance regular graphs (see Biggs [4]).

A *distance matrix* (Neumaier [12]) is a nonzero real symmetric $v\times v$-matrix $C=(c_{xy})$ with nonnegative entries, indexed by a set X with v elements, such that the distance function $d(x, y):=c_{xy}{}^{1/2}$ satisfies $d(x, x)=0$ and $d(x, z)\leqq d(x, y)+d(y, z)$ for all $x, y, z\in X$. If C has no repeated rows then this turns X into a metric space, and conversely, for every metric d on a finite set X, the matrix $C:=(d(x, y)^2)$ is a distance matrix. A distance matrix C has *degree s* if C has precisely s distinct nonzero entries, and *strength* (at least) t if for all nonnegative integers i, j with $i+j\leqq t$, the (x, y)-entry of the product of $C^{(i)}=(c_{xy}^i)$ and $C^{(j)}=(c_{xy}^j)$ can be written as a polynomial f_{ij} of degree $\leqq\min(i, j)$ in c_{xy}; we write this as $C^{(i)}C^{(j)}=f_{ij}\circ C$. A distance matrix C is *spherical* if $G=\gamma J-C$ is positive semidefinite for $\gamma>0$. Distance matrices are closely related to *spherical t-designs* defined by Delsarte et al. [9].

Proposition 4. *Let C be a distance matrix with v rows.*

(i) *C has strength* 1 *iff*

$$CJ = kJ \quad \textit{for some} \quad k > 0. \tag{1}$$

(ii) *C has strength* 2 *iff* (1) *holds and* $G:=\dfrac{k}{v}J-C$ *satisfies*

$$G^2 = nG \quad \textit{for some} \quad n \geqq 0; \tag{2}$$

in this case C is spherical.

(iii) *C has strength* 3 *iff* (1), (2) *hold, and*

$$(G\circ G)G = mG \quad \textit{for some} \quad m \geqq 0. \tag{3}$$

(iv) *C is the distance matrix of a spherical* 1*-design iff C is spherical and* (1) *holds, of a spherical* 2*-design iff* (1) *and* (2) *hold, and of a spherical* 3*-design iff* (1), (2), *and* (3) *hold with* $m=0$.

Proof. See Neumaier [12]. ∎

A distance matrix which has strength t for all t is called a *Delsarte matrix* ([12]).

A *distance regular graph* is a graph Γ such that the distance relation turns Γ into an association scheme. It is well-known (Biggs [4]) that a graph Γ of diameter δ is distance regular iff for all $i\leqq\delta$ the distance-i-adjacency matrix A_i is a polynomial of degree i in the (distance-1-)adjacency matrix A of Γ.

For *symmetric* matrices $A\in V$ with constant row sums $k=k_A$, we define $C_A:=k_AJ-A^\tau$. If A is a symmetric (0, 1)-matrix, denote by $\Gamma(A)$ the graph whose adjacency matrix is A.

Theorem 2. *Let $A\in V$ be a symmetric matrix with nonnegative diagonal and constant row sums k. Then*

(i) *C_A is a spherical distance matrix of strength* 1 *iff A is nonnegative.*

(ii) *C_A has strength* 2 *iff A is a* (0, 1)*-matrix, i.e. the adjacency matrix of a regular graph of valency k.*

(iii) *C_A has strength* 3 *iff $\Gamma(A)$ is an edge-regular graph, i.e. every edge of $\Gamma(A)$ is in a constant number of triangles.*

(iv) *C_A is the distance matrix of a spherical 3-design iff $\Gamma(A)$ is a triangle-free graph.*
(v) *C_A is a Delsarte matrix iff $\Gamma(A)$ is distance regular.*
(vi) *C_A has repeated rows iff $\Gamma(A)$ is disconnected.*

Proof.

(i) A is nonnegative $\Rightarrow A^\tau$ positive semidefinite $\Rightarrow k_A J - C_A$ is positive semidefinite $\Rightarrow C_A$ is a spherical 1-design. Conversely, C_A is a spherical 1-design $\Rightarrow \gamma J - C_A$ is positive semidefinite for some $r > 0 \Rightarrow (\gamma - k_A)J + A^\tau$ is positive semidefinite $\Rightarrow (\gamma - k_A)I + A$ is nonnegative $\Rightarrow A$ is nonnegative (since A has nonnegative diagonal).
(ii) C_A has strength $2 \Leftrightarrow A^\tau A^\tau = vA^\tau \Leftrightarrow A \circ A = A \Leftrightarrow A$ is a (0, 1)-matrix.
(iii) C_A has strength $3 \Leftrightarrow (A^\tau \circ A^\tau)A^\tau = \text{const.}\ A_\tau \Leftrightarrow A^2 \circ A = \text{const}\ A \Leftrightarrow \Gamma(A)$ is edge regular.
(iv) Same as (iii) with constant$=0$.
(v) C_A is a Delsarte matrix
$\Leftrightarrow E_i = Q_i \circ A$ (where Q_i is a polynomial of degree i)
$\Leftrightarrow D_i = Q_i(vk_A I - vA) = \varphi_i(A)$ for some polynomial φ_i of degree $i \Leftrightarrow \Gamma(A)$ is distance regular (for the last equivalence see Biggs [4]).

(vi)
$$C_A = \sum_i a_i D_i = k_A J - A^\tau \Leftrightarrow A = k_A I - \sum_i \frac{a_i}{v} D_i^\tau \Leftrightarrow$$
$$\Leftrightarrow A \text{ has eigenvalues } k_A - \frac{a_i}{v},$$

so C_A has repeated rows iff C_A has some off diagonal elements zero iff some $a_k \neq 0$ (for $k \neq 0$) iff A has multiple eigenvalue k_A iff $\Gamma(A)$ is disconnected. ∎

Remarks. 1. Similarly (cf. [15]), C_A has strength 4 iff in $\Gamma(A)$ two points at distance zero, one, or two have the same number k, λ, or μ of common neighbours, respectively.

2. The dual of a spherical distance matrix is a *Fiedler matrix,* i.e. a symmetric matrix with zero row sums and off-diagonal entries $\leqq 0$. It is possible to dualize the concept of strength to arbitrary Fiedler matrixes (not only to those inside a self-dual coherent algebra). Then the Fiedler matrices of unbounded strength are essentially the matrices $kI - A$ where A is the adjacency matrix of a distance regular graph of valency k. See Neumaier [15].

3. In general, the actual duality in case of matrices inside a self-dual coherent algebra is replaced by a formal duality between ordinary matrix product and elementwise product, between ranks and valencies, between Gram matrices and nonnegative matrices, etc. There are more such properties which look like duals, but the analogies are not yet understood. We outline the basic analogies.

For the following facts see e.g. Bannai and Ito [3]. Let Γ be a graph of diameter $s^* = \max\limits_{x,y} d(x, y)$, and girth $t^* + 1 =$ minimal size of a circuit. Then $t^* \leqq 2s^*$, and equality holds iff Γ is a Moore graph. (This can be taken as the definition of a Moore graph; cf. [11]). Every Moore graph is distance regular, and the Moore graphs of valency 2 are the polygons; there are only a few Moore graphs of valency > 2, and in particular, there is none for $s^* \geqq 3$. A graph Γ of diameter s^* and valency k contains $v \leqq v_{s^*}(k) = \dfrac{k(k-1)^{s^*}-2}{k-2}$ points, with equality iff Γ is a Moore graph; $v_{s^*}(k)$ is a polynomial of degree s^* in k. Finally, if the valencies of a primitive association scheme are ordered such that $k_1 \leqq k_2 \leqq \ldots \leqq k_n$ then $k_{i+1} \leqq k_i(k_1-1)$, and equality implies that the graph with valency k_1 corresponding to the first relation has girth t^*+1 with $t^* \geqq 4$.

Compare this with properties of a spherical set X of points, discussed in Delsarte, Goethals and Seidel [9]. Suppose that X has s distances and is a spherical t-design. Then $t \leqq 2s$, and equality holds iff X is a tight design. Every tight design is a Delsarte space; and the tight designs with $f=2$ are the polygons; there are only a few tight designs with $f>2$, and in particular, there is none for $s \geqq 12$. (See Bannai and Damerell [1], [2]). A set X with s distances in the unit sphere of $\mathbf{R}^f$ contains

$$w \leqq w_s(f) = \binom{f+s-1}{s} + \binom{f+s-2}{s-1}$$

points, with equality iff Γ is a tight design; $w_s(f)$ is a polynomial of degree s in f. Finally, if the multiplicities of a primitive association scheme are ordered such that $f_1 \leqq f_2 \leqq \ldots \leqq f_n$ then it can be shown that $f_2 \leqq \dfrac{1}{v}(f_1+2)(f_1-1)$, and equality implies that the set of points corresponding to the idempotent of rank f_1 is spherical 4-design.

Some immediate questions arise: How to explain the analogies? Why are the formulae slightly different in the two cases? Is there a spherical analogue of $k_{i+1} \leqq k_i(k_1-1)$ for $i \geqq 2$? Is there a Delsarte matrix analogue of the inequalities $k_{i+1}k_{i-1} \leqq k_i^2$ obtained by Taylor and Levingston [17] for distance regular graphs?

I want to close with some remarks on Delsarte's setting for duality. The basic observation is that if G is a $v \times v$-matrix such that $GG^* = G^*G = vI$, and $G = (G_0|G_1|\ldots|G_s)$ is a partition of the columns then the matrices $E_i = \dfrac{1}{v} G_i G_i^*$ are mutually orthogonal idempotents, and hence define an algebra. Hence we might call two algebras of $v \times v$-matrices *dual* if there is a matrix G with $GG^* = G^*G = vI$ such that a suitable partition of the columns gives the first algebra and a suitable partition of the rows gives the second algebra. Finally, we call two association schemes *dual* (in Delsarte's sense) if their Bose—Mesner algebras are dual with respect to a matrix G, and if, in addition, the column partition is given by the associates of a suitable point of the first scheme. Then it can be shown that the parameters of a dual pair of association schemes again satisfies the relations $k_i' = f_{i'}$, $f_i' = k_{i'}$, $P' = Q$ and $Q' = P$.

In the case of a regular abelian group, Delsarte's duality is equivalent with our duality, since we may take $G = (\langle x, y \rangle)_{x, y \in X'}$ and the spherical point defining the partition can be taken to be zero.

Acknowledgements. This paper is a polished account of one of a series of lectures I gave in 1980 at the Technische Hogeschool Eindhoven. I recall with pleasure the time I spent there, supported by a grant No. B—61—99 of the Netherlands Organization for the Advancement of Pure Research (Z.W.O.), the many discussions with

Prof. Seidel, Aart Blokhuis, and the audience of my lectures who encouraged me to organize my ideas on the subject to a coherent whole. I also want to thank René Klerx for providing a set of lecture notes [16], whose last chapter formed the basis for the present paper.

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