SOLVING MINIMAX PROBLEMS BY INTERVAL METHODS

SHEN ZUHE*, A. NEUMAIER and M. C. EIERMANN

Mathematics Department
Nanjing University
Nanjing
People's Republic of China

Institut für Angewandte Mathematik
Universität Freiburg
Hermann-Herder-Str. 10
7800 Freiburg i.Br.
Germany

Abstract.

Interval methods are used to compute the minimax problem of a twice continuously differentiable function $f(y,z)$, $y\in\mathbb{R}^m$, $z\in\mathbb{R}^n$ of $m+n$ variables over an $m+n$-dimensional interval. The method provides bounds on both the minimax value of the function and the localizations of the minimax points. Numerical examples, arising in both mathematics and physics, show that the method works well.

AMS Classification: Primary 65K10, secondary 49A40, 90C99, 65G10.

Keywords: Minimax problems, interval methods.

1. Introduction.

Let $f(x)=f(y,z)$ be a real C^2-function defined on the box $X^{(0)}=(Y^{(0)},Z^{(0)})\subseteq\mathbb{R}^m\times\mathbb{R}^n$, with gradient $g:=(g_i)_{i=1,\dots,m+n}=(g_y,g_z)$. We say that $(y^*,z^*)\in(Y^{(0)},Z^{(0)})$ is a minimax point of f in $(Y^{(0)},Z^{(0)})$ if

$$f^*:=f(y^*,z^*)=\min_{z\in Z^{(0)}}\max_{y\in Y^{(0)}}f(y,z). \tag{1.1}$$

Then $f(y^*,z^*)=\max f(y,z^*)$ and either $y^*\in\{\underline{y}^{(0)},\bar{y}^{(0)}\}$, or

$$\frac{\partial f}{\partial y_i}(y^*,z^*)=0 \text{ and } \frac{\partial^2 f}{\partial y_i^2}(y^*,z^*)\le 0,\quad i=1,\dots,m.$$

The purpose of this paper is to propose a method for solving the minimax problem (1.1). Such problems arise in both mathematics and physics, as the following examples show.

* This paper has been written while the first author worked as a visiting professor at the Institut für Angewandte Mathematik of the University of Freiburg i.Br., West Germany. It has been sponsored by Stiftung Volkswagenwerk, number I/63 064. He wishes to thank Professor Dr. K. Nickel for helping him to make his stay possible.

Received June 1989. Revised May 1990.

1.1. *Chebyshev approximation.*

Given a function g: $Y^{(0)} \subseteq R^m \to R$ and a function space P_n of functions p_z: $\mathbb{R}^m \to \mathbb{R}$, the Chebyshev approximate p_z of g in p_n solves the following minimax problem

$$\min_{z} \max_{y \in Y^{(0)}} (g(y) - p_z(y))^2.$$

1.2. *Game theory.*

A game is a triple (Y, Z, k) where Y, Z denote the spaces of strategies for player I and II, respectively, and k is a real-valued pay-off function of $y \in Y$ and $z \in Z$. Under natural conditions, the optimal strategies for both players solve the saddle point problem

$$\min_{z \in Z} \max_{y \in Y} k(y, z) = \max_{y \in Y} \min_{z \in Z} k(y, z).$$

1.3. *Engineering design problems.*

In many engineering design problems, one is interested in minimizing the largest eigenvalue of an $n \times n$ symmetric matrix-valued function $A(y)$ of a variable y in $\mathbb{R}^n$. Let $\lambda_i(y)$, $i = 1, \ldots, n$ be the ith eigenvalue of $A(y)$ and put $f(i, n) := \lambda_i(y)$. We obtain the minimax problem

$$\min_{y \in Y^{(0)}} \max_{i = 1, \ldots, n} f(i, y).$$

Another example is a process which constructs electronic parts within certain tolerances and consider the problem of minimizing the error in manufacture. Specifically, suppose that when a state z is specified, the process actually produces the state $y + z$ for some y in the tolerance set Z and let $\theta(y + z)$ measure the resulting distortion. Since y is not known in advance, the worst-case distortion should be minimized, leading to the problem

$$\min_{z \in Z} \max_{y \in Y} \theta(y + z).$$

The Mandel'shtam problem which occurs in electrical circuit theory (see [3]) is of this kind. Specifically, let $z = (z_1, \ldots, z_n)$ and

$f(y, z) = \left| \sum_{k=1}^{n} \cos(ky + z_k) \right|$; of all $z \in \mathbb{R}^n$ we wish to select a vector z^* such that

$$\max_{y \in [0, 2\pi]} f(y^*, z) = \min_{z \in \mathbb{R}^n} \max_{y \in [0, 2\pi]} f(y, z).$$

2. Interval methods and results.

Methods of interval arithmetic can be used to reliably solve optimization problems. The reader may find a list of related references in [1], [12]. A thorough introduction to interval mathematics is given in the book of Moore [8]. In this section we briefly list some contents of interval methods and related results which are needed for the problem (1.1).

Denote by $\mathbb{IR}$, $\mathbb{IR}^n$, $\mathbb{IR}^{n\times n}$ the set of real intervals, n-dimensional interval vectors and $n \times n$ interval matrices respectively, by $\underline{x}$, $\bar{x}$, $\check{x}$ and $\mathrm{rad}(X)$ the lower bound, upper bound, midpoint and radius of $X \in \mathbb{IR}$, and by $\mathrm{int}(X)$ the interior of $X \in \mathbb{IR}$.

2.1. *Interval enclosure.*

We call a function $f(x)\colon \mathbb{IR}^n \to \mathbb{IR}^n$ a *Lipschitz interval extension* of $f\colon D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ on $X^{(0)} \subseteq D$ if

(i) $f(x) \in f(X)$ for all $x \in X$, $\quad X \subseteq X^{(0)}$,
(ii) $\mathrm{rad}(f(X)) \to 0$ when $\mathrm{rad}(X) \to 0$,
(iii) $\mathrm{rad}(f(X)) \le c\,\mathrm{rad}(X)$ for every $X \subseteq X^{(0)}$,

where c is a positive number. Clearly, such an extension is an inclusion of the range

$$f^*(X) := \{f(x) \mid x \in X\}.$$

Throughout this paper we assume that the first and second derivative of f have Lipschitz interval extensions $g(X) = f'(X)^T$ and $H(X) = f''(X)$ on $X^{(0)}$. Then we get higher order inclusions from the centered forms

$$f_c(X,x) := f(x) + f'(X)(X - x) \tag{2.1}$$

with a *center* $x \in X$, and

$$f_q(X) := \{f(x) + (x' - x)^T g(x) + \tfrac{1}{2}(x' - x)^T H(X)(x' - x) \mid x' \in X\}. \tag{2.2}$$

By Baumann [2], we can choose the center in (2.1) in an optimal manner. For this purpose write $F'(X) = [\underline{\ell}, \bar{\ell}]$ and define the vectors c^- and c^+ as

$$c_i^- := \begin{cases} \bar{x}_i & \text{if } \bar{\ell}_i \le 0 \\ \underline{x}_i & \text{if } \underline{\ell}_i \ge 0 \\ (\bar{\ell}_i \underline{x}_i - \underline{\ell}_i \bar{x}_i)/(\bar{\ell}_i - \underline{\ell}_i) & \text{otherwise} \end{cases} \tag{2.3}$$

$$c_i^+ := \begin{cases} \underline{x}_i & \text{if } \bar{\ell}_i \le 0 \\ \bar{x}_i & \text{if } \underline{\ell}_i \le 0 \\ (\underline{\ell}_i \underline{x}_i - \bar{\ell}_i \bar{x}_i)/(\underline{\ell}_i - \bar{\ell}_i) & \text{otherwise,} \end{cases} \tag{2.4}$$

then $x = c^-$ yields the best lower bound and $x = c^+$ the best upper bound in (2.1).

For (2.2), we use in fact modifications by Hansen [4], so that $H(X)$ is a lower triangular matrix instead of a symmetric matrix, and some of the occurences of the

variables in the expressions for the Hessian components use the center x instead of the box X, without violating the enclosure property.

2.2. *Discarding rules.*

As the algorithm for the problem (1.1) proceeds we dynamically subdivide $X^{(0)} = (Y^{(0)}, Z^{(0)})$ into subintervals. As each new subinterval X is generated we want to discard those points of X in which no minimax point is contained. In this section we will describe the discarding rules.

2.2.1. *Monotonicity test* (cf. Step 5 of the algorithm).

Suppose that for some interval $X = (Y, Z)$ and some index $i \in \{1, \ldots, m\}$, $g_i(X) > 0$ (<0), then f is strictly increasing (decreasing) with respect to the x_i-direction. Hence there is no minimax point in the interior of the interval X and we reduce X to $(X_1, \ldots, X_{i-1}, \bar{x}_i, X_{i+1}, \ldots, X_{m+n})$ if $\bar{x}_i = \bar{x}_i^{(0)}$ (to $(X_1, \ldots, X_{i-1}, \underline{x}_i, X_{i+1}, \ldots, X_{m+n})$ if $\underline{x}_i = \underline{x}_i^{(0)}$). Otherwise we discard X from our consideration for the problem (1.1). We call this discarding rule the monotonicity test. Hence this test allows us to automatically recognize if f is strictly monotonic with respect to the x_i-direction in an interval X, then X is discarded from our consideration or replaced by an edge piece of $X^{(0)}$.

2.2.2. *Nonconcavity test* (cf. Step 6 of the algorithm).

Suppose that for some interval i and some index $i \in \{1, \ldots, m\}$, $h_{ii}(X) > 0$, then there is no minimax point in the interior of the interval X. If $\bar{x}_i = \bar{x}_i^{(0)}$ we reduce X to $(X_1, \ldots, X_{i-1}, \bar{x}_i, X_{i+1}, \ldots, X_{m+n})$, and store $(X_1, \ldots, X_{i-1}, \underline{x}_i, X_{i+1}, \ldots, X_{m+n})$ if $\underline{x}_i = \underline{x}_i^{(0)}$. Otherwise we discard X or reduce it to $(X_1, \ldots, X_{i-1}, \underline{x}_i, X_{i+1}, \ldots, X_{m+n})$ if $\underline{x}_i = \underline{x}_i^{(0)}$. We call this rule the non-concavity test. It then allows us to recognize if f is not concave with respect to the x_i-direction in some interval X. We use this test to retain or discard X or replace X by one or two edge pieces $X^{(0)}$.

NOTE. If we know theoretically that there is no minimax point at the boundary of the initial interval $X^{(0)}$, we discard X if the monotonicity test or non-concavity test is satisfied.

2.2.3. *Evaluation test* (cf. Step 8 of the algorithm),

First we state the following lemma.

LEMMA 1. *Let* $(Y, Z) \subseteq (Y^{(0)}, Z^{(0)})$ *and*

$$f(y,z) > \bar{\varphi} \ge f^* := \min_{z \in Z^{(0)}} \max_{y \in Y^{(0)}} f(y,z)$$

for all $(y,z) \in (Y, Z)$, *then there is no minimax point in the strip* $(Y^{(0)}, Z)$.

PROOF. Suppose that $(\tilde{y}, z) \in (Y^{(0)}, Z)$ is a minimax point. For $y \in Y$ it follows

$$f(y', z) = \max_{\tilde{y} \in Y^{(0)}} f(\tilde{y}, z) \geq f(y, z) > \min_{z \in Z^{(0)}} \max_{y \in Y^{(0)}} f(y, z).$$

This contradicts the assumption that (y', z) is a minimax point of f.

NOTE. This is true even if Y is thin.

Hence, if we have an upper bound $\bar{\varphi}$ of f^* and an interval (Y, Z) like above, then we are able to discard the entire strip $(Y^{(0)}, Z)$.

In practice we want to retain points where $f(y, z) > \bar{\varphi}$ is not satisfied. Using the Taylor expansion, this is the case for points x if

$$f(x') + (x - x')^T g(x') + \tfrac{1}{2}(x - x')^T H(\xi)(x - x') \leq \bar{\varphi}.$$

Here we usually take x' as midpoint $\check{x}$ of X. Writing $E := \bar{\varphi} - f(x')$, $x'' := x - x'$ and $X'' := X - x'$ we have

$$x''^T g(x') + \tfrac{1}{2} x''^T H(\xi) x'' \leq E.$$

After collecting terms in x_i and replacing x_j'' by X_j'' $(j \neq i)$ and ξ by X we obtain the quadratic inequality

$$x_i''^2 h_{ii}(X) + x_i'' \left(g_i(x') + \tfrac{1}{2} \sum_{\substack{j=1 \\ j \neq i}}^{m+n} h_{ij}(X) X_j'' \right) + \sum_{\substack{j=1 \\ j \neq i}}^{m+n} g_j(x') X_j'' + \tfrac{1}{2} \sum_{\substack{j,k=1 \\ j,k \neq i}}^{m+n} h_{j,k}(X) X_j'' X_k'' - E \leq 0.$$

The solution set X_i'' of this inequality is either empty, a single interval or the union of two intervals and so is the intersection $X_i''' := (X_i'' + x_i') \cap X_i$.

For the case $i \in \{1, \ldots, m\}$, if $X_i''' \neq X_i$, then there exists an interval satisfying the lemma and we discard the strip $(Y^{(0)}, Z)$. (cf. Step 9 of the algorithm).

For the case $i \in \{m+1, \ldots, m+n\}$, if $X_i''' \neq X_i$, then reduce the ith component to X_i''' of all intervals with the same z-component as the interval X. I.e., we discard, reduce or split the strip $(Y^{(0)}, Z)$ (cf. Step 10 of the algorithm). In the same way if we have a lower bound $\underline{\varphi}$ of $\max_{y \in Y^{(0)}} f(y, z)$ for $z \in Z$ we retain points where $f(y, z) < \underline{\varphi}$ is not satisfied (cf. Step 8 of the algorithm).

2.3. *Interval Newton method* (cf. Step 7 of the algorithm).

Suppose that the minimax point (y^*, z^*) of f is an interior point of $(Y^{(0)}, Z^{(0)})$, then y^* is a stationary point of $f(y^*, z^*)$ with respect to y. This fact motivates us to bound the solution of $g_y(y, z) = 0$. By the Taylor expansion we have

$$g(x) = g(x') + H(\xi)(x - x').$$

Suppose that $X = (Y, Z)$ is a new interval generated from $X^{(0)} = (Y^{(0)}, Z^{(0)})$ at a certain step of the solution process, then we will solve the linear interval equation

$$H_{yy}(X)(y - y') + g_y(y', z') + H_{yz}(X)(Z - z') = 0.$$

To this end we execute one step of the preconditioned Gauss-Seidel iteration,

$$\bar{Y} - y' := \Gamma(CA, Cb, Y - y'), \quad \text{with}$$

$A := H_{yy}(X)$
$b := -g_y(y', z') - H_{yz}(X)(Z - z')$,
$C := \check{A}^{-1}$ (but with $C = 1$ when $m = 1$),

where $\Gamma(A, B, X)$ is the Gauss-Seidel operator defined to be the enclosure of the nonempty bounded set $\{x \in X \mid ax = b \text{ for some } a \in A, b \in B\}$, obtained by solving in turn the ith equation for the ith variable $(i = 1, \ldots, n)$, taking intersections with the previous enclosures. (cf. Hansen [5], Neumaier [9], [10]).

If the solution set $\bar{Y}$ is empty we discard the interval X. Otherwise we reduce the interval $X = (\ldots, Y_i, \ldots)$ to $(\ldots, \bar{Y}_i, \ldots)$.

2.4. *Data management and update* (cf. step 10 of the algorithm).

From the initial interval $X^{(0)} = (Y^{(0)}, Z^{(0)})$ we generate a list of sublists $\{S^{(k)}\}$. Each sublist $S^{(k)}$ contains intervals $X^{(j,k)} := (Y^{(j,k)}, Z^{(k)})$, i.e. they have a common second part $Z^{(k)}$. Hence it suffices to store $Y^{(j,k)}$ for each interval $X^{(j,k)}$ and $Z^{(k)}$ for each sublist $S^{(k)}$. We store furthermore

$$\varphi(X^{(j,k)}) := [\underline{f}(\tilde{y}^{(j,k)}, Z^{(k)}), \bar{f}(Y^{(j,k)}, \tilde{z}^{(k)})], \qquad j, k \geq 1,$$

$$\varphi(S^{(k)}) := \left[\max_j \underline{\varphi}(X^{(j,k)}), \max_j \bar{\varphi}(X^{(j,k)}) \right], \qquad k \geq 1,$$

$$\varphi(L) := \left[\min_k \underline{\varphi}(S^{(k)}), \min_k \bar{\varphi}(S^{(k)}) \right],$$

where
$\underline{f}(X) := \inf(f(c^-) + f'(X)(X - c^-))$,
$\bar{f}(X) := \sup(f(c^+) + f'(X)(X - c^+))$,
and c^-, c^+ are defined in (2.3), (2.4).

Simple estimates show that $f^* \in \varphi(L) = [\underline{\varphi}(L), \bar{\varphi}(L)]$.

The very first step of the algorithm is to evaluate $\underline{\varphi}(X)$ and $\bar{\varphi}(X)$ at the initial interval $X^{(0)} = (Y^{(0)}, Z^{(0)})$. These values serve as the first ones for $\underline{\varphi}(L)$ and $\bar{\varphi}(L)$. If $X^{(j,k)} = (Y^{(j,k)}, Z^{(k)})$ is a new interval generated from $X^{(0)} = (Y^{(0)}, Z^{(0)})$ at a certain step of the solution process, we immediately evaluate $\underline{\varphi}(Z^{(j,k)})$ and $\bar{\varphi}(X^{(j,k)})$ and update $\varphi(S^{(k)})$ as

$$\varphi(S^{(k)}) := \left[\max(\underline{\varphi}(S^{(k)}), \max_j \underline{\varphi}(X^{(j,k)})), \min(\bar{\varphi}(S^{(k)}), \max_j \bar{\varphi}(X^{(j,k)})) \right].$$

If an interval $X^{(j,k)}$ was not reduced enough in the solution process by the discarding rules and the interval Newton method, then we bisect the interval in the direction of maximal width of $X^{(j,k)}$. If $X^{(j,k)} = (Y^{(j,k)}, Z^{(k)})$ is bisected in the z-direction we get the intervals $([\underline{y}^{(j,k)}, \bar{y}^{(j,k)}], Z^{(k)})$ and $([\check{y}^{(j,k)}, \bar{y}^{(j,k)}], Z^{(k)})$. Otherwise we have to bisect the entire sublist $S^{(k)}$, i.e. we get the sublists S' and S'' with intervals $(Y^{(j,k)}, [\underline{z}^{(k)}, \check{z}^{(k)}])$ and $(Y^{(j,k)}, [\check{z}^{(k)}, \bar{z}^{(k)}])$, $j \geq 1$. We insert the resulting intervals and sublists in the list L.

Finally, at the end of each loop, if necessary, we update the value $\varphi(L)$ as

$$\varphi(L) := \left[\max\left(\underline{\varphi}(L), \min_k \underline{\varphi}(S^{(k)})\right), \min\left(\bar{\varphi}(L), \min_k \bar{\varphi}(S^{(k)})\right) \right].$$

Thus, $\underline{\varphi}(L)$ and $\bar{\varphi}(L)$ are always the greatest lower and smallest upper bound of f^* currently known.

3. Algorithm.

Step 1. Initialization. Set $X := X^{(0)}$. Define a tolerance $\varepsilon > 0$. To begin with the list contains one sublist with the interval X. Goto step 3.

Step 2. Begin loop. Select a sublist S and an interval X of this sublist. Set Reduce to false.

Step 3. Evaluation test: $\bar{f}(X) < \underline{\varphi}(X)$.
If $\bar{f}(X) < \underline{\varphi}(X)$, then discard the interval X and goto step 12.

Step 4. Evaluation test: $\underline{\varphi}(X) > \bar{\varphi}(L)$.
If $\underline{\varphi}(X) > \bar{\varphi}(L)$, then discard the sublist S and goto step 12.

Step 5. Monotonicity test: $g_y(X) > 0 \quad (< 0)$.
For $i = 1, \ldots, m$ do the following:
Evaluate $g_i(X)$.
If $g_i(X) > 0 \quad (< 0)$, then
set $X := (\ldots, \bar{x}_i, \ldots) \quad (:= (\ldots, \underline{x}_i, \ldots))$ and Reduce to true,
if $\bar{x}_i = \bar{x}_i^{(0)} \quad (\underline{x}_i = \underline{x}_i^{(0)})$,
discard the interval X and goto step 12 otherwise.

Step 6. Non-concavity test: $H_{yy}(X) > 0$.
For $i = 1, \ldots, m$ do the following:
Evaluate $h_{ii}(X)$.
If $h_{ii}(X) > 0$, then
set $X := (\ldots, \bar{x}_i, \ldots)$ and Reduce to true if $\bar{x}_i = \bar{x}_i^{(0)}$,
set $X := (\ldots, \underline{x}_i, \ldots)$ and Reduce to true if $\bar{x}_i \neq \bar{x}_i^{(0)}$ and $\underline{x}_i = \underline{x}_i^{(0)}$.

Insert the interval $(\ldots, x_i, \ldots)$ in the sublist S if $\bar{x}_i = \bar{x}_i^{(0)}$ and $\underline{x}_i = \underline{x}_i^{(0)}$.
Discard the interval X and goto Step 12 if $\bar{x}_i \neq \bar{x}_i^{(0)}$ and $\underline{x}_i \neq \underline{x}_i^{(0)}$.

Step 7. Interval Newton method: Solve $g_y(\check{y}, \check{z}) + H_{yx}(Y, Z)(y - \check{y}, Z - \check{z}) = 0$ for y.
Compute $C = \check{H}_{yy}(X)^{-1}$, $q = C(g_y(\check{Y}, \check{Z}) + H_{yz}(Z - z))$.
For $i = 1, \ldots, m$ do the following:
Set $X_i' = \check{x}_i + \Gamma_i(CH_{yy}(X), q, Y - \check{y})$.
If X_i' is empty, then discard the interval X and goto Step 12.
If X_i' is a single interval, then reduce the interval X to $(\ldots, X_i', \ldots)$ and set Reduce to true if $\operatorname{rad}(X_i') \leq \operatorname{rad}(x_i)/2$.
If X_i' is the union of the intervals X_i'' and X_i''', then reduce the interval X to $(\ldots, X_i'', \ldots)$, insert the reduced interval $(\ldots, X_i''', \ldots)$ to the sublist and set Reduce to true.

Step 8. Evaluation test: Solve $f(y, z) \geq \underline{\varphi}(s)$ for y.
For $i = 1, \ldots, m$ do the following:
Solve the quadratic inequality
$f(\check{x}) + (X - \check{x})^T g(\check{x}) + \frac{1}{2}(X - \check{x})^T H(X)(X - \check{x}) \geq \underline{\varphi}(s)$ in the x_i-direction and obtain the solution set X_i'.
If $X_i' \cap X_i$ is empty, then discard the interval X and goto step 12.
If $X_i' \cap X_i$ is the single interval X_i'', then reduce the interval X to $(\ldots, X_i'', \ldots)$. If $\operatorname{rad}(X_i'') \leq \operatorname{rad}(X_i)/2$, then set Reduce to true.
If $X_i' \cap X_i$ is the union of the intervals X_i'' and X_i''', then reduce the interval X to $(\ldots, X_i'', \ldots)$, insert the reduced interval $(\ldots, X_i''', \ldots)$ in the sublist S and set Reduce to true.

Step 9. Evaluation test: Solve $f(y, z) \leq \bar{\varphi}(L)$ for y.
For $i = 1, \ldots, m$ do the following:
Solve the quadratic inequality
$f(\check{x}) + (X - \check{x})^T g(\check{x}) + \frac{1}{2}(X - \check{x})^T H(X)(X - \check{x}) \leq \bar{\varphi}(L)$
in the x_i-direction and obtain the solution set X_i'.
If $X_i' \cap X_i$ is empty, then discard the sublist S and goto Step 12.

Step 10. Evaluation test: solve $f(\check{y}, z) \leq \bar{\varphi}(L)$ for z.
For $i = m + 1, \ldots, m + n$ do the following:
Solve the quadratic inequality
$f(\check{y}, Z) + (Z - \check{z})^T g_z(\check{x}) + \frac{1}{2}(Z - \check{z})^T H_{zz}(X)(Z - \check{z}) \leq \bar{\varphi}(L)$
in the x_i-direction and obtain the solution set X_i'.
If $X_i' \cap X$ is empty then discard the sublist S and goto Step 12.
If $X_i' \cap X$ is the single interval X_i'', then reduce the ith component of the intervals of the sublist S to X_i'' and set Reduce to true if $\operatorname{rad}(X_i'') \leq \operatorname{rad}(X_i)/2$.
If $X_i' \cap X_i$ is the union of the intervals X_i'' and X_i''', then reduce the ith component of the intervals of the sublist S to X_i'', insert a sublist S' with the reduced intervals of the sublist S to X_i''' and set Reduce to true.

Step 11. Bisection.
If in the Steps 3–10 the interval X was not enough reduced (Reduce is false), then bisect the sublist S in the direction of maximal width of X and insert the resulting intervals or sublists in the list L.

Step 12. End loop.
Update the value $\varphi(L)$. If $\operatorname{rad}(\varphi(L)) > \varepsilon|\varphi(L)|$, then goto Step 2.

Step 13. Print solution.
Print $\varphi(L)$ is enclosure of minimax value, print remaining intervals of sublists of the list L is the enclosure of the minimax point(s).

If a reasonable tolerance ε of the algorithm is given, the computation will always stop after a finite number of steps. This holds since the order of the list is of decreasing width and the bisection rule of the interval is in the direction of maximum width, so that the width of the interval generated by the algorithm above from the initial interval $X^{(0)}$ tends to zero as the process proceeds. Through the discarding rules and the interval Newton method, locally only a few intervals and sublists remain in the list. It is not difficult to show that this implies that $\bar{\varphi}(L) - f^*$ must become arbitrarily small if the tolerance ε is small enough.

4. Numerical examples.

4.1. *Example.*
Solve the minimax problem with

$$f(y,z) = y(1-y)(y-z)^4, \qquad Y^{(0)} = Z^{(0)} = [0,1]$$

which has the exact solution

$$x_{\pm}^* = (1/2, 1/2(1 \pm \sqrt{(2/3)}), \qquad f(x_{\pm}^*) = 1/432 = 0.00231\overline{48}.$$

With a specified error tolerance of $\varepsilon = .1E-16$, the minimax value was enclosed by the interval $.2314814814814815\frac{52}{46}E-2$ (with a width of 11 machine numbers); the total width of the boxes enclosing the two minimax points was $.363E-27$, 121 loops were taken with maximally 23 boxes and 4 sublists at a time.

4.2. *Example.*
(Mandel'shtam Problem). Solve the minimax problem with

$$f(y,z) = \left(\cos y + \sum_{k=1}^{m} \cos((k+1)y + z_k)\right)^2$$

$$y \in Y^{(0)} = [-\pi, \pi]. \qquad z = (z_1, \ldots, z_m) \in Z^{(0)} = ([-\pi, \pi])_{k=1,\ldots,m},$$

(see [3]).

With the same error tolerance, the minimax value was enclosed by the interval $.309820755731058\frac{76}{47}E+1$ (with a width of 13 machine numbers); the total width of the boxes enclosing the four minimax points was $.131E-26$, 248 loops were taken with maximally 99 boxes and 14 sublists at a time.

REFERENCES

1. Basso, P.: *Optimal search for the global maximum of functions with bounded seminorm.* SIAM J. Numer. Anal., 9, 888–903 (1985).
2. Baumann, E.: *Optimal centered form*, BIT 28, 80–87 (1987).
3. Dem'yanov, V. F. and F. N. Malozemov: *Introduction To Minimax*. John Wiley & Sons, 1974.
4. Hansen, E.: *Global optimization using interval analysis – the multidimensional case*. Numer. Math., 34, 247–270 (1980).
5. Hansen, E. and S. Sengupta: *Bounding solutions of systems of equations using interval analysis*, BIT 21, 203–211 (1981).
6. Krawczyk, R. und K. Nickel: *Die zentrische Form in der Intervallarithmetik, ihre quadratische Konvergenz und ihre Inklusionsisotonie*, Computing, 28, 117–137 (1982).
7. Krawczyk, R. and A. Neumaier: *Interval slopes for rational functions and associated centered forms*. SIAM J. Numer. Anal., 22, 604–616 (1985).
8. Moore, R. E.: *Methods and Applications of Interval Analysis*, SIAM. Philadelphia, 1979.
9. Neumaier, A.: *Overestimation in linear interval equations*, SIAM J. Numer. Anal., 24, 207–214 (1987).
10. Neumaier, A.: *Interval Methods for Systems of Equations*, Cambridge University Press. Cambridge, 1990.
11. Nickel, K.: *Optimization using interval mathematics*, Freiburger Intervall-Berichte, 1 (1986).
12. Ratschek, H. and J. Rokne: *Computer Methods for the Range of Functions*, Horwood, Chichester, (1984).
13. Shubert, B. O.: *A sequential method seeking the global maximum of a function*. SIAM J. Numer. Anal., 9, 379–388 (1972).