

## EXCEPTIONAL GRAPHS WITH SMALLEST EIGENVALUE $-2$ AND RELATED PROBLEMS

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**ABSTRACT.** This paper summarizes the known results on graphs with smallest eigenvalue around  $-2$ , and completes the theory by proving a number of new results, giving comprehensive tables of the finitely many exceptions, and posing some new problems. Then the theory is applied to characterize a class of distance-regular graphs of large diameter by their intersection array.

### INTRODUCTION

This paper presents a theory of graphs with smallest eigenvalue around  $-2$  (in §§1 and 2, with tables in the appendix of the microfiche section) and their application to a characterization problem for distance-regular graphs (§3).

Apart from the classical line graph theorem of Cameron, Goethals, Seidel, and Shult [10]—which is introduced here in a new way by means of root lattices—and consequences of it observed by Bussemaker, Cvetković, Doob, Kumar, Rao, Seidel, Simić, Singhi, and Vijayan [8, 16, 18, 20, 21, 30, 35, 44], we obtain a number of new results, namely

- (i) a classification of graphs  $\Gamma$  with smallest eigenvalue  $-2$  such that  $\Gamma$  or its complement are edge-regular (Theorem 1.2),
- (ii) a complete list of minimal graphs with smallest eigenvalue  $-2$  (Theorem 1.7 and Table 3),
- (iii) a complete list of minimal forbidden subgraphs for the class of graphs with smallest eigenvalue  $\geq -2$  (Table 4), and
- (iv) the computation of the eigenvalue gap at  $-2$  (Theorem 2.4).

The importance of the eigenvalue gap is demonstrated by the characterization of a class of distance-regular graphs (folded cubes, folded half-cubes, and folded Johnson graphs of large diameters) by their intersection arrays, in the spirit of earlier work of Terwilliger [41] and Neumaier [34].

The proofs for (i)–(iv) are based on extensive computer calculations which enumerate the finitely many exceptions arising from the exceptional root lattices (or root systems)  $E_6$ ,  $E_7$ , and  $E_8$ . We challenge the reader at several places to provide conceptual proofs of some remarkable observations deduced here from lists of graphs generated by computer. We also point out a number of open questions.

*Notation.* If  $\Gamma$  is a graph and  $S$  a set of vertices of  $\Gamma$ , we denote by  $\Gamma \setminus S$  the

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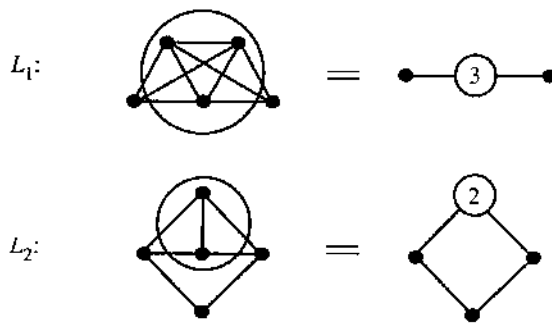


FIGURE 1. The two minimal forbidden line graphs with five vertices.

graph obtained by deleting from  $\Gamma$  the vertices in  $S$  and all edges containing a vertex of  $S$ . A *subgraph* of  $\Gamma$  always refers to an induced subgraph, i.e., a graph of the form  $\Gamma \setminus S$ . For a vertex  $\gamma \in \Gamma$ ,  $\Gamma(\gamma)$  denotes the subgraph induced on the set of neighbors of  $\gamma$ , and  $\gamma^\perp$  denotes the subgraph induced on the set consisting of  $\gamma$  and its neighbors. The relation  $\equiv$  defined by  $\gamma \equiv \delta$  if and only if  $\gamma^\perp = \delta^\perp$  is an equivalence relation on the set of vertices, and if we identify equivalent vertices, we obtain a *reduced graph*  $\bar{\Gamma}$ .  $\Gamma$  can be recovered from  $\bar{\Gamma}$  as a *clique extension*, i.e., by replacing each vertex  $\bar{\gamma}$  of  $\bar{\Gamma}$  by a suitable clique  $C_\gamma$ , and joining the vertices of  $C_\gamma$  with the vertices of  $C_\delta$  when  $\bar{\gamma}$  and  $\bar{\delta}$  are adjacent. We shall draw a clique extension of  $\bar{\Gamma}$  by drawing vertices of  $\bar{\Gamma}$  replaced by an  $i$ -clique as circles with label  $i$  if  $i > 1$ , and as black dots if  $i = 1$  (cf. Figure 1). The *eigenvalues* of a graph  $\Gamma$  are the eigenvalues of its  $(0, 1)$ -adjacency matrix; the *spectrum* of  $\Gamma$  is the collection of its eigenvalues (together with their multiplicities). For a general discussion of graph spectra, see the book by Cvetković, Doob, and Sachs [17]. We denote the largest eigenvalue of  $\Gamma$  by  $\lambda_{\max}(\Gamma)$  and the smallest eigenvalue of  $\Gamma$  by  $\lambda_{\min}(\Gamma)$ . By interlacing (cf. [17]), we have for a subgraph  $\Gamma'$  of  $\Gamma$  the relations

$$\lambda_{\min}(\Gamma) \leq \lambda_{\min}(\Gamma'), \quad \lambda_{\max}(\Gamma') \leq \lambda_{\max}(\Gamma).$$

The minimal valency of a graph  $\Gamma$  is denoted by  $k_{\min}(\Gamma)$ . A graph  $\Gamma$  is called *regular* if every vertex has the same valency  $k$ , *edge-regular* (*coedge-regular*) if  $\Gamma$  is regular and any two adjacent (nonadjacent) vertices have the same number  $\lambda$  ( $\mu$ ) of common neighbors, *amply regular* if  $\Gamma$  is edge-regular and any two vertices at distance 2 have the same number of common neighbors, and *strongly regular* if it is edge-regular and coedge-regular.

Since isomorphic graphs have the same spectrum, we do not distinguish between different isomorphic graphs.

## 1. GRAPHS WITH SMALLEST EIGENVALUE $\geq -2$

The well-known fact that all line graphs have smallest eigenvalue  $\geq -2$  prompted a great deal of interest in the characterization of certain classes of graphs  $\Gamma$  with  $\lambda_{\min}(\Gamma) \geq \lambda^*$  for  $\lambda^*$  around  $-2$ . The work done on this problem culminated in a beautiful theory of Cameron, Goethals, Seidel, and Shult [10] who related the question to root systems. Together with computer calculations by Bussemaker, Cvetković, and Seidel [8], this theory implies a complete

classification of all regular graphs with smallest eigenvalue  $\geq -2$ , and, as noted by Doob and Cvetković [21], a classification of all graphs (whether regular or not) with smallest eigenvalue  $> -2$ . In this section we summarize these results, and give some numerical information on the exceptional graphs.

A *root lattice* is an additive subgroup  $L$  of  $\mathbb{R}^n$  generated by a set  $X$  of vectors such that  $(x, x) = 2$  and  $(x, y) \in \mathbb{Z}$  for all  $x, y \in X$ ; here,  $(x, y) = \sum x_i y_i$  is the standard inner product in  $\mathbb{R}^n$ . The vectors in  $L$  of norm  $(x, x) = 2$  are called the *roots* of  $L$ . A root lattice is called *irreducible* if it is not a direct sum of proper sublattices. Every irreducible root lattice is isomorphic to one of the lattices

$$A_n = \{x \in \mathbb{Z}^{n+1} \mid \sum x_i = 0\} \quad (n \geq 1),$$

$$D_n = \{x \in \mathbb{Z}^n \mid \sum x_i \text{ even}\} \quad (n \geq 4),$$

$$E_8 = D_8 \cup (c + D_8), \quad \text{where } c = \frac{1}{2}(1, 1, 1, 1, 1, 1, 1, 1),$$

$$E_7 = \{x \in E_8 \mid \sum x_i = 0\},$$

$$E_6 = \{x \in E_7 \mid x_7 + x_8 = 0\}$$

(cf. Witt [45], Cameron et al. [10], Neumaier [33]). In terms of basis vectors  $e_1, \dots, e_n$  of  $\mathbb{Z}^n$ , the roots of  $A_n$  are the  $n(n+1)$  vectors

$$e_i - e_j \quad (1 \leq i < j \leq n+1),$$

those of  $D_n$  are the  $2n(n-1)$  vectors

$$\pm e_i \pm e_j \quad (1 \leq i < j \leq n),$$

and those of  $E_8$  are the  $240 = 112 + 128$  vectors

$$\pm e_i \pm e_j \quad (1 \leq i < j \leq 8)$$

and, with an even number of  $+$  signs,

$$\frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4 \pm e_5 \pm e_6 \pm e_7 \pm e_8).$$

From this, one finds that  $E_7$  contains  $126 = 56 + 70$  roots and  $E_6$  contains  $72 = 32 + 40$  roots.

If  $\Gamma$  is a connected graph with  $\lambda_{\min}(\Gamma) > -2$  and adjacency matrix  $A$ , then  $G = A + 2I$  is a symmetric positive semidefinite matrix. Thus,  $G$  is the Gram matrix of a set  $X$  of vectors of  $\mathbb{R}^n$ , i.e., there is a bijection  $-: \Gamma \rightarrow X$  such that

$$(\gamma, \delta) = \begin{cases} 2 & \text{if } \gamma = \delta, \\ 1 & \text{if } \gamma, \delta \text{ are adjacent,} \\ 0 & \text{otherwise.} \end{cases}$$

Such a mapping is called a spherical  $(2, 1, 0)$ -representation, and in this paper simply a *representation* of  $\Gamma$ . The additive subgroup  $L^+(\Gamma)$  generated by  $X$  is a root lattice (whose isomorphism type depends on  $\Gamma$  but not on  $X$ ), and since  $\Gamma$  is connected,  $L^+(\Gamma)$  is irreducible. This implies the basic observation of Cameron et al. [10] that every connected graph  $\Gamma$  with  $\lambda_{\min}(\Gamma) \geq -2$  has a representation by roots of  $A_n$  ( $n \geq 1$ ),  $D_n$  ( $n \geq 4$ ), or  $E_n$  ( $n = 6, 7, 8$ ). Conversely, if  $\Gamma$  is represented by roots of  $A_n$ ,  $D_n$ , or  $E_n$ , then the Gram matrix  $A + 2I$  of the image of  $\Gamma$  is positive semidefinite, so that  $\lambda_{\min}(\Gamma) \geq -2$ .

The *line graph* of a graph  $\Delta$  is the graph  $L(\Delta)$  whose vertices are the edges of  $\Delta$ , two edges being adjacent if they intersect. *Generalized line graphs*  $L(\Delta, a_1, \dots, a_m)$ , introduced by Hoffman [25], are obtained from the line graph  $L(\Delta)$  of a graph  $\Delta$  with vertex set  $\{1, \dots, m\}$  by adding the vertices  $(i, \pm l)$  ( $i = 1, \dots, m; l = 1, \dots, a_i$ ) and joining  $(i, l)$  with all vertices  $i, j$  of  $L(\Delta)$  and all  $(i, l')$ ,  $l' \neq \pm l$ . We get a representation of a generalized line graph by representing the vertices  $ij$  of  $L(\Delta)$  by  $e_i + e_j$  and the adjoined vertices  $(i, \pm l)$  by  $e_i \pm e_{(i,l)}$ ,  $l = 1, \dots, a_i$ , where the  $e_i, e_{(i,l)}$  form a set of orthonormal vectors. Moreover, if  $\Delta$  is bipartite with bipartite parts  $A, B$ , then  $L(\Delta)$  has also a representation by roots of  $A_{m-1}$  obtained by representing an edge  $ij$  with  $i \in A, j \in B$  by  $e_i - e_j$ .

By analyzing the possible representations by roots of  $A_n$  and  $D_n$ , Cameron, Goethals, Seidel, and Shult [10] arrived at the following result.

1.1. **Theorem.** *Let  $\Gamma$  be a connected graph with smallest eigenvalue  $\geq -2$ . Then one of the following holds:*

- (i)  $\Gamma$  is a generalized line graph.
- (ii)  $\Gamma$  has a representation by roots of  $E_8$ . The number  $v$  of vertices, and the average valency  $k$ , are restricted by  $v \leq \min(36, 2k + 8)$ . Moreover, every vertex has valency at most 28.

An as yet unsolved problem is the characterization of the graphs under (ii). Since every subgraph of such a graph is again represented by roots of  $E_8$ , it suffices to determine the (finitely many) maximal graphs under (ii) which are not generalized line graphs. It seems that most graphs under (ii) can be obtained by switching.

*Switching* a graph  $\Gamma$  with respect to a set  $S$  of vertices (or with respect to its complement  $S^c$ ) is the operation of removing all edges of  $\Gamma$  between  $S$  and  $S^c$  and adding the new edges  $\gamma_1, \gamma_2$  ( $\gamma_1 \in S, \gamma_2 \in S^c, \gamma_1 \neq \gamma_2$ ) (cf. Seidel [36]). If  $\Gamma^1$  is obtained from  $\Gamma$  by switching with respect to  $S_1$ , and  $\Gamma^2$  is obtained from  $\Gamma^1$  by switching with respect to  $S_2$ , then  $\Gamma^2$  can be directly obtained from  $\Gamma$  by switching with respect to the symmetric difference  $(S_1 \cap S_2) \cup (S_1^c \cap S_2^c)$ . Therefore, switching defines an equivalence relation on the set of graphs with a given vertex set. If  $\Gamma$  is the line graph of a graph  $\Delta$  with vertex set  $\{1, 2, \dots, 8\}$ , then the graph  $\Gamma'$  obtained from  $\Gamma$  by switching with respect to  $S$  can be represented in  $E_8$  by the roots  $e_i + e_j$  (if  $ij$  is an edge  $\notin S$ ) and  $c - e_i - e_j$  (if  $ij$  is an edge in  $S$ ); here,  $c = \frac{1}{2}(e_1 + \dots + e_8)$ . Therefore,  $\Gamma'$  has smallest eigenvalue  $\geq -2$ . The maximal graphs obtainable from this construction are the graphs which are switching-equivalent to the triangular graph  $T(8)$ , the line graph of the complete graph on eight vertices.

A graph with 36 vertices, maximal valency 28, and smallest eigenvalue  $-2$  which is not a generalized line graph can, e.g., be obtained by adding to  $K_8 + L(K_8)$  edges joining  $i \in K_8$  with  $jk \in L(K_8)$  whenever  $i \notin \{j, k\}$ ; a  $(2, 1, 0)$ -representation in  $E_8$  is given by the vectors  $\frac{1}{2}(f_1 + \dots + f_8) - f_i$  ( $i \leq 8$ ) and  $f_i + f_j$  ( $i < j \leq 8$ ), where  $f_1, \dots, f_8$  are obtained from  $e_1, \dots, e_8$  by reversing the sign of one  $e_i$ . Thus, examples satisfying equality in (ii) of the theorem exist.

If we restrict ourselves to regular graphs, sharper results are possible. Bussemaker, Cvetković, and Seidel [8], supported by a computer, used this theorem to show that, up to isomorphism, there are precisely 187 regular graphs with

smallest eigenvalue -2 (and none with  $\lambda_{\min}(\Gamma) > -2$ ) which are not generalized line graphs, namely

- 4 graphs generating  $E_6$  (nos. 5, 185-187 in [8]),
- 24 graphs generating  $E_7$  (nos. 19, 69, 164-184 in [8]),
- 159 graphs generating  $E_8$  (the remaining ones).

Reference [8] also contains explicit adjacency matrices and representations by roots of  $E_8$ . To simplify the application of their results, we describe here the most important ones, and give (in Table 1) a list of relevant numerical invariants of these graphs. As shown in [8], all 187 graphs are subgraphs of the Gosset graph  $E_7(1)$  with 56 vertices  $e_i + e_j, c - e_i - e_j$  ( $1 \leq i < j \leq 8$ ), two vertices being adjacent if their inner product is 1. (Note that this defines not a representation in our sense, since  $(e_i + e_j, c - e_i - e_j) = -1$ ; indeed, the smallest eigenvalue of  $E_7(1)$  is -9. However, all subgraphs not containing such an antipodal pair of vertices are switching-equivalent to a line graph and hence have smallest eigenvalue  $\geq -2$ .) The Gosset graph is the skeleton of the Gosset polytope  $3_{21}$  (cf. Coxeter [15]), and is related to the 28 bitangents of a quartic surface (cf. Dickson [32]). A modern description of the structure of  $E_7(1)$  is given by Taylor [39] in terms of a regular two-graph with 28 points. We are interested in the graphs  $E_n(1)$  ( $n = 1, \dots, 6$ ), the subgraphs of  $E_7(1)$  induced on the set of common neighbors of  $e_i + e_8$  ( $i = n + 1, \dots, 7$ ), and some other graphs.

(i) The Schläfli graph  $E_6(1)$  (no. 184 in Table 1) has the 27 vertices  $e_i + e_7, e_i + e_8$  ( $i \leq 6$ ),  $c - e_i - e_j$  ( $i < j \leq 6$ ) and valency 16. The graph  $E_6(1)$  is the complement of the point graph of the generalized quadrangle of order (2, 4) with 27 points, and is related to the 27 lines on a cubic surface (see Baker [1]; one easily recognizes a double six in the description given).

(ii) The Clebsch graph  $E_5(1)$  (no. 187 in Table 1) has the 16 vertices  $e_6 + e_7, e_i + e_8$  ( $i \leq 5$ ),  $c - e_i - e_j$  ( $1 < j \leq 5$ ) and valency 10. The complement is a triangle-free graph obtained by identifying antipodal points of the 5-dimensional cube. The graph  $E_5(1)$  contains two regular proper subgraphs which are not line graphs (nos. 185, 186 in Table 1); namely a graph with the 12 vertices  $e_i + e_8$  ( $i = 2, 3, 4$ ),  $c - e_i - e_j$  ( $i < j \leq 5, (i, j) \neq (1, 5)$ ) and valency 7, and a graph with the eight vertices  $e_6 + e_7, e_i + e_8$  ( $i = 2, 3, 4$ ),  $c - e_i - e_{i+1}$  ( $i \leq 4$ ) and valency 4 (cf. Figure 2).

(iii) The graph  $E_4(1)$  is isomorphic to the triangular graph  $T(5)$  with ten vertices and valency 6.

(iv) The Petersen graph (no. 5 in Table 1) has ten vertices and valency 3. It is obtained from the triangular graph  $T(5)$  with vertices  $e_i + e_j$  ( $i < j \leq 5$ ) by switching with respect to  $\{e_i + e_j \mid i, j \leq 5, j \equiv i + 1 \pmod{5}\}$ . This graph is strongly regular with parameters  $(\nu, k, \lambda, \mu) = (10, 3, 0, 1)$ .

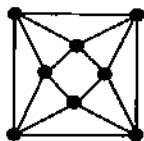


FIGURE 2. A subgraph of the Clebsch graph.

(v) The Shrikhande graph  $L_2^2(4)$  (no. 69 in Table 1) has 16 vertices and valency 6. It is obtained from the  $(4 \times 4)$ -grid  $L_2(4)$  with vertices  $e_i + e_j$  ( $i \leq 4 < j$ ) by switching with respect to  $\{e_i + e_{i+4} \mid i \leq 4\}$ . This graph is strongly regular with the same parameters  $(\nu, k, \lambda, \mu) = (16, 6, 2, 2)$  as  $L_2(4)$  (cf. Shrikhande [37]). It is a quotient of the triangular lattice in  $\mathbb{R}^2$ .

(vi) The three Chang graphs  $T'(8)$ ,  $T''(8)$ ,  $T'''(8)$  (nos. 161–163 in Table 1) have 28 vertices and valency 12. They are obtained from the triangular graph  $T(8)$  with vertices  $e_i + e_j$  ( $i < j \leq 8$ ) by switching with respect to one of the sets

$$\begin{aligned} &\{e_i + e_{i+4} \mid 1 \leq i \leq 4\} \quad \text{for } T'(8), \\ &\{e_i + e_j \mid 1 \leq i \leq 8, j \equiv i + 1 \pmod{8}\} \quad \text{for } T''(8), \\ &\{e_1 + e_2, e_2 + e_3, e_3 + e_1, e_4 + e_5, e_5 + e_6, e_6 + e_7, e_7 + e_8, e_8 + e_4\} \\ &\hspace{15em} \text{for } T'''(8). \end{aligned}$$

These graphs are all strongly regular with the same parameters  $(\nu, k, \lambda, \mu) = (28, 12, 6, 4)$  as  $T(8)$  (cf. Chang [11, 12], and Seidel [36] for equivalence under switching).

The information collected by Bussemaker et al. [8] about the 187 regular exceptional graphs can be summarized together with the analysis of regular generalized line graphs in Cameron et al. [10] in the following theorem.

**1.2. Theorem.** *Let  $\Gamma$  be a connected regular graph with  $v$  points, valency  $k$ , and smallest eigenvalue  $\geq -2$ . Then one of the following holds:*

- (i)  $\Gamma$  is the line graph of a regular or a bipartite semiregular connected graph  $\Delta$ .
- (ii)  $v = 2(k + 2) \leq 28$ , and  $\Gamma$  is a subgraph of  $E_7(1)$ , switching-equivalent to the line graph of a graph  $\Delta$  on eight vertices, where all valencies of  $\Delta$  have the same parity (graphs nos. 1–163 in Table 1).
- (iii)  $v = \frac{3}{2}(k + 2) \leq 27$ , and  $\Gamma$  is a subgraph of the Schläfli graph (graphs nos. 164–184 in Table 1).
- (iv)  $v = \frac{4}{3}(k + 2) \leq 16$ , and  $\Gamma$  is a subgraph of the Clebsch graph (graphs nos. 185–187 in Table 1).
- (v)  $v = k + 2$ , and  $\Gamma \cong K_{m \times 2}$  for some  $m \geq 3$ .

Moreover,  $\mathbb{L}^+(\Gamma) \cong A_n$  if and only if (i) holds with a bipartite graph  $\Delta$  with  $n + 1$  vertices, and  $\mathbb{L}^+(\Gamma) \cong D_n$  if and only if either (i) holds with a graph  $\Delta$  with  $n$  vertices with is not bipartite or (v) holds with  $m = n - 1$ .

New and computer-free proofs of Theorems 1.1 and 1.2 are contained in Brouwer, Cohen and Neumaier [6].

A glance through Table 1, together with a straightforward analysis of line graphs, leads to the following application of the preceding result, which generalizes the characterization of strongly regular graphs with smallest eigenvalue  $-2$  by Seidel [36].

**1.3. Theorem.** *Let  $\Gamma$  be a connected regular graph with smallest eigenvalue  $-2$ .*

- (i) *If  $\Gamma$  is strongly regular, then  $\Gamma$  is a triangular graph  $T(n)$ , a square grid  $n \times n$  (also called a lattice graph  $L_2(n)$ ), a complete multipartite graph  $K_{n \times 2}$ , or one of the graphs of Petersen, Clebsch, Schläfli, Shrikhande, or Chang.*

- (ii) If  $\Gamma$  is edge-regular, then  $\Gamma$  is strongly regular or the line graph of a regular triangle-free graph.
- (iii) If  $\Gamma$  is amply regular, then  $\Gamma$  is strongly regular or the line graph of a regular graph of girth  $\geq 5$ .
- (iv) If  $\Gamma$  is coedge-regular, then  $\Gamma$  is strongly regular, an  $(m \times n)$ -grid, or one of the two regular subgraphs of the Clebsch graph with eight and 12 vertices, respectively.

The multiplicity of the eigenvalue  $-2$  of a graph with  $\lambda_{\min}(\Gamma) \geq -2$  can be found quite easily from the following results of Doob [19] (case (i)) and Cvetković, Doob, and Simić [18] (case (ii)).

**1.4. Theorem.** Let  $\Gamma$  be a connected graph with  $\lambda_{\min}(\Gamma) \geq -2$ .

- (i) If  $\Gamma$  is the line graph of a graph  $\Delta$  with  $n$  vertices and  $e$  edges, then  $-2$  is an eigenvalue of  $\Gamma$  with multiplicity  $e - n + 1$  if  $\Delta$  is bipartite, and  $e - n$  otherwise.
- (ii) If  $\Gamma = L(\Delta; a_1, \dots, a_n)$  ( $\sum a_i > 0$ ) is a generalized line graph of a graph  $\Delta$  with  $n$  vertices and  $e$  edges, then  $-2$  is an eigenvalue of multiplicity  $e - n + \sum a_i$ .
- (iii) If  $L^+(\Gamma) \cong E_n$  ( $n = 6, 7, 8$ ), then  $-2$  is an eigenvalue of multiplicity  $|\Gamma| - n$ .

The case when the multiplicity of  $-2$  is zero corresponds to the graphs  $\Gamma$  with  $\lambda_{\min}(\Gamma) > -2$ . Since  $\lambda_{\min}(\Gamma) > -2$  implies that  $A + 2I$  is positive definite, so that  $\Gamma$  is represented by a linearly independent set of roots, we get the following results of Doob and Cvetković [21].

**1.5. Theorem.** Let  $\Gamma$  be a connected graph with  $\lambda_{\min}(\Gamma) > -2$ . Then  $\Gamma$  is one of the following cases:

- (i) The line graph of a connected graph without cycles of even length and with at most one cycle of odd length.
- (ii) The generalized line graph  $L(\Delta; 1, 0, \dots, 0)$  obtained from the line graph of a tree  $\Delta$  by adding two nonadjacent vertices  $\infty^+, \infty^-$  which are adjacent with all edges of  $\Delta$  containing a fixed vertex  $\infty$  of  $\Delta$ .
- (iii) A graph represented by a set of  $n \in \{6, 7, 8\}$  linearly independent roots generating  $E_n$ .

**1.6. Corollary.** A connected regular graph with smallest eigenvalue  $> -2$  is a complete graph or a polygon with an odd number of vertices.

Calculations of the first author (quoted in [21]) imply that, up to isomorphism, there are precisely 573 graphs of the form (iii), namely

- 20 graphs with six vertices generating  $E_6$ ,
- 110 graphs with seven vertices generating  $E_7$ ,
- 443 graphs with eight vertices generating  $E_8$ .

Their adjacency matrices and smallest eigenvalues are listed in Table 2. The 20 graphs with six vertices generating  $E_6$  are drawn in Figure 3 (see next page). It is a useful fact that every graph with  $n$  vertices generating  $E_n$  ( $n = 7, 8$ ) contains a subgraph with  $n-1$  vertices generating  $E_{n-1}$ . It would be interesting to have a simple explanation of this fact.

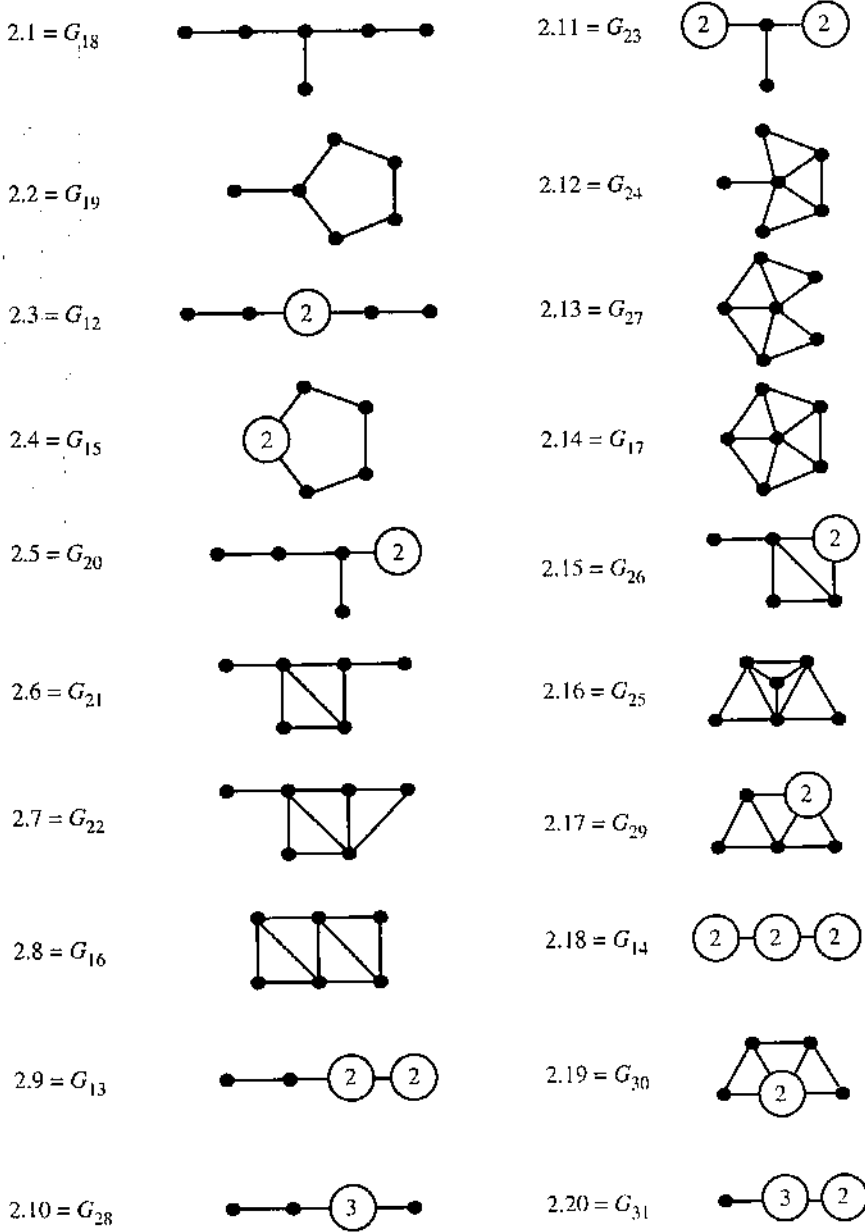


FIGURE 3. The graphs with six vertices generating  $E_6$ . ( $2.i$  is graph number  $i$  in Table 2;  $G_i$  is the notation of [18].)  $G_{12}$ – $G_{17}$  are the minimal forbidden line graphs with six vertices.

As another consequence of Theorem 1.4 we determine the minimal graphs with smallest eigenvalue  $-2$ . The proof is straightforward and left to the reader.

**1.7. Theorem.** *Let  $\Gamma$  be a connected graph with  $\lambda_{\min}(\Gamma) = -2$  such that  $\lambda_{\min}(\Gamma') > -2$  for all proper subgraphs  $\Gamma'$  of  $\Gamma$ . If  $\Gamma$  is a generalized line*



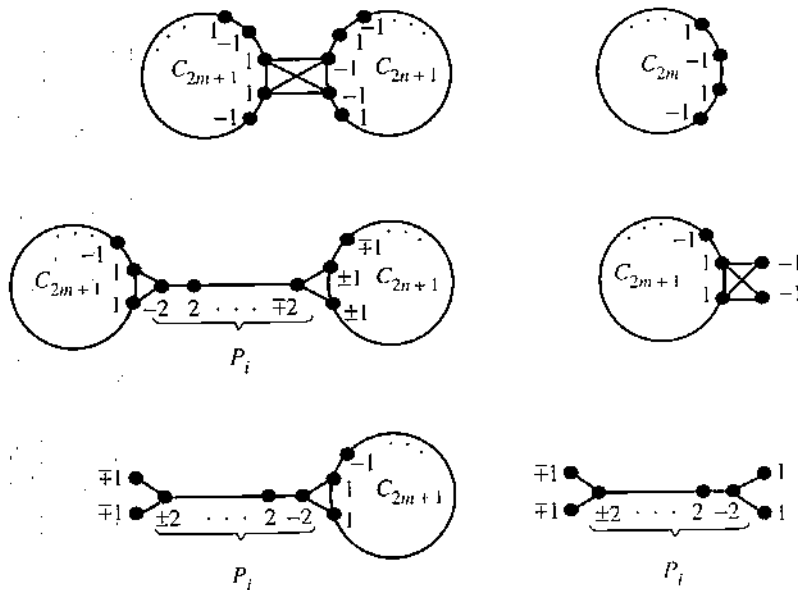


FIGURE 4. The minimal generalized line graphs with smallest eigenvalue  $-2$  and associated eigenvectors ( $C_i$  is a cycle with  $i \geq 3$  vertices,  $P_i$  a path with  $i \geq 1$  vertices).

graph, then  $\Gamma$  is one of the graphs drawn in Figure 4. Otherwise,  $\Gamma$  contains  $n + 1$  vertices ( $n \in \{6, 7, 8\}$ ) and has a representation by roots of  $E_n$ .

There are precisely 777 minimal graphs  $\Gamma$  with smallest eigenvalue  $-2$  which are not generalized line graphs, namely

- 12 graphs with seven vertices generating  $E_6$ ,
- 79 graphs with eight vertices generating  $E_7$ ,
- 686 graphs with nine vertices generating  $E_8$ .

Their adjacency matrices are listed in Table 3 together with an eigenvector belonging to the eigenvalue  $-2$ , normalized such that its absolutely smallest entries have the value  $\pm 1$ . It is a useful fact that the normalized eigenvectors (belonging to  $\lambda = -2$ ) of all minimal graphs with smallest eigenvalue  $-2$  are integral, and it implies that one can delete a vertex whose normalized eigenvector coefficient is  $\pm 1$  without changing the lattice generated. It would be interesting to have a simple explanation of this fact.

A reader who wants to check the information given for the exceptional graphs in Theorem 1.5 and Theorem 1.7 can use the fact that a quadrangle has smallest eigenvalue  $-2$ ; thus it is sufficient to check all graphs  $\Gamma$  with  $v \leq 9$  vertices and without quadrangles for their smallest eigenvalue, and if  $\lambda_{\min}(\Gamma) = -2$  to determine the multiplicity of  $-2$  ( $\Gamma$  is minimal if and only if  $-2$  is a simple eigenvalue and the corresponding eigenvector contains no zero entry).

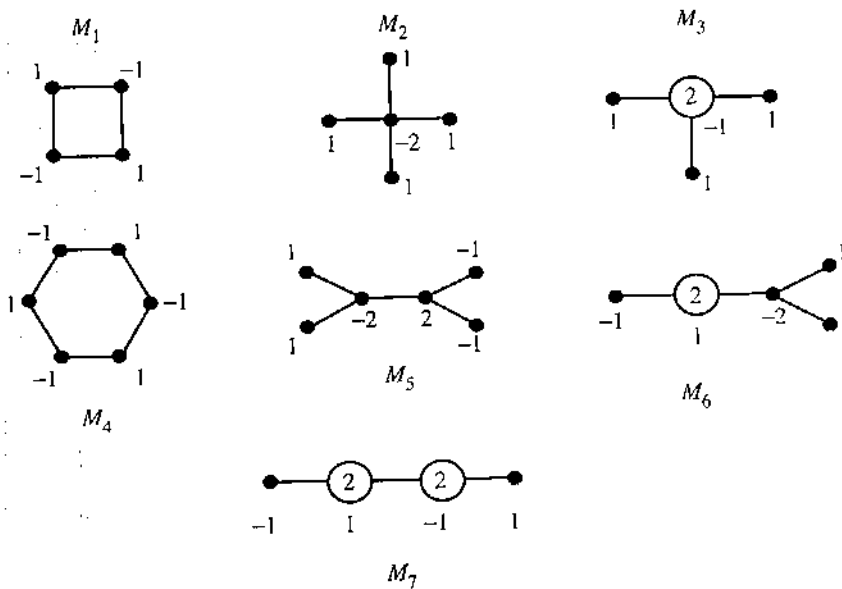


FIGURE 5. The minimal graphs with smallest eigenvalue  $-2$  and  $v \leq 6$  vertices, and associated eigenvectors.

2. MINIMAL FORBIDDEN SUBGRAPHS

Let  $\mathcal{G}$  be a class of graphs such that if  $\Gamma$  is in  $\mathcal{G}$  then every subgraph of  $\Gamma$  is also in  $\mathcal{G}$ . A *minimal forbidden subgraph* for  $\mathcal{G}$  is a graph  $\Gamma \notin \mathcal{G}$  all of whose proper subgraphs are in  $\mathcal{G}$ . A *complete list of minimal forbidden subgraphs* is a list  $\mathcal{G}^\#$  of pairwise nonisomorphic minimal forbidden subgraphs for  $\mathcal{G}$  such that every graph  $\Gamma \notin \mathcal{G}$  contains a subgraph isomorphic to some graph of  $\mathcal{G}^\#$ . Thus,  $\Gamma \in \mathcal{G}$  if and only if  $\Gamma$  contains no subgraph isomorphic to some graph of  $\mathcal{G}^\#$ ; in particular, if  $\mathcal{G}^\#$  is a known finite list, then we have an obvious finite algorithm for deciding whether a given graph is in  $\mathcal{G}$  or not.

A complete list of minimal forbidden subgraphs for the class  $\mathcal{L}$  of line graphs has been found by Beineke [3] (who also gives credit to unpublished work by N. Robertson). The list  $\mathcal{L}^\#$  consists of nine graphs, the 3-claw  $K_{1,3}$  (with four vertices), the two graphs drawn in Figure 1 (with five vertices), and the graphs  $G_{12}-G_{17}$  in Figure 3 (with six vertices).

A complete list of minimal forbidden subgraphs for the class  $\mathcal{L}_0$  of generalized line graphs has been found independently by Rao, Singhi, and Vijayan [35] and Cvetković, Doob, and Simić [18]. The list  $\mathcal{L}_0^\#$  consists of 31 graphs, namely the 20 graphs drawn in Figure 3 and the 11 graphs drawn in Figure 6. Their adjacency matrices and smallest eigenvalues are given as the first 20 entries of Table 2 and the first 11 entries of Table 4.

We discuss some properties of the list  $\mathcal{L}_0^\#$ .

1. The minimal forbidden subgraphs  $\Gamma$  for  $\mathcal{L}_0$  with  $\lambda_{\min}(\Gamma) > -2$  are precisely the graphs represented by a set of linearly independent generators for the lattice  $E_6$ . Indeed,  $L^+(\Gamma) \not\cong A_n$  or  $D_n$ , since  $\Gamma$  is not a generalized line graph. Moreover,  $L^+(\Gamma) \not\cong E_7$  or  $E_8$ , since (as observed above) any set of linearly independent generators for  $E_7$  and  $E_8$  contains a subset generating  $E_6$ .

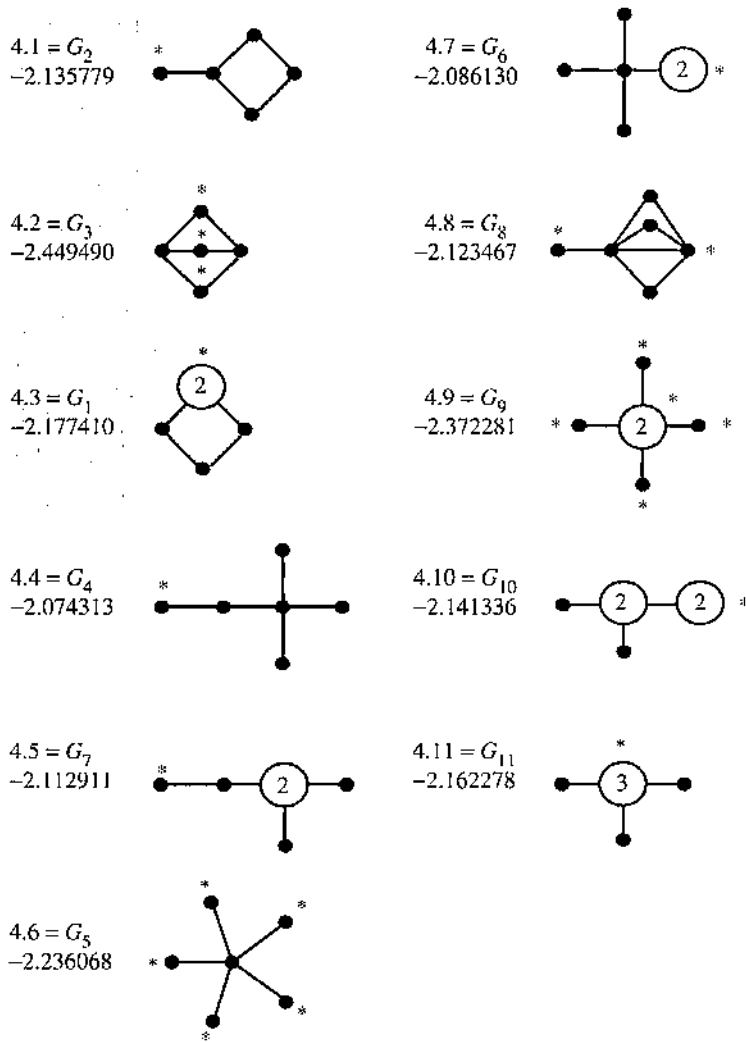


FIGURE 6. The minimal graphs with smallest eigenvalue  $< -2$  and up to six vertices, and their smallest eigenvalue. A vertex is starred when its deletion leaves a graph with smallest eigenvalue  $-2$ . ( $4.i$  is graph number  $i$  in Table 4;  $G_i$  is the notation of [18].)

2. There is no minimal forbidden subgraph for  $\mathcal{L}_0$  with  $\lambda_{\min}(\Gamma) = -2$ . Indeed, as observed above, the minimal graphs  $\Gamma$  with smallest eigenvalue  $-2$  have a proper subgraph generating the same lattice as  $\Gamma$ . (Note that the argument given in Cvetković et al. [18, Corollary 4.2] to prove  $\lambda_{\min}(\Gamma) \neq -2$  for  $\Gamma \in \mathcal{L}_0^\#$  is incorrect, since it does not cover the case where some  $\Gamma \setminus \{\gamma\}$  is disconnected or generates  $A_n$ ; however, their argument can be replaced by the simple fact that such a  $\Gamma$  would have at most nine vertices and thus is ruled out by McKay's computer search mentioned in Proposition 4.5 of [18].)

3. The set of minimal forbidden subgraphs for  $\mathcal{L}_0$  with  $\lambda_{\min}(\Gamma) < -2$  coincides with the set of minimal graphs with smallest eigenvalue  $< -2$  and at

most six vertices. Indeed, if  $\Gamma$  is a graph with  $\lambda_{\min}(\Gamma) < -2$  and at most six vertices, then  $\Gamma \setminus \{\gamma\}$  generates a lattice of dimension  $\leq 5$ , hence  $A_n$  or  $D_n$ , so that each  $\Gamma \setminus \{\gamma\}$  is a generalized line graph and  $\Gamma \in \mathcal{L}_0^\#$ . However, the remarkable fact that there are no graphs in  $\mathcal{L}_0$  with  $\lambda_{\min}(\Gamma) < -2$  and more than six vertices has not yet found a simple explanation.

4. Every minimal forbidden subgraph for  $\mathcal{L}_0$  with  $\lambda_{\min}(\Gamma) < -2$  contains one of the minimal graphs with smallest eigenvalue  $-2$  and  $\leq 5$  vertices (cf. Figure 5). Again, a simple explanation is missing.

Rao et al. [35] observed that the complete list of minimal forbidden subgraphs for the class  $\mathcal{G}_{-2}$  of graphs with smallest eigenvalue  $\geq -2$  is finite, since  $\mathcal{L}_0^\#$  is finite and there are only finitely many graphs in  $\mathcal{G}_{-2} \setminus \mathcal{L}_2$  (by Theorem 1.1). In particular,  $\mathcal{G}_{-2}^\# \setminus \mathcal{L}_0^\#$  consists of graphs with at most 37 vertices. Kumar, Rao, and Singhi [30] improved this estimate by showing that the maximal number of vertices of a graph in  $\mathcal{G}_{-2}^\#$  is ten. (Note, however, that the graph with ten vertices they give is *not* in  $\mathcal{G}_{-2}^\#$ .) They also determine the graphs in  $\mathcal{G}_{-2}^\#$  with at most seven vertices (see Figures 6 and 7), but incorrectly state that  $\mathcal{G}_{-2}^\#$  contains more than 100 graphs with eight vertices. Their complicated arguments were simplified in Vijayakumar [44]. We shall give a new proof of the results in [35] and [30], together with a complete list  $\mathcal{G}_{-2}^\#$ , based on the following variation of Lemma 4.3 of Cvetković, Doob, and Simić [18].

**2.1. Proposition.** *Let  $\Gamma$  be a minimal forbidden subgraph for the class  $\mathcal{G}_{-m}$  of graphs with smallest eigenvalue  $\geq -m$  ( $m \geq 1$ ). Then for any two distinct vertices  $\gamma, \delta \in \Gamma$ , the graph  $\Gamma \setminus \{\gamma, \delta\}$  has smallest eigenvalue  $> -m$ .*

*Proof.* Let  $v$  be the number of vertices of  $\Gamma$ , and denote by  $p(x)$  and  $p_\alpha(x)$  ( $\alpha \in \Gamma$ ) the characteristic polynomials of  $\Gamma$  and  $\Gamma \setminus \{\alpha\}$ , respectively. By Clarke [14], the derivative  $p'(x)$  can be expressed as  $p'(x) = \sum_{\alpha \in \Gamma} p_\alpha(x)$ . Since  $\Gamma \in \mathcal{G}_{-m}^\#$ , all proper subgraphs of  $\Gamma$  have smallest eigenvalue  $\geq -m$ ; in particular,  $(-1)^{v-1} p_\alpha(x)$  is positive for all  $x < -m$ . Hence,  $(-1)^{v-1} p(x)$  is strictly increasing for  $x < -m$ , and since  $\lambda_{\min}(\Gamma) < -m$ , it follows that  $\lambda_{\min}(\Gamma)$  is a simple eigenvalue of  $\Gamma$  and all other eigenvalues are  $> -m$ .

Now suppose that  $\Gamma' = \Gamma \setminus \{\gamma, \delta\}$  has smallest eigenvalue  $\leq -m$  (and hence equal to  $-m$ ). Let  $z = (z_\alpha \mid \alpha \in \Gamma')$  be a corresponding eigenvector, and denote by  $x_{c,d}$  the vector  $x = (x_\alpha \mid \alpha \in \Gamma)$  with  $x_\gamma = c$ ,  $x_\delta = d$ ,  $x_\alpha = z_\alpha$  for  $\alpha \in \Gamma'$ . Writing  $A$  for the adjacency matrix of  $\Gamma$ , we have  $x_{0,0}^T (A + mI)x_{0,0} = 0$ , and therefore

$$x_{c,0}^T (A + mI)x_{c,0} = c^2 m + 2c \sum_{\substack{\alpha \in \Gamma(\gamma) \\ \alpha \neq \delta}} x_\alpha.$$

Since  $\Gamma \setminus \{\gamma\}$  has smallest eigenvalue  $\geq -m$ , this expression must be nonnegative for all  $c \in \mathbb{R}$ , and this is possible only if  $\sum_{\alpha \in \Gamma(\gamma), \alpha \neq \delta} x_\alpha = 0$ . By the same reasoning we find that  $\sum_{\alpha \in \Gamma(\delta), \alpha \neq \gamma} x_\alpha = 0$ . Now, by construction of  $z$ , we find the relation  $(A + mI)x_{0,0} = 0$ , which is impossible, since  $-m$  is not an eigenvalue of  $\Gamma$ . Therefore, the smallest eigenvalue of  $\Gamma \setminus \{\gamma, \delta\}$  is  $> -m$ .  $\square$

In the special case  $-m = -2$ , this result can be combined with the results of §1 and yields the following restrictions on graphs in  $\mathcal{G}_{-2}^\#$ .

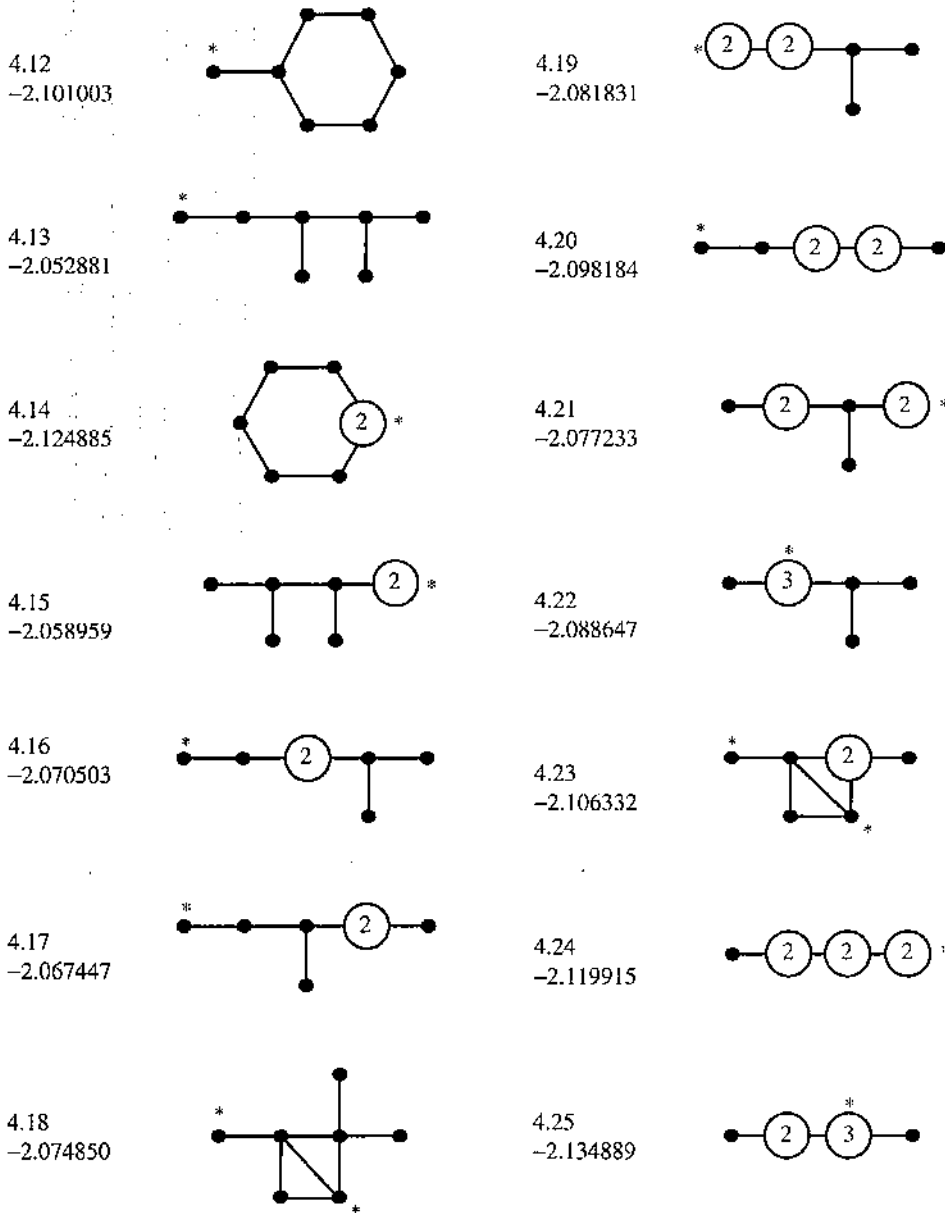


FIGURE 7. The minimal graphs with smallest eigenvalue  $< -2$  and seven vertices, and their smallest eigenvalue. A vertex is starred when its deletion leaves a graph with smallest eigenvalue  $-2$ . (4. $i$  is graph number  $i$  in Table 4.)

2.2. **Theorem.** Let  $\Gamma$  be a graph with  $v$  vertices, and suppose that  $\Gamma$  is a minimal forbidden subgraph for the class  $\mathcal{G}_{-2}$  of graphs with smallest eigenvalue  $\geq -2$ . Then every subgraph of  $\Gamma$  with smallest eigenvalue  $-2$  has  $v-1$  vertices, and one of the following holds:

(i)  $v \leq 10$ , and there exists a vertex  $\gamma \in \Gamma$  such that  $\Gamma \setminus \{\gamma\}$  is a minimal graph with smallest eigenvalue  $-2$ .

(ii)  $v \in \{7, 8, 9\}$ ,  $\lambda_{\min}(\Gamma \setminus \{\gamma\}) > -2$  for every vertex  $\gamma \in \Gamma$ , and for some  $\gamma \in \Gamma$ , the graph  $\Gamma \setminus \{\gamma\}$  has a representation by  $n = v - 1$  linearly independent roots generating  $E_n$ .

*Proof.* We showed already that subgraphs with smallest eigenvalue  $-2$  have  $v - 1$  vertices. To prove the remainder, we distinguish two cases.

*Case 1.* All proper subgraphs of  $\Gamma$  with  $v - 1$  vertices are generalized line graphs. Then  $\Gamma$  is a minimal forbidden subgraph for  $\mathcal{L}_0$  with smallest eigenvalue  $< -2$ , and hence one of the graphs of Figure 6. This implies that  $\Gamma$  satisfies (i).

*Case 2.* The graph  $\Gamma$  contains a subgraph  $\Gamma \setminus \{\delta\}$  which is not a generalized line graph. Then  $L^+(\Gamma \setminus \{\delta\}) \cong E_{v-1}$ ; in particular,  $\Gamma \setminus \{\gamma, \delta\}$ , having smallest eigenvalue  $> -2$ , is represented by  $v - 2$  linearly independent roots of  $E_n \subseteq E_8$ , so that  $v \leq 10$ . If  $\Gamma$  contains a vertex  $\gamma \in \Gamma$  such that  $\lambda_{\min}(\Gamma \setminus \{\gamma\}) = -2$ , then  $\Gamma \setminus \{\gamma\}$  is a minimal graph with this property and (i) holds. Otherwise,  $\Gamma \setminus \{\delta\}$  is represented by  $v - 1$  linearly independent roots generating  $E_{v-1}$ , and  $\Gamma$  satisfies (ii).  $\square$

In particular, a comparison with Figure 5 yields:

2.3. **Corollary.** Let  $\Gamma$  be a graph in  $\mathcal{G}_{-2}^{\#}$  with  $v$  vertices. If  $v \geq 5$ , then  $\Gamma$  contains no quadrangle; if  $v \geq 6$ , then  $\Gamma$  contains no subgraph of the form  $M_i$  ( $i \leq 3$ ); and if  $v \geq 7$ , then  $\Gamma$  contains no subgraph of the form  $M_i$  ( $i \leq 7$ ).

Theorem 2.2 and the corollary now allow a reasonably fast determination of a complete list of forbidden subgraphs for  $\mathcal{G}_{-2}$  by computer. We have already seen that a graph  $\Gamma \in \mathcal{G}_{-2}^{\#}$  with at most six vertices is a minimal forbidden subgraph and hence one of the 11 graphs in Figure 6. For graphs with more than six vertices, the fact that  $\Gamma$  contains no quadrangle drastically restricts the possibilities for extending subgraphs of  $\Gamma$  so that a systematic extension process together with checks on the smallest eigenvalues of  $\Gamma$  and the  $\Gamma \setminus \{\gamma\}$  yields a complete list in a reasonable time. (Several earlier trials to get a complete list turned out to be much too time consuming. The breakthrough was when Aart Blokhuis noticed that no minimal forbidden subgraph with six or seven vertices contained a quadrangle. After further experiments, this finally led to the corollary and then to the above theorem.)

The result of the computer search was that a complete list of minimal forbidden subgraphs for the class  $\mathcal{G}_{-2}$  of graphs with smallest eigenvalue  $\geq -2$  consists of 1812 graphs; cf. the following statistics ( $\#$  = number of graphs in  $\mathcal{G}_{-2}^{\#}$  with  $v$  vertices).

$v$	5	6	7	8	9	10	total
$\#$	3	8	14	67	315	1405	1812

The adjacency matrices and smallest eigenvalues of the 1812 graphs in  $\mathcal{G}_{-2}^{\#}$  are listed in Table 4. The graphs in  $\mathcal{G}_{-2}^{\#}$  with up to seven vertices are drawn in Figures 6 and 7.

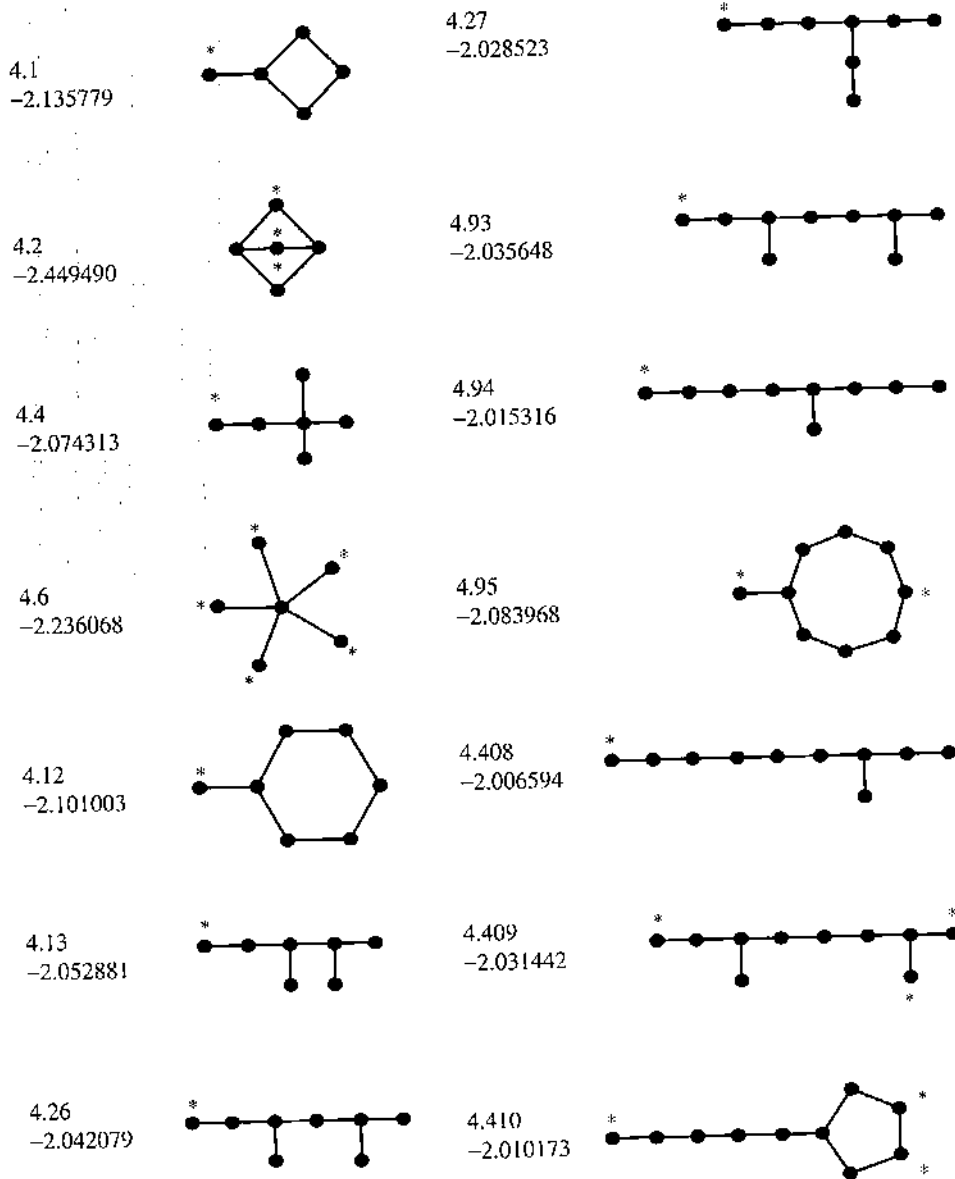


FIGURE 8. Minimal triangle-free graphs with smallest eigenvalue  $< -2$  and their smallest eigenvalue. A vertex is starred when its deletion leaves a graph with smallest eigenvalue  $-2$ . ( $4.i$  is graph number  $i$  in Table 4.)

An inspection of Table 4 shows that there are only 14 graphs in  $\mathcal{G}_{-2}^{\#}$  without triangles; they are drawn in Figure 8. The completeness of the list of triangle-free graphs in  $\mathcal{G}_{-2}^{\#}$  can be established easily by hand on the basis of Theorem 2.2 and its corollary.

2.4. **Theorem** (Doob [20]). *Let  $\Gamma$  be a graph with  $\lambda_{\min}(\Gamma) < -2$ . Then  $\Gamma$  contains a minimal graph with smallest eigenvalue  $-2$  and at most nine vertices.*

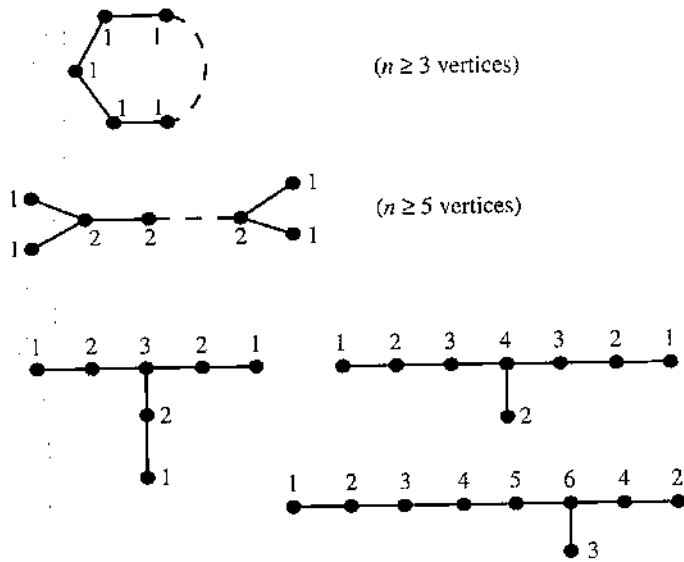


FIGURE 10. The graphs with largest eigenvalue 2, with a corresponding eigenvector.

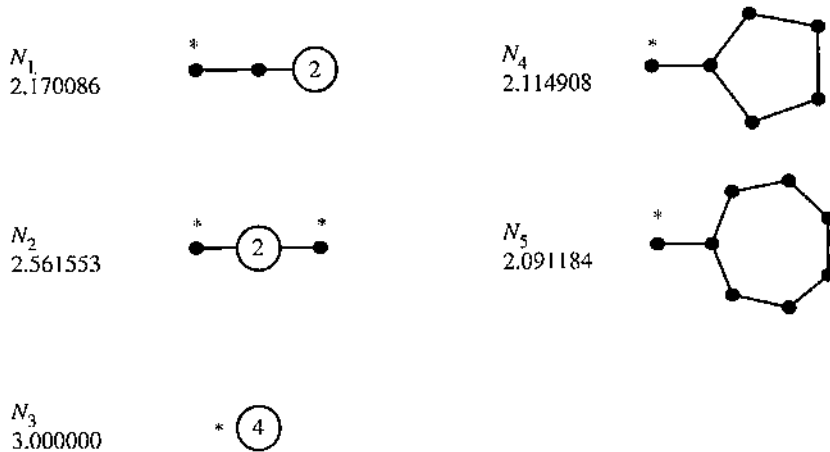
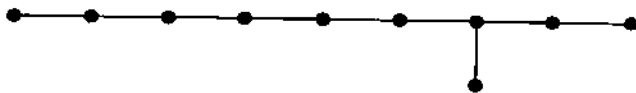


FIGURE 11. The minimal graphs with largest eigenvalue  $> 2$  which are not bipartite, and their largest eigenvalue. A vertex is starred when its deletion leaves a graph with largest eigenvalue  $-2$ .

$\epsilon_{2i} = \epsilon_{i2} = 3$  for  $i > 3$ ,  $\epsilon_{33} = \epsilon_{34} = \epsilon_{43} = 2$ ,  $\epsilon_{3i} = \epsilon_{i3} = \epsilon_{44} = \epsilon_{45} = \epsilon_{54} = 1$ , and  $\epsilon_{ik} = 0$  otherwise). It is remarkable that  $\hat{m} = \sup\{\lambda_{\max}(\Gamma) \mid \Gamma \in \mathcal{E}_m\}$  although no graph with  $\lambda_{\max}(\Gamma) = \hat{m}$  exists; in particular, this shows that the set of maximal eigenvalues of graphs is not closed. As observed by Hoffman [27],  $\mathcal{E}_m^\#$  is infinite, since it contains all subgraphs obtained by adding a vertex of valency 1 to the vertices of a polygon (and  $\hat{m}$  is maximal with this property). It would be interesting to know the set of numbers  $m, -m$  such that  $\mathcal{E}_m^\#$  or  $\mathcal{E}_{-m}^\#$  are finite; however, these seem to be very difficult problems.



FIGURE 12. The extremal graph  $Y_{621}$ .

The fact that  $\mathcal{G}_{+2}^\#$  and  $\mathcal{G}_{-2}^\#$  are finite implies the existence of "eigenvalue gaps" at  $\pm 2$  in the following sense (cf. [23] for  $\lambda_{\max}$ ).

**2.5. Theorem.** Let  $\rho \doteq 2.006594$  be the largest solution of the equation  $(\rho^3 - \rho)^2(\rho^2 - 3)(\rho^2 - 4) = 1$ . Then there is no graph  $\Gamma$  such that  $-\rho < \lambda_{\min}(\Gamma) < -2$  or  $2 < \lambda_{\max}(\Gamma) < \rho$ . Moreover, every graph with  $\lambda_{\min}(\Gamma) = -\rho$  or  $\lambda_{\max}(\Gamma) = \rho$  is isomorphic to the graph  $Y_{621}$ .

*Proof.* Inspection of Table 5 shows that the only graph in  $\mathcal{G}_{-2}^\#$  with  $\lambda_{\min}(\Gamma) \geq -\rho$  is  $Y_{621}$ , which has  $\lambda_{\min}(\Gamma) = -\rho$ . Now every graph with  $\lambda_{\min}(\Gamma) < -2$  not in  $\mathcal{G}_{-2}^\#$  contains a proper subgraph  $\Gamma' \in \mathcal{G}_2^\#$ ; hence,  $\lambda_{\min}(\Gamma) \leq \lambda_{\min}(\Gamma') \leq -\rho$ . If equality holds, then  $\Gamma' = Y_{621}$ , and all subgraphs of  $\Gamma$  strictly containing  $\Gamma'$  have  $-\rho$  as a multiple eigenvalue. In particular, the subgraph obtained by adjoining to  $\Gamma'$  one further vertex of  $\Gamma$  and deleting one of the vertices of  $\Gamma'$  again has  $-\rho$  as smallest eigenvalue, and hence must be isomorphic to  $Y_{621}$ . But no graph with 11 vertices has this property. Hence,  $\lambda_{\min}(\Gamma) = -\rho$  implies  $\Gamma \cong Y_{621}$ . The statement about  $\lambda_{\max}$  follows immediately, since  $Y_{621}$  is bipartite and the nonbipartite minimal graphs with largest eigenvalue  $> 2$  (Figure 11) have largest eigenvalue  $> \rho$ .  $\square$

For graphs with large minimum valency, the eigenvalue gap at  $-2$  is considerably larger. The following highly nontrivial result was proved by Hoffman [26] using Ramsey's theorem.

**2.6. Theorem.** Let  $\hat{\lambda}_k = \sup\{\lambda_{\min}(\Gamma) \mid k_{\min}(\Gamma) \geq k, \lambda_{\min}(\Gamma) < -2\}$ . Then  $\hat{\lambda}_k$  is a monotonic decreasing sequence with limit  $-1 - \sqrt{2} \doteq 2.414214$ .

Theorem 2.5 implies the value  $\hat{\lambda}_1 = -\rho \doteq -2.006594$ . Lower bounds for the values of  $\hat{\lambda}_i$  can be obtained from particular graphs with minimal valency  $k$ . In particular, we get

$$\hat{\lambda}_2 \geq \tilde{\lambda}_2 = \frac{1 - 2\sqrt{13}}{3} \doteq -2.070368,$$

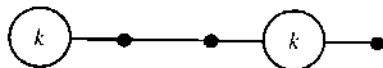
since the clique extension of an  $n$ -cycle, where a single vertex is replaced by a 2-clique, provides a sequence of graphs  $\Gamma_i$  with minimal valency 2 and  $\lambda_{\min}(\Gamma_i) \rightarrow \tilde{\lambda}_2$  for  $i \rightarrow \infty$ . The graph of Figure 13 gives the lower bound

$$\hat{\lambda}_k \geq \tilde{\lambda}_k \doteq -(1 + \sqrt{2})(1 - k^{-1} + O(k^{-2})) \quad \text{for } k \geq 3,$$

where  $\tilde{\lambda}_k$  is the smallest solution of the equation

$$(x + 1)^2(x + 2)(x + 3) + k(2x^3 + 9x^2 + 10x + 1) + k^2(x^2 + 2x - 1) = 0.$$

The reader is challenged to provide more extreme examples or to prove that the examples given are extremal. An explicit upper bound for  $\hat{\lambda}_k$  which tends to  $-1 - \sqrt{2}$  for  $k \rightarrow \infty$  would also be of considerable interest. In particular, for

FIGURE 13. A graph with minimal valency  $k$ .

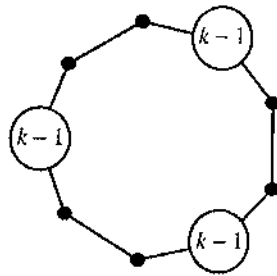


FIGURE 14. A regular graph with valency  $k$ .

applications to distance-regular graphs (see §3), we would like to know whether

$$\hat{\lambda}_k < -2.4 \quad \text{if } k \geq 64;$$

note that  $\hat{\lambda}_k < -2.4$  if  $k > 29.4$ .

If we require that  $\Gamma$  is regular of large valency, then  $-1 - \sqrt{2}$  seems no longer to be the right limit. Among the regular graphs of valency  $k$ , the largest value of  $\lambda_{\min}(\Gamma)$  observed in a limited number of test cases occurred for the graph in Figure 14, where  $\lambda_{\min}(\Gamma) = -1 - \alpha_k$  with the largest zero  $\alpha_k$  of the equation  $x^3 + 2x^2 + x - 3 - k(x^2 + 2x - 2) = 0$ , and  $\lim_{k \rightarrow \infty} (-1 - \alpha_k) = -1 - \sqrt{3}$ .

The result corresponding to Theorem 2.5 for the largest eigenvalue is trivial for minimal valency  $k > 2$ , since (by Perron-Frobenius theory) the largest eigenvalue of a graph  $\Gamma$  with minimal valency  $k$  is at least  $k$ , with equality if and only if the graph is regular of valency  $k$ . The more relevant value

$$\begin{aligned} & \inf \{ \lambda_{\max}(\Gamma) \mid k_{\min}(\Gamma) \geq k, \lambda_{\max}(\Gamma) > k \} \\ & = \inf \{ \lambda_{\max}(\Gamma) \mid \Gamma \text{ not regular, } k_{\min}(\Gamma) \geq k \} \end{aligned}$$

is not known, not even for  $k = 2$ .

### 3. APPLICATIONS TO DISTANCE-REGULAR GRAPHS

In this section we apply the preceding results to a characterization problem in the theory of distance-regular graphs. A connected graph  $\Gamma$  is called *distance-regular* if for any two vertices  $\gamma$  and  $\delta$  at distance  $i = d(\gamma, \delta)$ , there are precisely  $c_i$  neighbors of  $\delta$  at distance  $i - 1$  from  $\gamma$ , and  $b_i$  neighbors of  $\delta$  at distance  $i + 1$  from  $\gamma$  (see Biggs [4], Bannai and Ito [2]). The sequence

$$(1) \quad i(\Gamma) = \{b_0, b_1, \dots, b_{d-1}; c_1, c_2, \dots, c_d\},$$

where  $d$  is the diameter of  $\Gamma$ , is called the *intersection array* of  $\Gamma$ . A fundamental problem in the theory of distance-regular graphs is the characterization of known graphs by their intersection array. Recently, Paul Terwilliger and the second author achieved a breakthrough in this direction by utilizing the classification of graphs with smallest eigenvalue  $-2$  to settle this problem for a large class of intersection arrays containing those for the Hamming graphs, the Johnson graphs, and the half-cubes (Terwilliger [41], Neumaier [34]). Here we show that knowledge of the eigenvalue gap in Theorem 2.4 allows the characterization of distance-regular graphs for another class of intersection arrays, at least for large diameter. We begin by summarizing the background needed.

The adjacency matrix  $A$  of a distance-regular graph  $\Gamma$  of diameter  $d$  has precisely  $d+1$  distinct eigenvalues  $\theta_0 > \theta_1 > \dots > \theta_d$ ; the largest eigenvalue  $\theta_0$  is the valency  $k = b_0$  of  $\Gamma$  and has multiplicity 1. To each eigenvalue  $\theta$  of  $\Gamma$  there corresponds a unique idempotent matrix  $E$  in the algebra of polynomials in  $A$  satisfying the equation  $AE = \theta E$ , and the rank  $f$  of  $E$  agrees with the multiplicity of  $\theta$ . The  $(\gamma, \delta)$ -entries  $E_{\gamma\delta}$  of  $E$  depend only on the distance of  $\gamma$  and  $\delta$ ,

$$(2) \quad E_{\gamma\delta} = u_i \quad \text{if } d(\gamma, \delta) = i,$$

and the  $u_i$  satisfy the recurrence relations

$$(3) \quad \begin{aligned} u_0 &= 1, & u_1 &= \theta/k, \\ c_i u_{i-1} + a_i u_i + b_i u_{i+1} &= \theta u_i & (i &= 1, \dots, d-1), \\ c_d u_{d-1} + a_d u_d &= \theta u_d, \end{aligned}$$

where  $a_i = k - b_i - c_i$ . Conversely, if (3) holds, then  $\theta$  is an eigenvalue of  $A$  and (2) defines the corresponding idempotents. These facts can be found, e.g., in [2, 4, 6]. Since an idempotent symmetric matrix is positive semidefinite,  $E$  can be considered as the Gram matrix of a set of vectors in  $\mathbb{R}^f$ ; hence, there is a mapping  $-\cdot: \Gamma \rightarrow \mathbb{R}^f$  such that the images  $\bar{\gamma}, \bar{\delta}$  of two vertices  $\gamma, \delta \in \Gamma$  have inner product

$$(\bar{\gamma}, \bar{\delta}) = u_i \quad \text{if } d(\gamma, \delta) = i.$$

From this graph representation in  $\mathbb{R}^f$  it is possible to deduce the following result concerning the smallest eigenvalues of the local subgraphs  $\Gamma(\gamma)$ :

**3.1. Proposition** (Terwilliger [43]). *Let  $\Gamma$  be a distance-regular graph with intersection array (1), and suppose that  $\theta$  is an eigenvalue of  $\Gamma$  with multiplicity  $f$ . If  $-1 < \theta < k$ , then*

$$(4) \quad \lambda_{\min}(\Gamma(\gamma)) \geq -b_1/(\theta + 1) \quad \text{for all } \gamma \in \Gamma.$$

Moreover, if  $f < k$ , then (4) holds with equality,  $\theta$  is the second-largest eigenvalue of  $\Gamma$ , and either  $\theta + 1$  is an integer dividing  $b_1$ , or  $\theta + 1$  and  $b_1/(\theta + 1)$  are irrational quadratic algebraic integers.

*Proof.* Inequality (4) is essentially Theorem 1(2) in [43]. The second assertion is part of Theorem 5 in [43], apart from the statement about equality in (4), which derives from the proof of that theorem.  $\square$

The following result of Terwilliger is also relevant in the present context.

**3.2. Proposition** (Terwilliger [40]). *Let  $\Gamma$  be a distance-regular graph with intersection array (1).*

(i) *If  $\Gamma$  contains a quadrangle, then*

$$(5) \quad c_i - b_i \geq c_{i-1} - b_{i-1} + a_1 + 2 \quad (i = 1, \dots, d).$$

(ii) *If  $c_2 - b_2 = c_1 - b_1 + a_1 + 1$ , then every 2-claw of  $\Gamma$  is in at most one quadrangle.*

*Proof.* (i) is in Terwilliger [40], and (ii) is a simple consequence of the simplified proof of (i) in Terwilliger [42].  $\square$

Terwilliger classified in [42] the distance-regular graphs satisfying (5) with equality for all  $i$ . Here we consider a class of intersection arrays which have equality in (5) for all  $i \neq d$ , namely the arrays (1) defined by

$$\begin{aligned}
 & b_i = \mu \binom{m-i}{2} - (m-i)(m-i-2) \quad (i = 0, \dots, d-1), \\
 (6) \quad & c_i = \mu \binom{i}{2} - i(i-2) \quad (i = 1, \dots, d-1), \\
 & c_d = \gamma \left( \mu \binom{d}{2} - d(d-2) \right),
 \end{aligned}$$

where

$$(d, \gamma) \in \left\{ \left( \frac{m}{2}, 2 \right), \left( \frac{m-1}{2}, 1 \right) \right\}, \quad d > 1.$$

Note that  $\mu = c_2$  and  $m \in \{2d, 2d+1\}$  are positive integers,  $m \geq 4$ . There are three families of distance-regular graphs realizing these arrays:

(i) The *folded  $m$ -cube* with  $v = \frac{1}{2}2^m$  vertices is the graph obtained by identifying antipodal vertices in the  $m$ -cube, and realizes (6) with  $\mu = 2$ . It can be described as the graph whose vertices are the partitions of an  $m$ -set into two sets, where two such partitions are adjacent whenever their common refinement contains two singletons.

(ii) The *folded Johnson graph* with  $v = \frac{1}{2} \binom{2m}{m}$  vertices is the graph whose vertices are the partitions of a  $2m$ -set into two  $m$ -sets, with adjacency defined as before. Its intersection array realizes (6) with  $\mu = 4$ .

(iii) The *folded half  $2m$ -cube* with  $v = \frac{1}{2}2^{2m-1}$  vertices is the graph whose vertices are the partitions of a  $2m$ -set into two sets of even size, where two such partitions are adjacent whenever their common refinement contains two sets of size 2. Its intersection array realizes (6) with  $\mu = 6$ .

In view of these examples, we call a distance-regular graph with intersection array (6) a *pseudopartition graph*. We shall prove the following characterization theorem.

**3.3. Theorem.** *Let  $\Gamma$  be a pseudopartition graph with diameter  $d$ .*

(i) *If  $\mu = 2$ , then either  $\Gamma$  is a folded cube, or  $d = 3$  and  $\Gamma$  is the incidence graph of a  $(16, 6, 2)$ -biplane.*

(ii) *If  $d \geq 3$ , then  $\mu \in \{2, 4, 6\}$ .*

(iii) *If  $d \geq 154$ , then  $\Gamma$  is a folded cube, a folded Johnson graph, or a folded half-cube.*

*Proof.* We proceed in several steps.

*Step 1.*  $\Gamma$  has an eigenvalue  $\theta = m - 4 + (\mu - 2) \binom{m-2}{2}$  with multiplicity

$$(7) \quad f = \frac{m(m-1)(2 + (\mu - 2)(m-1))(4 + (\mu - 2)(2m-5))}{(4 + (\mu - 2)(m-2))(4 + (\mu - 2)(m-3))}.$$

To show this, we note that the intersection array belongs to the family of  $Q$ -polynomial intersection arrays of type II discussed in Bannai and Ito [2], with

parameters (in the notation of [2])

$$r_1 = -\frac{m+1}{2}, \quad r_2 = -\frac{m+2}{2}, \quad r_3 = -m - \frac{2}{\mu-2},$$

$$h = 2\mu - 4, \quad s = -m - \frac{1}{2} - \frac{2}{\mu-2}, \quad s^* = -m - 1.$$

Therefore, the eigenvalues of  $\Gamma$  are given by

$$\theta_i = k - 4i + (\mu - 2)i(2i + 1 - 2m) \quad (i = 0, \dots, d),$$

and their multiplicity is

$$(8) \quad f_i = \prod_{j=1}^i q_j,$$

where

$$q_j = \frac{b_{j-1}^*}{c_j^*} = \frac{(j+s)(2j+1+s)}{j(2j-1+s)} \cdot \frac{(j+r_1)(j+r_2)(j+r_3)}{(j+s-r_1)(j+s-r_2)(j+s-r_3)}.$$

In particular, since  $k = b_0 = \mu \binom{m}{2} - m(m-2) = \frac{m}{2}(2 + (\mu-2)(m-1))$ , we get for  $i = 1$  by simplification the above values  $\theta$  for  $\theta_1$  and  $f$  for  $f_1$ .

*Step 2.* If  $d \geq 3$ , then  $\mu \neq 1, 3, 5$ .

$c_3 > 0$  excludes  $\mu = 1$ . For  $\mu = 3$ , (7) reduces to

$$f = \frac{m(m-1)(2m-1)}{(m+2)},$$

so that  $m+2 \mid 30$ . For  $d \geq 3$  ( $m \geq 6$ ) this leaves the cases  $m = 8, 13, 28$ , and (8) yields a nonintegral  $f_3, f_4, f_5$  in the respective cases. And for  $\mu = 5$ , (7) becomes

$$f = \frac{m(m-1)(3m-1)(6m-11)}{(3m-2)(3m-5)},$$

which is nonintegral for all  $m > 3$ , hence for  $d \geq 2$ .

*Step 3.* If  $d \geq 3$ , then  $\mu \in \{2, 4, 6\}$ .

To get this, we apply Proposition 3.1; note that  $\theta < k$  and  $\theta + 1 = \frac{m-3}{m-1}b_1 > 0$ . Since  $m \geq 2d \geq 6$ ,  $\theta + 1$  is no divisor of  $b_2$ , and since  $\theta$  is rational, we must have  $f \geq k = \frac{m}{2}(2 + (\mu-2)(m-1))$ . This implies

$$2(m-1)(4 + (\mu-2)(2m-5)) \leq (4 + (\mu-2)(m-2))(4 + (\mu-2)(m-3)),$$

which simplifies to  $(\mu-6)(m-3)(2 + (\mu-2)(m-2)) \leq 0$ . Therefore,  $\mu \leq 6$  and thus  $\mu \in \{2, 4, 6\}$  by Step 2.

*Step 4.* If  $\mu = 2$ , then the conclusion of (i) holds.

For  $m \geq 7$  this follows from Egawa [22]. For  $m = 6$ ,  $\Gamma$  has  $v = 32$  vertices and intersection array  $\{6, 5, 4; 1, 2, 6\}$ ; hence,  $\Gamma$  is bipartite of diameter 3 and must be the incidence graph of a  $2 - (\frac{v}{2}, k, \mu)$ -design, i.e., of a  $(16, 6, 2)$ -biplane. For  $m = 4, 5$ , the graph  $\Gamma$  is easily seen to be  $K_4$  and  $K_{4,4}$ , respectively, and hence a folded  $m$ -cube.

*Step 5.* For any two nonadjacent vertices  $\alpha, \beta \in \Gamma(\gamma)$ , the number  $\mu(\alpha, \beta)$  of common neighbors of  $\alpha$  and  $\beta$  in  $\Gamma(\gamma)$  is  $\mu - 1$  or  $\mu - 2$ .

For, if  $\mu(\alpha, \beta) \leq \mu - 3$ , then there are two distinct vertices  $\delta, \delta' \in \Gamma_2(\gamma)$  adjacent with  $\alpha$  and  $\beta$  so that the 2-claw  $\alpha\gamma\beta$  is in two distinct quadrangles, contradicting Proposition 3.2(ii). Since  $\mu(\alpha, \beta) \leq \mu - 1$ , this forces  $\mu(\alpha, \beta) \in \{\mu - 1, \mu - 2\}$ .

*Step 6.* Each neighborhood  $\Gamma(\gamma)$  has smallest eigenvalue  $\geq -2 - \frac{2}{m-3}$ .

This follows from Proposition 3.1 since

$$\frac{b_1}{\theta + 1} = \frac{m - 1}{m - 3} = 1 + \frac{2}{m - 3}.$$

*Step 7.* If  $d \geq 154$ , then each neighborhood  $\Gamma(\gamma)$  is a line graph.

In this case,  $m \geq 308$  so that  $\lambda_{\min}(\Gamma(\gamma)) \geq -2 - \frac{2}{305} > -2.00656 > -\rho$ , and by Theorem 2.5,  $\lambda_{\min}(\Gamma(\gamma)) \geq -2$ . Now  $\Gamma(\gamma)$  is a regular graph with  $k = \frac{m}{2}(2 + (\mu - 2)(m - 1)) \geq m^2 > 28$  (since we may assume  $\mu \geq 4$  by Steps 3 and 4) vertices and valency  $a_1 = k - 1 - b_1 < k - 2$ , and by Theorem 1.2, each  $\Gamma(\gamma)$  must be a line graph.

*Step 8.* If  $\mu = 4$ , then each neighborhood  $\Gamma(\gamma)$  which is a line graph is in fact an  $(m \times m)$ -grid.

By Step 5, the hypothesis of Proposition 5 of Neumaier [34] is satisfied with  $c = 2$  for  $\Gamma(\gamma) = L(\Delta)$ , and part (iii) of that proposition, together with the fact that  $\Gamma(\gamma)$  contains  $k = m^2$  vertices and is regular of valency  $a_1 = k - 1 - b_1 = 2(m - 1)$ , only leaves the case  $\Delta = K_{m,m}$ . Therefore,  $\Gamma(\gamma) = L(K_{m,m})$  is an  $(m \times m)$ -grid.

*Step 9.* If  $d \geq 154$  and  $\mu = 4$ , then  $\Gamma$  is a folded Johnson graph.

For  $\Gamma$  is locally an  $(m \times m)$ -grid by Steps 7 and 8, and has the same intersection array, hence the same number  $\frac{1}{2} \binom{2m}{m}$  of vertices as a folded Johnson graph. By Blokhuis and Brouwer [5],  $\Gamma$  is a quotient of a Johnson graph  $J(2m, m)$ , and distance regularity forces that antipodal vertices (corresponding to complementary  $m$ -sets) must be identified, so that  $\Gamma$  is a folded Johnson graph.

*Step 10.* If  $\mu = 6$ , then each neighborhood  $\Gamma(\gamma)$  which is a line graph is in fact a triangular graph  $T(2m)$ .

By Step 5, the hypothesis of Proposition 5 of Neumaier [34] is satisfied with  $c = 4$  for  $\Gamma(\gamma) \cong L(\Delta)$ , and part (i) of that proposition, together with the fact that  $\Gamma(\gamma)$  contains  $k = m(2m - 1)$  vertices, implies  $\Delta = K_{2m}$  and  $\Gamma(\gamma) \cong T(2m)$ .

*Step 11.* If  $d \geq 154$  and  $\mu = 6$ , then  $\Gamma$  is a folded half-cube.

For  $\Gamma$  is locally  $T(2m)$  by Steps 7 and 10, and has the same intersection array, hence the same number  $\frac{1}{2} 2^{2m-1}$  of vertices as the folded half  $m$ -cube. Since  $d \geq 3$  and  $m \geq 6$ , the vertices and  $m$ -cliques of  $\Gamma$  form a semiplane, i.e., distinct vertices are in precisely zero or two blocks ( $m$ -cliques), and distinct blocks intersect in zero or two vertices. The incidence graph  $\Gamma^*$  of this semiplane is an amply regular graph with  $\lambda = 0$  and  $\mu = 2$ . Application of Egawa [22] shows that  $\Gamma^*$  is a folded  $2m$ -cube, so that  $\Gamma$  is a folded half-cube.  $\square$

Together with the results of Terwilliger [41] (which inspired the first three steps of the preceding proof), this implies that for large diameters ( $d \geq 154$ ), all  $Q$ -polynomial distance-regular graphs of type II are known.

The diameter bound seems much too pessimistic, and there should be no exceptions for  $d \geq 4$ . In order to obtain assertion 3.3(iii) for  $d \geq 4$  in place of  $d \geq 154$ , the argument of Step 7 has to be improved; since  $m \geq 8$  for  $d \geq 4$ , we would need a result like

$$\Gamma \text{ regular of valency } k \geq 64 \Rightarrow \lambda_{\min}(\Gamma) < -2.4 \text{ or } \lambda_{\min}(\Gamma) \geq -2.$$

For  $d \leq 3$ , there are many exceptions: Husain [28] shows that there are precisely three (16, 6, 2)-biplanes, and since each of them is self-dual, their incidence graphs give three nonisomorphic distance-regular graphs with intersection array  $\{6, 5, 4; 1, 2, 6\}$ , one of which is the folded 5-cube. Bussemaker et al. [9] show that there are at least 1853 strongly regular graphs with the same intersection array  $\{16, 9; 1, 4\}$  as the folded Johnson graph with  $v = \frac{1}{2} \binom{8}{4} = 35$  vertices. Any pair of orthogonal Latin squares of order 8 gives a Latin square graph  $LS_4(8)$  with the same intersection array  $\{28, 25; 1, 6\}$  as the folded half 8-cube. Any Latin square of order 16 gives a Latin square graph  $LS_3(16)$  with the same intersection array  $\{45, 28; 1, 6\}$  as the folded half 10-cube.

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