

Coherent Quantization I:

Coherent spaces for linear fields

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*Lecture given December 11, 2023 at the
University of Erlangen, Germany*

For abstracts, slides and preprints (once available) see
<https://arnold-neumaier.at/cohErlangen2023.html>

For details on the mathematics see
<https://arnold-neumaier.at/cohSpaces.html>

Coherent spaces and coherent manifolds

The notion of a **coherent space** is a nonlinear version of the notion of a complex Hilbert space:

The vector space axioms are dropped while the notion of inner product, now called a **coherent product**, is kept.

Coherent spaces combine the rich, often highly characteristic variety of symmetries of traditional geometric structures with the computational tractability of traditional tools from numerical analysis and statistics.

A **Euclidean space** is a complex vector space $\mathbb{H}$ with a binary operation that assigns to $\phi, \psi \in \mathbb{H}$ the **Hermitian inner product** $\phi^* \psi \in \mathbb{C}$, antilinear in the first and linear in the second argument, such that

$$\overline{\phi^* \psi} = \psi^* \phi, \tag{1}$$

$$\psi^* \psi > 0 \quad \text{for all } \psi \in \mathbb{H} \setminus \{0\}. \tag{2}$$

$\mathbb{H}$ is naturally embedded into its antidual $\mathbb{H}^\times$, the space of all antilinear functionals on $\mathbb{H}$, and has a natural locally convex topology in which these are continuous.

The Hilbert space completion $\overline{\mathbb{H}}$ sits between these two spaces,

$$\mathbb{H} \subseteq \overline{\mathbb{H}} \subseteq \mathbb{H}^\times.$$

A **coherent space** is a nonempty set Z with a distinguished function $K : Z \times Z \rightarrow \mathbb{C}$, called the **coherent product**, such that

$$\overline{K(z, z')} = K(z', z), \quad (3)$$

and for all $z_1, \dots, z_n \in Z$, the $n \times n$ matrix G with entries $G_{jk} = K(z_j, z_k)$ is positive semidefinite.

The coherent space Z is called **nondegenerate** if

$$K(z'', z') = K(z, z') \quad \text{for all } z' \in Z \quad \Rightarrow \quad z'' = z.$$

Any subset Z of a Euclidean space is a coherent space with coherent product $K(z, z') := z^* z'$.

Coherent spaces provide a setting for the study of geometry in a different direction than traditional metric, topological, and differential geometry.

Just as it pays to study the properties of manifolds independent of their embedding into a real Euclidean space, so it is fruitful to study the properties of coherent spaces independent of their embedding into a complex Euclidean space.

Let Z be a coherent space. A map $A : Z \rightarrow Z$ is called **coherent** if there is an **adjoint map** $A^* : Z \rightarrow Z$ such that

$$K(z, Az') = K(A^*z, z') \quad \text{for } z, z' \in Z \quad (4)$$

If Z is nondegenerate, the adjoint is unique, but not in general.

A **symmetry** of Z is an invertible coherent map on Z with an invertible adjoint.

Coherent maps form a semigroup $\text{Coh } Z$ with identity; the symmetries form a group.

An **isometry** is a coherent map A that has an adjoint satisfying $A^*A = 1$. An invertible isometry is called **unitary**.

A **coherent manifold** is a smooth ($= C^\infty$) real manifold Z with a smooth coherent product $K : Z \times Z \rightarrow \mathbb{C}$ with which Z is a coherent space.

A nondegenerate coherent manifold has a canonical Riemannian metric induced by the **distance** (PARTHASARATHY & SCHMIDT 1972)

$$d(z, z') := \sqrt{K(z, z) + K(z', z') - 2 \operatorname{Re} K(z, z')} \quad (5)$$

of two points $z, z' \in Z$.

The distance is a metric precisely when the coherent space is nondegenerate. In the resulting topology, the coherent product is continuous.

Quantization

In the interesting examples, the coherent space is an extended classical phase space, and there is a quantization functor that turns the symmetries of the coherent space into unitary operators in the corresponding quantum space.

Thus the quantum space is a representation space for quantum dynamics.

This provides a universal framework for quantization, extending the traditional geometric quantization of finite-dimensional symplectic manifolds to more general situations, and in particular to the quantization of certain classical field theories.

A **quantum space** $\mathbb{Q}(Z)$ of a coherent space Z is a Euclidean space spanned (algebraically) by a distinguished set of vectors $|z\rangle$ ($z \in Z$) called **coherent states** satisfying

$$\langle z|z'\rangle = K(z, z') \quad \text{for } z, z' \in Z \quad (6)$$

with the linear functionals

$$\langle z| := |z\rangle^*$$

acting on $\mathbb{Q}(Z)$.

Coherent states with distinct labels are distinct iff Z is nondegenerate.

A construction by ARONSZAJN 1943 (attributed by him to MOORE), usually phrased in terms of reproducing kernel Hilbert spaces, proves the following basic result.

Moore–Aronszajn Theorem:

Every coherent space has a quantum space.

It is unique up to isometry.

The antidual $\mathbb{Q}^\times(Z) := \mathbb{Q}(Z)^\times$ of the quantum space $\mathbb{Q}(Z)$ is called the **augmented quantum space**.

It contains the **completed quantum space** $\overline{\mathbb{Q}}(Z)$, the Hilbert space completion of $\mathbb{Q}(Z)$,

$$\mathbb{Q}(Z) \subseteq \overline{\mathbb{Q}}(Z) \subseteq \mathbb{Q}^\times(Z).$$

The completed quantum spaces of coherent spaces may be viewed as reproducing kernel Hilbert spaces, but constructed without starting from a Hilbert space conventionally defined using a measure.

Let $\mathbb{Q}(Z)$ be a quantum space of the coherent space Z . Then for any coherent map A on Z , there is a unique linear map $\Gamma(A) : \mathbb{Q}(Z) \rightarrow \mathbb{Q}^\times(Z)$ such that

$$\Gamma(A)|z\rangle = |Az\rangle \quad \text{for all } z \in Z \quad (7)$$

(NEUMAIER & GHANI FARASHAHI 2022).

We call $\Gamma(A)$ the **quantization** of A and Γ the **quantization map**.

The quantization map furnishes a representation of the semigroup of coherent maps on Z on the quantum space of Z .

In particular, this gives a **unitary representation** of the group of unitary coherent maps on Z .

Example. The **Möbius space** $Z = \{z \in \mathbb{C}^2 \mid |z_1| > |z_2|\}$ is a coherent manifold with coherent product

$$K(z, z') := (\bar{z}_1 z'_1 - \bar{z}_2 z'_2)^{-1}.$$

A quantum space is the Hardy L^2 space of analytic functions on the complex upper half plane with square integrable limit on the real line.

The Möbius space has a large semigroup of coherent maps (a semigroup of compressions, OLSHANSKI 1981) consisting of the matrices $A \in \mathbb{C}^{2 \times 2}$ such that

$$\alpha := |A_{11}|^2 - |A_{21}|^2, \quad \beta := \bar{A}_{11} A_{12} - \bar{A}_{21} A_{22}, \quad \gamma := |A_{22}|^2 - |A_{12}|^2$$

satisfy the inequalities

$$\alpha > 0, \quad |\beta| \leq \alpha, \quad \gamma \leq \alpha - 2|\beta|.$$

It contains as a group of symmetries the group $GU(1, 1)$ of matrices preserving the Hermitian form $|z_1|^2 - |z_2|^2$ up to a positive factor.

This example generalizes to central extensions of all **semisimple Lie groups** and associated **symmetric spaces** or **symmetric cones** and their **line bundles**.

These provide many interesting examples of coherent manifolds.

This follows from work on coherent states discussed in monographs by PERELOMOV 1986, by FARAUT & KORÁNYI 1994, and by NEEB 2000.

In particular, the nonclassical states of light in quantum optics called **squeezed states** are described by coherent spaces corresponding to the **metaplectic group**, and collective states in solid state physics are described by coherent spaces corresponding to the **metagonal (spin) group**.

For a coherent manifold, the symmetry group is usually a Lie group generated by first order differential operators called **coherent differential operators**.

Under weak conditions, every coherent differential operator X can be quantized by promoting it to the self-adjoint operator

$$d\Gamma(X) := \left. \frac{d}{dt} \Gamma(e^{\iota t X}) \right|_{t=0}.$$

In the quantum physics applications, $\iota = i/\hbar$.

The **infinitesimal quantization map** $d\Gamma$ has natural properties; for example, it is linear and the relation $e^{\iota t d\Gamma(X)} = \Gamma(e^{\iota t X})$ holds.

By taking limits of linear combinations of coherent states for Z we may construct **excited coherent states** corresponding to coherent products for certain bundles $\text{Ex}_N Z$ over Z , with the same completed quantum space.

In the simplest case, the first order excited coherent states are the

$$|z, X(z)\rangle := \lim_{s \rightarrow 0} s^{-1}(|e^{sX} z\rangle - |z\rangle),$$

where X is a vector field on Z . These are singly excited coherent states for the space TZ with coherent inner product

$$K([z, X(z)], [z', X'(z')]) := \langle z, X(z) | z', X'(z') \rangle = L_X R_{X'} K(z, z'),$$

where L_X and $R_{X'}$ are the directional derivatives along the vector fields X and X' in the left and right argument, respectively.

Thus coherent manifolds are a new, geometric way of working with concrete Hilbert spaces in which they are embeddable.

In place of measures and integration, differentiation is the basic tool for evaluating inner products.

This makes many calculations easy that are difficult in Hilbert spaces whose inner product is defined through an integral.

Klauder spaces and Fock spaces

Bosonic Fock spaces arise as the quantum spaces of Klauder spaces.

We call a coherent space Z **slender** if any finite set of linearly dependent, nonzero coherent states in a quantum space $\mathbb{Q}(Z)$ of Z contains two parallel coherent states.

In quantum spaces of slender coherent spaces it is possible to define generalized **annihilation operators** and generalized **normal ordering** with similar properties as in Fock spaces. For details see NEUMAIER & GHANI FARASHAHI 2022.

Slenderness proofs are quite nontrivial and exploit very specific properties of the coherent states of the coherent spaces. For example, the Möbius space is slender.

The **Klauder space** $KL[V]$ over the Euclidean space V is the coherent manifold $Z = \mathbb{C} \times V$ of pairs $z := [z_0, \mathbf{z}] \in \mathbb{C} \times V$ with coherent product $K(z, z') := e^{\bar{z}_0 + z'_0 + \mathbf{z}^* \mathbf{z}'}$.

$KL[\mathbb{C}]$ is essentially in KLAUDER 1963. Its coherent states are precisely the scalar multiples of the coherent states discovered by SCHRÖDINGER 1926.

Klauder spaces are slender. Their quantum spaces are essentially the bosonic **Fock spaces** introduced by FOCK 1932 in the context of **quantum field theory**; they were first presented by SEGAL 1960 in a form equivalent to the above.

The quantum space of $KL[\mathbb{C}^n]$ was systematically studied by BARGMANN 1963.

Klauder spaces have a large semigroup of coherent maps, which contains a large unitary subgroup. The **oscillator semigroup** over V is the semigroup $Os[V]$ of matrices

$$A = [\rho, p, q, \mathbf{A}] := \begin{pmatrix} 1 & p^* & \rho \\ 0 & \mathbf{A} & q \\ 0 & 0 & 1 \end{pmatrix} \in \text{Lin}(\mathbb{C} \times V \times \mathbb{C})$$

with $\rho \in \mathbb{C}$, $p \in V^\times$, $q \in V$, and $\mathbf{A} \in \text{Lin}(V, V^\times)$; one easily verifies the formulas for the product

$$[\rho, p, q, \mathbf{A}][\rho', p', q', \mathbf{A}'] = [\rho' + \rho + p^* q', \mathbf{A}'^* p + p', q + \mathbf{A} q', \mathbf{A} \mathbf{A}']$$

and the identity $1 = [0, 0, 0, 1]$.

With the adjoint

$$[\rho, p, q, \mathbf{A}]^* = [\bar{\rho}, q, p, \mathbf{A}^*],$$

$Os[V]$ is a $*$ -semigroup consisting of coherent maps acting on $[z_0, \mathbf{z}] \in Kl[V]$ as

$$[\rho, p, q, \mathbf{A}][z_0, \mathbf{z}] := [\rho + z_0 + p^* \mathbf{z}, q + \mathbf{A} \mathbf{z}]. \quad (8)$$

The subset of coherent maps of the form

$$W_\lambda(q) := [\tfrac{1}{2}(q^*q + i\lambda), -q^*, q, 1] \quad (q \in V, \lambda \in \mathbb{R})$$

is the **Heisenberg group** $H(V)$ over V . We have

$$W_\lambda(q)W_{\lambda'}(q') = W_{\lambda+\lambda'+\sigma(q,q')}(q+q'),$$

$$W_\lambda(q)[z_0, \mathbf{z}] = [\tfrac{1}{2}(q^*q + \lambda) + z_0 - q^*z, q + \mathbf{z}]$$

with the symplectic form

$$\sigma(q, q') = 2 \operatorname{Im} q^* q'. \tag{9}$$

The n -dimensional **Weyl group** is the subgroup of $H(\mathbb{C}^n)$ consisting of the $W_\lambda(q)$ with real λ and q .

Theorem For a Klauder space Z :

(i) Every linear operator $A : \mathbb{Q}(Z) \rightarrow \mathbb{Q}^\times(Z)$ can be written uniquely in normally ordered form $A = :F(a^*, a):$.

(ii) The map $F \rightarrow :F:$ is linear, with $:1: = 1$ and

$$:f(a)^* F(a^*, a)g(a): = f(a)^* :F(a^*, a): g(a).$$

In particular,

$$:f(a)^* g(a): = f(a)^* g(a).$$

(iii) The quantized coherent maps satisfy

$$\Gamma(A) = :e^{\rho + p^* a + a^* q + a^* (\mathbf{A} - 1)a}: \quad \text{for } A = [\rho, p, q, \mathbf{A}].$$

(iv) The **Weyl relations**

$$e^{p^* a} e^{a^* q} = e^{p^* q} e^{a^* q} e^{p^* a}$$

and the **canonical commutation relations**

$$(p^* a)(a^* q) - (a^* q)(p^* a) = \sigma(p, q)$$

hold with the symplectic form (9).

Coherent quantization of linear field equations

The coherent quantization of linear field equations for bosons (and fermions) is done in terms of infinite-dimensional symplectic (and orthogonal) **Hua spaces**, a new class of coherent spaces based on the geometric analysis by LOO-KENG HUA 1945 (for the finite-dimensional case).

Their symmetry groups are infinite-dimensional metaplectic and metagonal (spin) groups. They allow one to describe the full quantum scattering behavior of linear field equations in terms of classical scattering and Maslov corrections for the phase of the S-matrix.

Traditionally, the fermionic Fock space construction is based on a distinguished vacuum state – the 0-particle state.

In quantum field theory, many vectors in a fermionic Fock space may serve as a potential **vacuum state**; the vacuum of a Fock space depends on the Hamiltonian under consideration.

In time-dependent systems, the Hamiltonian and hence the corresponding vacuum state changes with time. Thus a Fock space description is inadequate.

We conclude that the structure of interest in quantum field theory is not the Fock space itself but a mathematically precise version of *”what is left from Fock space when no vacuum state is distinguished”*.

In quantum mechanics (quantizing a finite-dimensional classical dynamics), this "*Fock space stripped of a special vacuum state*" is, as a Hilbert space, the same object as the Fock space itself, but each choice of a vacuum defines a different associated excitation structure, anticommuting algebra, and exterior algebra, related to each other by Bogoliubov transformations.

In quantum field theory (quantizing infinite-dimensional classical dynamics), the excitation structure becomes the multi-particle structure of the theory.

Moreover, problems appear in infinite dimensions due to the existence of inequivalent representations of the canonical commutation relations.

As we may use affine spaces where all points are on equal footing in place of vector spaces with a distinguished origin, so symplectic and orthogonal **Hua spaces** are bundles over symmetric spaces without any distinguished origin, equipped with a coherent product related to the Kähler structure of these spaces.

In group theoretic terms, we have bundles over the symmetric spaces $Sp(\mathbb{V}^2)/U(\mathbb{V})$ in the symplectic case and $SO(\mathbb{V}^2)/U(\mathbb{V})$ in the orthogonal case.

In finite dimensions, HUA 1945 gave a description of these spaces in terms of linear operators A from $\mathbb{V}^2$ to $\mathbb{V}$ satisfying the bilinear equation $z^T J z = 0$, where

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad J = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

in the symplectic and orthogonal case, respectively.

Hua showed that the set of these operators has the structure of a metric space with a distance function taking integer values only.

The points form an infinite graph, in modern terminology a distance regular graph called a **dual polar graph**.

The symmetry group acts as a distance transitive group of automorphism.

Incidentally, I ended my career as a pure mathematician by coauthoring an encyclopedic book (BROUWER et al. 1989) on the beautiful subject of distance regular graphs.

These constitute a vast generalization of the concept of platonic solids, featuring computationally very useful geometric representations for most finite simple groups.

But I discovered Hua's contribution to this only many years after the publication of that book

A central extension of the distance transitive symmetry groups of Hua acts on the corresponding line bundles.

In finite dimensions, the coherent states of PERELOMOV 1986 associated with these line bundles lead to a coherent product on these bundles, turning them into coherent spaces whose symmetry groups are the metaplectic and metagonal groups.

The construction may be extended to infinite dimensions.

The explicit description of the coherent product involves questions of how to specify the cocycles of the central extensions; for the applications one needs as explicit formulas as possible.

Since there is no distinguished origin, the quantum space associated with a Hua space has no distinguished state vector.

Only the whole class of coherent states is distinguished, and any of them may serve as a potential vacuum state.

Thus the quantum spaces are examples of **infra Fock spaces**, a generalization of Fock spaces that will be discussed in Lecture III.

The case of charged fermionic matter interacting solely through a time-dependent external electromagnetic field is the only practically relevant field theory in 4 dimensions whose quantization is truly understood (GRACIA-BONDÍA & VÁRILLY et al. 1994).

In this case, the single-particle dynamics is given by a linear Dirac equation in a complex Euclidean space with (in general) time-dependent Hamiltonian $H(t)$, called a **Dirac operator**.

In the language of second quantization, this Euclidean space is called the 1-particle space, although its classical meaning is that of a space of half-densities for charged fermionic matter.

As is well-known, a Dirac operator has a real spectrum unbounded in both directions.

At all times where this Dirac operator has no generalized eigenstate with zero energy, the invariant subspace spanned by the positive eigenfunctions of the Dirac operator is a maximal isotropic subspace of the bilinear form defined by the real part of the inner product.

Unless the Dirac operator happens to be time independent, this subspace changes with time, and defines the coherent state specifying the correct vacuum state at each time t .

All this can be discussed with full mathematical details.

I hope that a preprint will be available by the end of the year.

If you want to see a preview, please let me know.

Then I'll give you a preliminary copy (with a number of gaps and probably some inaccurate or even faulty statements), in exchange for your comments on the contents.

This is ongoing work in progress.

What I showed you is just the tip of a huge iceberg waiting to be charted and explored. Two more views of this beautiful iceberg:

Coherent Quantization II:

Causal groups and quantum fields

Thursday, December 14, 2023, 16:15-18:00, AG Mathematische Physik, Übungsraum Ü1, 01.250, Cauerstr. 11, Erlangen

Coherent Quantization III:

Infra Fock spaces and nonlinear fields

Monday, December 18, 2023, 14:15-15:45, AG Lie-Gruppen, Übungsraum Ü2, 01.251, Cauerstr. 11, Erlangen

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Thank you for your attention!

For the discussion of questions concerning coherent spaces,
please use the discussion forum

<https://www.physicsoverflow.org>

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