

Coherent Quantization II:

Causal groups and quantum fields

Arnold Neumaier

*Fakultät für Mathematik
Universität Wien, Österreich*

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University of Erlangen, Germany*

For abstracts, slides and preprints (once available) see
<https://arnold-neumaier.at/cohErlangen2023.html>

In particular, a preprint on causal groups is already available.

Great are the works of the Lord; they are pondered by all who delight in them.

(Psalm 111:2)

For us mortals, understanding the Creation is a fascinating process.

For me, part of the fascination is the discovery of the mathematical structure of physics.

It may be viewed as part of God's thoughts when He designed the universe.

Whereas quantum mechanics quickly matured into a meaningful theory with solid mathematical foundations, the consolidation of quantum field theory took several decades and, as a matter of fact, has not yet come to a fully satisfactory end.

Buchholz & Fredenhagen, 2020

Why is quantum mechanics "*a meaningful theory with solid mathematical foundations*"?

Beginning with the birth of modern quantum mechanics (1925), perturbative quantum mechanics and perturbative quantum field theory were understood as **applied Lie algebras**.

Beginning with WEYL (1928), nonperturbative quantum mechanics is widely understood to be mostly **applied Lie groups**.

Why not quantum field theory?

To regard nonperturbative quantum field theory as **applied Lie groups** is a much less travelled path, discovered by BOGOLIUBOV & SHIRKOV (1955) in their **causal S-matrix** theory for relativistic quantum field theory.

But the latter was used by them – and later – almost exclusively for what today is called **causal perturbation theory**!

Independently, GOLDIN (1971) introduced the group approach for the nonperturbative study of the **current algebras** arising in nonrelativistic quantum field theory.

Relativistic developments of current algebra culminated in the discovery of the Kac–Moody groups and related 2-dimensional field theories. But the techniques successful there did not seem to generalize to the physical spacetime dimension 4.

In my coherent quantum physics program we study fields from the general point of view of symmetry and causality.

Symmetry is described in terms of infinite-dimensional groups.

Causality (responses are later than their cause) is described by **causal spaces**.

Both merge in the concept of a **causal group**.

In the present lecture, the nonperturbative group-based path to quantum field theory will be enriched and better paved by introducing the concept of a causal group.

It provides solid infrastructure for handling relativistic quantum field theory, and might be the path leading to its mathematically fully satisfactory consolidation.

Special causal groups defined by certain generators and relations arose in the recent (2020+) work by BUCHHOLZ & FREDENHAGEN and BRUNETTI, DÜTSCH, FREDENHAGEN & REJZNER as ingredients for the construction of dynamical C^* -algebras for quantum field theory over Minkowski spaces.

My generalization of their approach to arbitrary causal groups frees the causal S-matrix setting from its S-matrix connotations and extends it to arbitrary causal spaces.

Causal spaces

My concept of a causal space simultaneously covers

- the physical notion of causality that assume a manifold structure but not hyperbolicity,
- the causal spaces of KRONHEIMER & PENROSE (1967) – our hyperbolic causal spaces –, and
- the causal sets of BOMBELLI, LEE, MEYER & SORKIN (1987) – our discrete hyperbolic causal spaces.

One advantage of this generality is that one can get motivation and intuition from the consideration of much simpler toy problems.

A **causal space** is a Hausdorff space $\mathbb{M}$, whose elements are called **points**, with two transitive relations $\leq$ and $<$ on $\mathbb{M}$ such that there is a continuous **precedence function** $\tau : \mathbb{M} \times \mathbb{M} \rightarrow \mathbb{R}$ with

$$\tau(x, x) = 0 \quad \text{for } x \in \mathbb{M},$$

$$x \leq y \quad \Leftrightarrow \quad \tau(x, y) \leq 0 \leq \tau(y, x).$$

$$x < y \quad \Leftrightarrow \quad \tau(x, y) < 0 < \tau(y, x).$$

The precedence function τ is not unique; the composition $\kappa \circ \tau$ with any strictly increasing continuous function $\kappa : \mathbb{R} \rightarrow \mathbb{R}$ with $\kappa(0) = 0$, is again a precedence function.

Due to the properties of the precedence function, $\leq$ is a preorder (reflexive and transitive), the **causal preorder** of $\mathbb{M}$, and $<$ is a strict partial order (irreflexive, asymmetric, and transitive), the **strict causal order** of $\mathbb{M}$.

Interpretation:

Strict law of causality: $x < y$ iff a change of forces (**cause**) at x can possibly influence a response (**effect**) at y .

The **past** of a subset S of $\mathbb{M}$ is the open set

$$\text{Past } S := \{y \in \mathbb{M} \mid y < x \text{ for some } x \in S\}. \quad (1)$$

The **closed past** of S is the closed set

$$\text{cPast } S := \{y \in \mathbb{M} \mid y \leq x \text{ for some } x \in \overline{S}\}.$$

The **present** of S is the closed set

$$\text{Present } S := \text{cPast } S \setminus \text{Past } S.$$

The **closed future** of S is the closed set

$$\text{cFuture } S := \{y \in \mathbb{M} \mid y \geq x \text{ for some } x \in \overline{S}\}.$$

If S, S' are subsets of $\mathbb{M}$ we say that S is **prior** to S' and write $S \wedge \vee S'$ if the closed past of S is disjoint from the closed future of S' .

We call $S, S' \subseteq \mathbb{M}$ **causally unrelated** and write $S \times S'$ if $S \wedge \vee S'$ and $S' \wedge \vee S$.

*** drawings ***

We call a causal space $\mathbb{M}$

- **linear** if the causal order is a strict linear order, i.e., for all $x, y \in \mathbb{M}$, exactly one of the relations $x = y$, $x < y$, and $x > y$ holds.
- **elliptic** if $x \leq y$ implies $x = y$ (so that $x < y$ never holds),
- **parabolic** if there is a **parabolic time map**, a nonconstant function $t : \mathbb{M} \rightarrow \mathbb{R}$ such that $\tau(x, y) := t(x) - t(y)$ is a precedence function, and
- **hyperbolic** if $\leq$ is antisymmetric and hence a partial order.

*** drawings ***

In applications to physics, a **spacetime** is a causal space $\mathbb{M}$, usually with a manifold structure.

Of main interest in physics are the following translation invariant causal spaces:

- $\mathbb{M} = \mathbb{R}^{d+1}$ (**Euclidean spacetime**, elliptic, modeling equilibrium physics).
- $\mathbb{M} = \mathbb{R} \times \mathbb{R}^d$ (**Galilei spacetime**, parabolic, modeling nonrelativistic physics).
- $\mathbb{M} = \mathbb{R}^{1,d}$ (**Minkowski spacetime**, hyperbolic, modeling relativistic physics).

Their common special case is $d = 0$, where $\mathbb{M}$ is the real timeline $\mathbb{R}$.

In Galilei and Minkowski spacetime, points are denoted by $x = \begin{pmatrix} x_0 \\ \mathbf{x} \end{pmatrix}$ with $x_0 \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^d$.

The causal order in Galilei spacetime is defined by the precedence function

$$\tau(x, y) := x_0 - y_0$$

corresponding to the parabolic time map $t(x) := x_0$.

The causal order in Minkowski spacetime is defined by the precedence function

$$\tau(x, y) := x_0 - y_0 + |\mathbf{x} - \mathbf{y}|,$$

where $|\mathbf{x}| = \sqrt{\mathbf{x}^T \mathbf{x}}$.

In general relativity and quantum gravity, spacetime is a curved smooth orientable Lorentzian manifold.

The causal law

The law of causality takes its simplest mathematical (and physically very useful) form in **linear response theory**, which describes how a change of forces applied to an arbitrary physical system changes its response after sufficiently short times.

A **causal function space** with **base** $\mathbb{M}$ is a Euclidean vector space $\mathbb{H} \subseteq L^2(\mathbb{M}, \mathbb{F})$ of continuous, with respect to some Borel measure square integrable functions from a causal space $\mathbb{M}$ to a complex Euclidean space $\mathbb{F}$, with Hermitian inner product

$$f^*g := \int dx f(x)^*g(x) \quad \text{for } f, g \in \mathbb{H}.$$

We call $f \in \mathbb{H}$ **prior** to $g \in \mathbb{H}$ and write $g \wedge \vee f$ if the closed past of the support of g and the closed future of the support of f are disjoint. We call two functions $f, g \in \mathbb{H}$ **causally unrelated** and write $f \times g$ if $f \wedge \vee g$ and $g \wedge \vee f$.

The causal properties of $\mathbb{H}$ inherited from the causal space become visible in the algebra $\text{Lin } \mathbb{H}$ of everywhere defined linear operators on $\mathbb{H}$:

In **linear response theory**, we interpret function values $f(x)$ of a function $f \in \mathbb{H}$ as a **generalized force** applied to a physical system at the point x and assume that the response $q(x) \in \mathbb{F}$ of the system at the point $y \geq x$ to the force depends linearly on f .

Linearity means that $q = Gf$ for some linear operator $G \in \text{Lin } \mathbb{H}$. The **causal law** expresses the observed fact that a change in the response cannot appear before the force changes. In mathematical terms, if $f, h \in \mathbb{H}$ and $z \in \mathbb{M}$ are such that

$$f(x) = g(x) \quad \text{for all } x \leq z$$

then

$$Gf(x) = Gg(x) \quad \text{for all } x \leq z.$$

By linearity and continuity, the causal law can be expressed equivalently as

$$\text{Supp } f \subseteq \text{Future}(z) \Rightarrow \quad \text{Supp } Gf \subseteq \text{Future}(z). \quad (2)$$

In terms of the prior relation, the causal law takes the equivalent weak form

$$g \wedge \vee f \Rightarrow \quad g^* Gf = 0. \quad (3)$$

This weak form makes sense in the more general case where the responses Gf belong to the typically larger antidual space $\mathbb{H}^\times$, so that $G \in \text{Lin}^\times \mathbb{H} := \text{Lin}(\mathbb{H}, \mathbb{H}^\times)$. (This is a way of saying that G is a bilinear form on $\mathbb{H}$.)

We call $G \in \text{Lin}^\times \mathbb{H}$ a **causal map** if the **causality condition** (3) holds for all $f, g \in \mathbb{H}$.

This is necessary and sufficient for the causal behavior of the response. A system with a linear response $q = Gf$ to the force f is called **causal** if G is causal.

Causal maps generalize **lower block triangular matrices** to the operator level.

For example, it is easy to see that on a finite timeline $[1 : n]$, with functions f regarded as vectors $f \in \mathbb{R}^n$ with components $f_x = f(x)$, the causal maps are given by multiplication with lower triangular matrices.

If $\mathbb{M}$ is Minkowski space and the measure used to define $\mathbb{H}$ is the Lebesgue measure then the linear map G whose kernel is the retarded Green's function of a linear hyperbolic differential equation $D\psi = 0$ is an important example of a causal map with a direct physical meaning.

From such a G we may form the metric

$$\Pi := \frac{1}{2\pi i}(G - G^*) = \delta(D).$$

This implies that

$$\langle f, g \rangle := f^* \Pi g$$

is the natural inner product on the space of solutions of $D\psi = 0$.

Causal groups for discrete models

To motivate the concept of a causal group we consider a class of nonlinear dynamical models simple enough that all functional analytic difficulties are absent.

Our causal space is the parabolic space $\mathbb{M} = \mathbb{Z} \times \mathbb{S}$ with points $x = (t, \mathbf{x})$, where the parabolic time map projects to the first coordinate. Our **fiber of field values** is a vector space $\mathbb{V}$ of complex-valued functions on $\mathbb{S}$, corresponding to a complex scalar field.

We define the causal function space $\mathbb{H}$ consisting of all sequences f with values in $C^\infty(\mathbb{V}, \mathbb{V})$, indexed by $t \in \mathbb{Z}$ with bounded temporal support, i.e., $f_t = 0$ for sufficiently large $|t|$. We call the elements $f \in \mathbb{H}$ **acceleration sequences**.

The correspondence with fields is given by

$$q(t, \mathbf{x}) = q_t(\mathbf{x}), \quad F(t, \mathbf{x}) = f_t(\mathbf{x}) \quad \text{for } q_t \in \mathbb{V}, \mathbf{x} \in \mathbb{S}. \quad (4)$$

We consider the discrete dynamical system

$$q_{t+1} - 2q_t + q_{t-1} = f_t(q_t), \quad (5)$$

with **discrete time** t , **dynamical variables** $q_t \in \mathbb{V}$, and a given acceleration sequence $f \in \mathbb{H}$.

The **phase space** associated with (5) consists of the

$$\mathbf{z}_t := \begin{pmatrix} q_t \\ q_{t-1} \end{pmatrix} \in \mathbb{V}^2.$$

Given $\mathbf{z}_t$ for some t , the dynamics (5) can be uniquely solved for all t by recursion in both time directions,

$$\begin{aligned} \mathbf{z}_{t+1} = A_t(f)(\mathbf{z}_t) &:= \begin{pmatrix} f_t(q_t) + 2q_t - q_{t-1} \\ q_t \end{pmatrix}, \\ \mathbf{z}_t = A_t(f)^{-1}(\mathbf{z}_{t+1}) &:= \begin{pmatrix} q_t \\ f_t(q_t) + 2q_t - q_{t+1} \end{pmatrix}, \end{aligned}$$

Since $f \in \mathbb{H}$, the acceleration f_t vanishes for large $|t|$.

Therefore the asymptotic dynamics at large negative and large positive times is linear in t .

Thus it is possible to discuss asymptotic **scattering**.

We parameterize the linear asymptotic solutions for large negative time by $\begin{pmatrix} u \\ v \end{pmatrix}$, where

$$q_t = u - tv \quad \text{for } t < \text{Supp } f, \quad (6)$$

so that

$$\mathbf{z}_t = \begin{pmatrix} q_t \\ q_{t-1} \end{pmatrix} = \Omega_t \begin{pmatrix} u \\ v \end{pmatrix} := \begin{pmatrix} u - tv \\ u - (t-1)v \end{pmatrix}. \quad (7)$$

Similarly, we parameterize the linear asymptotic solutions for large positive time by $\begin{pmatrix} u' \\ v' \end{pmatrix}$, where

$$q_t = u' - tv' \quad \text{for } t > \text{Supp } f. \quad (8)$$

The **classical S-matrix** $S(f)$ is not a matrix but a nonlinear map of the asymptotic phase space to itself, the classical analogue of the quantum S-matrix in quantum mechanical scattering.

$S(f)$ maps the phase space at sufficiently large negative time to the phase space at sufficiently large positive time by

$$S(f) \begin{pmatrix} u \\ v \end{pmatrix} := \Omega_T^{-1} \mathbf{z}_T = \begin{pmatrix} u' \\ v' \end{pmatrix},$$

where $T > \text{Supp } f$ and $\mathbf{z}_T$ is obtained by solving the dynamics (5) started with (6).

Theorem. The transformations

$$U(f, g) := S(f)S(g)^{-1} \quad \text{for } f, g \in \mathbb{H} \quad (9)$$

satisfy the **transition relation**

$$U(f, g)U(g, h) = U(f, h) \quad \text{for } f, g, h \in \mathbb{H} \quad (10)$$

and the **causality relation**

$$U(f + e, g + e) = U(f, g) \quad \text{if } e \wedge \vee g - f. \quad (11)$$

General causal groups

Let $\mathbb{G}$ be a group and $\mathbb{H}$ be a causal function space with base $\mathbb{M}$. A **causal transition map** on $\mathbb{G}$ with **carrier space** $\mathbb{H}$ is a map $U : \mathbb{H} \times \mathbb{H} \rightarrow \mathbb{G}$ with (10) satisfying (11) for all $f, g, e \in \mathbb{H}$ such that e and $g - f$ have compact support.

We call $\mathbb{G}$ a **causal group** over $\mathbb{M}$ with **carrier space** $\mathbb{H}$ if it has a causal transition map.

In the most important case where the causal space is Minkowski spacetime, the causality condition arose first in the context of the causal S-matrix theory for quantum field theory by BOGOLIUBOV & SHIRKOV (1959).

Using (9) to express the transition map in terms of S , (11) may be viewed as a condition

$$S(f + e)S(g + e)^{-1} = S(f)S(g)^{-1} \quad \text{if } e \wedge \vee g - f \quad (12)$$

restricting the possibilities for S . Fully formal versions equivalent to (12) were given independently by EPSTEIN & GLASER (1973) and SLAVNOV (1977).

This relation is the basis of **causal perturbation theory**, now a quite extensive branch of algebraic quantum field theory.

The transition relation is equivalent to the statement that, restricted to an arbitrary path $f : \mathbb{R} \rightarrow \mathbb{H}$, the transition operators $U(f(s), f(t))$ define a time-dependent flow on $\mathbb{G}$.

Informally, the causality relation says that the same additive changes e of the *common* past of f and g in a localized region do not affect the transition behavior restricted to compact regions.

From (10) we find for $g = g' = f$ that

$$U(f, f) = 1, \tag{13}$$

and then for $g' = f$

$$U(f, g)^{-1} = U(g, f). \tag{14}$$

Given an arbitrary map $S : \mathbb{H} \rightarrow \mathbb{G}$, the map $U : \mathbb{H} \times \mathbb{H} \rightarrow \mathbb{G}$ defined by (9) satisfies (10).

Conversely, given (10), the map $S_e : \mathbb{H} \rightarrow \mathbb{G}$ defined for any fixed element $e \in \mathbb{H}$ by

$$S_e(f) := U(f, e) \tag{15}$$

determines U through (9) with $S = S_e$ since (10) implies

$$S_e(f)S_e(g)^{-1} = U(f, e)U(g, e)^{-1} = U(f, g).$$

Let $\mathbb{M}$ be a causal space and $\mathbb{H}$ be a causal function space with base $\mathbb{M}$. Let ω be a real symplectic form on $\mathbb{H}$ such that

$$\omega(f, g) = 0 \quad \text{if } f \wedge \vee g. \quad (16)$$

(For example, this is always the case for $\omega(f, g) := \lambda \operatorname{Im} f^* g$ with fixed real λ .)

Then the **Heisenberg group** $\mathbb{G}_\omega(\mathbb{H})$, consisting of all $W_\alpha(f)$ with $\alpha \in \mathbb{R}$, $f \in \mathbb{H}$ with product

$$W_\alpha(f)W_\beta(g) := W_{\alpha+\beta-\omega(f,g)}(f+g), \quad (17)$$

is a causal group with carrier space $\mathbb{H}$ and causal transition map

$$U(f, g) := W_{\omega(f,g)}(f - g).$$

The transition relation follows from the multiplication law (17) and the causality relation follows from (16).

The **free causal group** on $\mathbb{H}$ is a group defined by generators and relations. The generators are the symbols $U(f, g)$ with $f, g, \in \mathbb{H}$, and the defining relations are (10) and (11).

Every causal group is a homomorphic image of the free causal group with the same carrier space.

Certain causal groups defined by generators and relations also arise via (12) as ingredients for the construction of dynamical C^* -algebras in work by BUCHHOLZ & FREDENHAGEN (2020+) for bosonic fields. and by BRUNETTI, DÜTSCH, FREDENHAGEN & REJZNER (2022+) for fermionic fields and fields in curved spacetimes. In their treatments, the free causal group is subject to additional relations related to a specific Lagrangian.

Locality and quantum fields

Causal groups over d -dimensional elliptic, parabolic, and hyperbolic spacetimes with $d \leq 4$ are relevant for Euclidean, nonrelativistic, and relativistic (classical and quantum) field theories, respectively.

They give rise to the well-established local commutation relations characterizing the local quantum physics approach of algebraic quantum field theory.

From the point of view of my coherent approach to field theory,

- a **local classical field theory** is a representation of a causal group by diffeomorphisms of a manifold, and
- a **local quantum field theory** is a unitary highest weight representation of a causal group on a Euclidean vector space.

Here 'highest weight' has to be understood in a generalized sense (whose precise meaning I haven't yet fixed).

If the causal group is represented classically on a coherent manifold, coherent field quantization produces a corresponding quantum field theory.

Conjecture.

(i) For every quantum field theory $\mathcal{T}_q$ (in the traditional sense) there is a coherent space Z with a causal group of symmetries whose classical limit is a classical field theory $\mathcal{T}_c$, and whose quantum space reproduces $\mathcal{T}_q$.

(ii) If $\mathcal{T}_q$ or $\mathcal{T}_c$ has an explicit construction then Z also has an explicit construction.

(The classical limit is a well-defined concept for any coherent space. It corresponds to a limit $\hbar \rightarrow 0$ in the traditional formulation of quantum mechanics.)

Proposition

$$U(f, g')U(f, g) = U(f, g' + g - f) \quad \text{if } g' - f \wedge \vee g - f, \quad (18)$$

$$[U(f, g'), U(f, g)] = 0 \quad \text{if } g' - f \times g - f \quad (19)$$

(19) is the group version of the traditional **causal commutation relation** for local quantum fields.

A **domain** of a causal space $\mathbb{M}$ is a nonempty, open subset $\mathcal{O}$ of $\mathbb{M}$ with compact closure.

A **local net of groups** over $\mathbb{M}$ is a collection of subgroups $\mathbb{G}(\mathcal{O})$ of a group $\mathbb{G}$, one for each domain $\mathcal{O}$ of $\mathbb{M}$, such that

$$\mathbb{G}(\mathcal{O}) \subseteq \mathbb{G}(\mathcal{O}') \quad \text{if } \mathcal{O} \subseteq \mathcal{O}' \quad (20)$$

and $\mathbb{G}(\mathcal{O})$ and $\mathbb{G}(\mathcal{O}')$ commute whenever $\mathcal{O} \times \mathcal{O}'$.

Proposition Let $f \in \mathbb{H}$. Then the collection of groups $\mathbb{G}_f(\mathcal{O})$ generated by all $U(f, g)$ with $\text{Supp } g \subseteq \mathcal{O}$ is a local net of groups.

This generalizes the locality properties of the causality relation first investigated by SLAVNOV (1977) and IL'IN & SLAVNOV (1978).

Using the standard machinery of functional analysis one can derive from a local net of groups corresponding local nets of C^* -algebras and local nets of von Neumann algebras.

We now assume that the causal group $\mathbb{G}$ is a Lie group and the transition map $U(f, g)$ is continuously differentiable in g .

We call the algebra $U(\mathbb{G})$ of linear differential operators on $C^\infty(\mathbb{G})$ the **universal group algebra** of $\mathbb{G}$. (It is isomorphic to a semidirect product of $\mathbb{G}$ and the universal enveloping algebra of the Lie algebra of $\mathbb{G}$, but this is not directly relevant here.)

We may treat $U(f, g)$ as multiplication operator in the universal group algebra $U(\mathbb{G})$.

For $f, g, e \in \mathbb{H}$ we define the (g -independent) operators

$$A_f(e) := i\hbar \frac{d}{d\tau} U(f + \tau e, g) \Big|_{\tau=0} U(f, g)^{-1}; \quad (21)$$

These are linear operators on $C^\infty(\mathbb{G})$ generating the **field algebra** of a local quantum field theory.

Using distribution-valued **quantum fields**

$$A_f(x) = i \frac{dU(f, g)}{df(x)} U(f, g)^{-1}, \quad (22)$$

familiar from causal perturbation theory, where the quotient on the right denotes functional differentiation we may rewrite (21) as

$$A_f(e) = \int_{\mathbb{M}} dx e(x) A_{f,g}(x).$$

In terms of (9), (22) reads

$$A_f(x) := i \frac{dS(f)}{df(x)} S(f)^{-1},$$

a formula going back to the early days of causal perturbation theory.

Causal commutation rules for the field operators follow from the commutation rule

$$[U(f, g), U(f, h)] = 0 \quad \text{if } \text{Supp } g \times \text{Supp } h$$

by differentiation.

Quantization of causal groups

We now show how to quantize the causal group constructed from our discrete dynamics, so that it produces the correct discrete quantum mechanical dynamics.

We treat the complex space $\mathbb{V}$ as a real Euclidean vector space with bilinear inner product $x^T y$.

Linear operators $A \in \text{Lin } \mathbb{V}^2$ are written as 2×2 block matrices

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \quad (23)$$

whose components $A_{jk} \in \text{Lin } \mathbb{V}$ are the operators defined by

$$A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} A_{11}x_1 + A_{12}x_2 \\ A_{21}x_1 + A_{22}x_2 \end{pmatrix} \quad \text{for } x_1, x_2 \in \mathbb{V}.$$

Examples of interest are the block matrices

$$J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad A(c) := \begin{pmatrix} c & -1 \\ 1 & 0 \end{pmatrix} \quad \text{for } c \in \text{Lin } \mathbb{V}.$$

They satisfy the relations

$$J^T = -J, \quad J^2 = -1, \quad A(c)^T J A(c) = J.$$

$A \in \text{Lin } \mathbb{V}^2$ is called **symplectic** if $A^T J A = J$.

The set of all symplectic matrices $A \in \text{Lin } \mathbb{V}^2$ is a group, the **symplectic group** $Sp(\mathbb{V}, \mathbb{R})$.

Examples of symplectic matrices are J and the $A(c)$.

A **canonical transformation** is a map $\mathbf{Q} \in \text{Diff}(\mathbb{V}^2)$ such that

$$\frac{\partial \mathbf{Q}(\mathbf{z})}{\partial \mathbf{z}} \in Sp(\mathbb{V}, \mathbb{R}) \quad \text{for } \mathbf{z} \in \mathbb{V}^2.$$

The canonical transformations form a group $\text{Can}(\mathbb{V}, \mathbb{R})$.

Since

$$A_t(f)'(\mathbf{z}_t) = \begin{pmatrix} f'(q_t) + 2 & -1 \\ 1 & 0 \end{pmatrix} \in Sp(\mathbb{V}, \mathbb{R}),$$

the time step maps $A_t(f)$ are canonical transformations.

$$\Omega_t = \begin{pmatrix} 1 & -t \\ 1 & 1 - t \end{pmatrix}$$

satisfies $\Omega_t^T J \Omega_t = J$, hence is symplectic, hence a linear canonical transformation.

As a product of canonical transformations, the classical S-matrix $S(f)$ is a canonical transformation.

To obtain physically relevant unitary representations via **geometric quantization** we lift these canonical transformations to quantomorphisms of a line bundle over the symplectic space. If the force in our discrete dynamical system has the form

$$f(q) = -\nabla V(q)$$

for some scalar potential $V(q)$, the required quantomorphisms maps a point (q, p, s) to (Q, P, S) with

$$Q = Q(q, p), \quad P = P(q, p), \quad S = s + \pi(q, p),$$

where

$$\begin{aligned} Q(q, p) &:= 2q - p - V'(q)/m, & P(q, p) &:= q, \\ \pi(q, p) &:= q^2 - pq + V(q)/m - qV'(q)/m. \end{aligned}$$

Thus in this case, the dynamics can be quantized by geometric quantization.

Tomonaga–Schwinger structures

As in our discrete example, classical field theories with a uniquely solvable initial-value problem may give rise to causal groups.

This is achieved on the classical level by geometric means that imitate the ideas of TOMONAGA (1946) and SCHWINGER (1948) for covariant quantum electrodynamics that earned them a Nobel prize.

A **Cauchy structure** on a causal set $\mathbb{M}$ is a collection $\mathcal{S}$ of subsets of $\mathbb{M}$ called **Cauchy slices** such that, for all compact subsets A of $\mathbb{M}$ and all closed subsets P and F of S with $P \wedge \vee F$, there are Cauchy slices Σ_- , Σ_0 , and Σ_+ satisfying

$$\Sigma_- \subseteq \text{Past } S, \quad S \subseteq \text{Past } \Sigma_+, \quad (24)$$

$$P \subseteq \text{cPast } \Sigma_0, \quad \Sigma_0 \subseteq \text{Past } F. \quad (25)$$

In a parabolic causal space, the collection $\mathcal{S}$ of **time slices**

$$\Sigma_\tau := \{x \in \mathbb{M} \mid t(x) = \tau\}$$

defines a Cauchy structure iff the parabolic time map is unbounded from below and from above.

In Minkowski space, the collection $\mathcal{S}$ of Cauchy surfaces defines a Cauchy structure.

A **Tomonaga–Schwinger structure** on $\mathbb{M}$ consists of

- a causal function space $\mathbb{H}$ with base $\mathbb{M}$,
- a Cauchy structure $\mathcal{S}$ on $\mathbb{M}$,
- sets $Z(\Sigma)$ ($\Sigma \in \mathcal{S}$) and Z_f ($f \in \mathbb{H}$), and
- for each $\Sigma \in \mathcal{S}$ and $f \in \mathbb{H}$ an invertible **Møller operator** $\Omega_f(\Sigma) : Z_f \rightarrow Z(\Sigma)$ such that

$$\Omega_0(\Sigma)^{-1}\Omega_f(\Sigma) = \begin{cases} \Omega_-(f) & \text{if } \Sigma \wedge \vee \text{ Supp } f, \\ \Omega_+(f) & \text{if } \text{Supp } f \wedge \vee \Sigma. \end{cases} \quad (26)$$

The maps $\Omega_+(f), \Omega_-(f) : Z_f \rightarrow Z_0$ are uniquely determined by (26).

$\mathbb{H}$ is called the **carrier space** of the Tomonaga–Schwinger structure.

The **transition maps**

$$U_f(\Sigma', \Sigma) := \Omega_f(\Sigma')\Omega_f(\Sigma)^{-1} : Z(\Sigma) \rightarrow Z(\Sigma')$$

describe the **many-fingered time flow** from $Z(\Sigma)$ to $Z(\Sigma')$.

Theorem. Every Tomonaga–Schwinger structure with carrier space $\mathbb{H}$ determines on the group $\text{Sym}(Z_f)$ of bijections of Z_f a causal transition map whose carrier space is the subspace $\mathbb{H}_c$ consisting of all functions in $\mathbb{H}$ with compact support.

Tomonaga–Schwinger structures can be constructed from certain classes of ordinary or partial differential equations with uniquely solvable initial value problem.

We assume that $\mathcal{S}$ is a Cauchy structure on the causal space $\mathbb{M}$ and $\mathbb{H}$ is a causal function space with base $\mathbb{M}$. Suppose that $\mathbb{M}$ and the value spaces $\mathbb{F}, \mathbb{F}', \mathbb{F}_{\text{phys}}$ are smooth manifolds,

$P : C^\infty(\mathbb{M}, \mathbb{F}) \rightarrow C^\infty(\mathbb{M}, \mathbb{F})$ and

$F : C^\infty(\mathbb{M}, \mathbb{F}) \times \mathbb{H} \rightarrow C^\infty(\mathbb{M}, \mathbb{F}_{\text{phys}})$ are smooth local maps, i.e., $(Pu)(x)$ and $F(u, f)(x)$ depend only on $u(x)$, $f(x)$ and their derivatives.

We only state the principle; to be really useful for 4-dimensional spacetimes, one probably needs to weaken the assumptions made.

Theorem. Suppose that for every $\Sigma \in \mathcal{S}$, every $f \in \mathbb{H}$, and every $u_0 \in C^\infty(\Sigma, \mathbb{F})$, the equations

$$F(u, f) = 0, \quad u|_\Sigma = u_0 \tag{27}$$

have a solution $u \in C^\infty(\mathbb{M}, \mathbb{F})$, and any two such solutions have the same Pu . Then

$$Z(\Sigma) := C^\infty(\Sigma, \mathbb{F}) \quad \text{for } \Sigma \in \mathcal{S},$$

$$Z_f := \{Pu \mid u \in C^\infty(\mathbb{M}, \mathbb{F}), F(u, f) = 0\} \quad \text{for } f \in \mathbb{H},$$

$$\Omega_f(\Sigma)[z] := z|_\Sigma \quad \text{for } z \in Z_f$$

defines a Tomonaga–Schwinger structure on $\mathbb{M}$ with carrier space $\mathbb{H}$.

In parabolic spacetimes, existence theorems for ODEs in Banach spaces apply.

Therefore ODEs and regular index 1 DAEs should produce in this way Tomonaga–Schwinger structures on parabolic spacetimes, thus recovering nonrelativistic field theories.

In particular, we recover results by BUCHHOLZ & FREDENHAGEN (2020) – who phrased their results in terms of dynamical C^* -algebras – on nonrelativistic quantum mechanics, considered as $1 + 0$ -dimensional quantum field theories.

All this can be discussed with full mathematical details.

A preprint of the contents of this second lecture is already available
at

<https://arnold-neumaier.at/cGroups.pdf>

This is ongoing work in progress.

What I showed you is just the tip of a huge iceberg waiting to be charted and explored. One more view of this beautiful iceberg:

Coherent Quantization III:

Infra Fock spaces and nonlinear fields

Monday, December 18, 2023, 14:15-15:45, AG Lie-Gruppen,
Übungsraum Ü2, 01.251, Cauerstr. 11, Erlangen

For abstracts, slides and preprints (once available) see

<https://arnold-neumaier.at/cohErlangen2023.html>

For the discussion of questions concerning my coherent approach to quantum theory, please use the discussion forum

<https://www.physicsoverflow.org>