Coherent Quantization III:

Infra Fock spaces and nonlinear fields

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For abstracts, slides and preprints (once available) see
`https://arnold-neumaier.at/cohErlangen2023.html`

(In particular, a preprint with the material for the Wednesday lecture is already available.)

This is the third and last lecture of my lecture series on

Coherent quantization and field theory

When I agreed in July to give these three lectures I had hoped to be able to announce today an existence proof for quantum electrodynamics (QED).

Unfortunately I was a bit too optimistic – spoilt by the increasing frequency of minor and major miracles that happened in my understanding of how the many pieces of the fundamental physics puzzle interrelate and match each other.

Mathematicians take such miracles as a sure sign that they are onto something extremely fruitful and interesting ....

Instead I'll present today concepts, partial results, and conjectures related to my coherent physics research program.

I believe that the concepts are the right ones for a nonperturbative understanding of quantum field theory.

I am still confident that in due time they lead to a mathematically impeccable proof of existence for QED, the standard model, and a unified quantum field theory that also incorporates gravity.

The goal of this lecture is to convince you that this is indeed a reasonable expectation.

This is work in progress.

Definitions and hence results are not yet completely stable
and might slightly change in the final version.

In particular, my notion of an infra Fock space kept changing,
as I made the whole conceptual framework more encompassing
and more useful.

The last change in its definition was made yesterday
during my final preparations for this lecture,
so this might still go on ....

Conventions

In the following,

• All our spaces are smooth, finite- or infinite-dimensional manifolds, and all our groups are smooth Lie groups.

• The causal space $\mathbb{M}$ is a smooth Lorentzian spacetime manifold of dimension d (not necessarily $d = 3$) admitting a spin structure.

The **space of field values** $\mathbb{F}$ is a smooth complex manifold expressing the field content of a theory.

The **field bundle** $\mathbb{W}$ is a bundle with base $\mathbb{M}$ and standard fiber $\mathbb{F}$.

A **field** is a section of the field bundle.

Locally on a bundle chart with base $\mathcal{O}$
(i.e., in the mainstream physicist's view),
a smooth field is given by a smooth map $\phi \in C^\infty(\mathcal{O}, \mathbb{F})$.

For ϕ^4 **theory**,

$$\mathbb{F} = \mathbb{C} \ni \phi(x)$$

with a Hermitan scalar field $\phi(x)$.

For **quantum electrodynamics** (**QED**), $d = 3$, $\mathbb{M} = \mathbb{R}^{1,3}$, and

$$\mathbb{F} = \mathbb{C}^4 \times \mathbb{R}^{1,3} \ni (\psi(x), A(x))$$

with

- the massive **electron-positron field** $\psi(x)$ of spin 1/2 and
- the massless **photon field** (= **electromagnetic vector potential**) $A(x)$ of spin 1.

Globally, $\psi(x)$ actually represents a spinor field and $A(x)$ a connection on a (spinor $\times$ line) bundle over $\mathbb{M}$.

For **thermal quantum electrodynamics**, the space of field values must be

$$\mathbb{F} = \mathbb{C} \times \mathbb{C}^4 \times \mathbb{R}^{1,3} \ni (s(x), \psi(x), A(x))$$

with an additional Hermitian field representing
• the **entropy density** $s(x)$.

To also include **gravity**, the space of field values must be

$$\mathbb{F} = \mathbb{C} \times \mathbb{C}^4 \times \mathbb{R}^{1,3} \times GL(\mathbb{R}^{1,3}) \ni (s(x), \psi(x), A(x), E(x))$$

with an additional frame field representing
• the **gravitational potential** $E(x)$ (= **tetrad** in the Palatini formalism of general relativity).

Quantum bundles

A **quantum bundle** is a bundle $\mathbb{W}$ with base $\mathbb{U}$
whose standard fiber is a complex Euclidean space $\mathbb{H}$,
together with a Lie $*$-group $\mathbb{B}$ acting as bundle automorphisms.

We refer to the elements w of a fiber $\mathbb{W}_u$ $(u \in \mathbb{U})$ as
state vectors with **infrastructure** u.

The elements of $B \in \mathbb{B}$ are called **B-transforms** (short for **Bogoliubov transforms**).

Each $B \in \mathbb{B}$ maps

$$u \in \mathbb{U} \to Bu \in \mathbb{U}$$

$$u \in \mathbb{U} \to B(u) \in \mathrm{Iso}(\mathbb{W}_u, \mathbb{W}_{Bu}),$$

$$(u, w) \in \mathbb{W} \to (Bu, B(u)w) \in \mathbb{W},$$

and $B(u)$ is an invertible linear isometry from $\mathbb{W}_u$ to $\mathbb{W}_{Bu}$.

The orbits $\mathbb{B}u$ of $\mathbb{B}$ on the space $\mathbb{U}$ of infrastructures $u \in \mathbb{U}$ play the role of (superselection) **sectors** of the quantum bundle.

The sections $\psi \in S(\mathbb{W})$ of the quantum bundle $\mathbb{W}$ are called **wave functions**.

They map $u \in \mathbb{U}$ to a state vector $\psi(u) \in \mathbb{W}_u$ with infrastructure u.

B-transforms $B \in \mathbb{B}$ act on wave functions $\psi \in S(\mathbb{W})$ as

$$(B\psi)(Bu) := B(u)\psi(u) \quad \text{for } u \in \mathbb{U}.$$

A quantum bundle may be viewed as a way of specifying a **nonseparable** Hilbert space of wave functions.

The latter arises as the Hilbert space completion of the direct sum of the fibers $\mathbb{W}_u$ with one representative infrastructure u from each sector of the quantum bundle.

As was observed by Borchers & Sen (1975), quantum bundles and nonseparable Hilbert spaces are fully equivalent.

But the bundle view offers substantial conceptual advantages.

A **coherent bundle** is a bundle Z with base $\mathbb{U}$ whose fibers Z_u $(u \in \mathbb{U})$ are coherent spaces, together with a Lie $*$-group $\mathbb{B}$ acting as bundle automorphisms.

Now a B-transform $B \in \mathbb{B}$ involves for each $u \in \mathbb{U}$ an invertible unitary coherent map $B(u) : Z_u \to Z_{Bu}$.

Associated (in the sense of bundles) to each coherent bundle is a quantum bundle $\mathbb{W} := \mathbb{Q}(Z)$ whose fibers are the quantum spaces $\mathbb{W}_u := \mathbb{Q}(Z_u)$ of the fibers Z_u $(u \in \mathbb{U})$.

We briefly consider two cases of special interest.

In **first quantization**:

• Each Z_u is the Cartesian product of the extended symplectic phases spaces of classical particles, with the particle contents specified by the field bundle.

The particles move with the dynamics given by a classical Lagrangian $L(u)$ determined by the infrastructure $u \in \mathbb{U}$.

• The corresonding quantum spaces $\mathbb{W}_u = \mathbb{Q}(Z_u)$ are the Euclidean spaces for the quantum dynamics of independent single particles with the same Lagrangian $L(u)$, including their antiparticles.

The quantum spaces may also be described directly as the spaces of solutions of a corresponding generalized Dirac equation

$$D(u)\psi = 0,$$

which, together with a positivity condition singles out the physical wave functions $\psi \in \mathbb{H}$ in a causal function space $\mathbb{H}$.

A covariant description is given in terms of the **metric operator**

$$\Pi(u) := \delta(D(u))$$

by letting $\mathbb{W}_u$ be the space $\mathbb{H}$ equipped with the u-inner product

$$\langle \phi, \psi \rangle_u := \phi^* \Pi(u) \psi.$$

In **second quantization**:

• Each Z_u is a coherent space of classical fields z over spacetime $\mathbb{M}$ satisfying a linear field equation $D(u)z = 0$ whose coefficients depend on the infrastructure.

• The corresponding quantum spaces $\mathbb{W}_u = \mathbb{Q}(Z_u)$ are Euclidean spaces for the quantum dynamics of linear fields whose dynamics depends on the infrastructure u.

Infra Fock spaces

In descriptions of many-particle fermionic quantum systems, many vectors in a fermionic Fock space may serve as a potential vacuum state; the vacuum of a Fock space depends on the Hamiltonian under consideration.

In time-dependent systems, the Hamiltonian and hence the corresponding vacuum state changes with time. Thus a Fock space description is inadequate.

Each choice of a vacuum defines a different associated excitation structure, anticommuting algebra, and exterior algebra, related to each other by Bogoliubov transformations.

This is naturally handled by the concept of infrastructure introduced above.

In quantum field theory (quantizing infinite-dimensional classical dynamics), the excitation structure becomes the multi-particle structure of the theory.

It is conventionally accounted for by Fock spaces with a distinguished vacuum state.

Thus to incorporate changes of the vacuum state we need a bundle of Fock spaces!

$\mathrm{Fock}\,\mathbb{V}$ (or $\mathrm{Fock}^R\mathbb{V}$ if the statistics R is specified) denotes the **Fock space** of

- **unsymmetrized** ($R = 0$),
- **symmetrized** ($R = 1$),
- **antisymmetrized** ($R = -1$), or
- **braided** (R a more general Yang–Baxter operator)

wave functions over the Euclidean vector space $\mathbb{V}$.

$\mathrm{Fock}_N\mathbb{V}$ denotes the subspace of wave functions with N arguments.

Motivated by the above, we define an **infra Fock space** as a quantum bundle whose fibers are Fock spaces.

Corresponding to each quantum bundle $\mathbb{W}$ (and each statistics R), there is an associated infra Fock space $\widehat{\mathbb{W}}$ whose fibers are the

$$\widehat{\mathbb{W}}_u = \operatorname{Fock} \mathbb{W}_u \quad (u \in \mathbb{U}).$$

Thus infra Fock spaces are just second quantized quantum bundles.

Examples of Infra Fock spaces are the quantum bundles $\mathbb{Q}(Z)$,
where Z is a coherent bundle
whose fibers Z_u are Klauder or Hua spaces.

These make the quantization process gewometrically accessible and computationally transparent.

The **field algebra** of an infra Fock space $\widehat{\mathbb{W}}$ is the algebra $\mathbb{E}_f$ of all bundle endomorphisms of $\widehat{\mathbb{W}}$.

The **observable algebra** of an infra Fock space $\widehat{\mathbb{W}}$ is the subalgebra $\mathbb{E}_o$ of all $\mathbb{B}$-invariant bundle endomorphisms of $\widehat{\mathbb{W}}$.

This definition explains the role of the group $\mathbb{B}$ of B-transforms.

This could have been done already on the level of quantum bundles, but the terminology introduced is colored by the quantum field case.

A **state** of an infra Fock space $\widehat{\mathbb{W}}$ is a positive linear functional $\langle\cdot\rangle$ on the observable algebra $\mathbb{E}_o$ of $\widehat{\mathbb{W}}$.

Each $\mathbb{B}$-orbit $\mathbb{B}\widehat{\psi}$ of $\mathbb{B}$ on wave functions $\widehat{\psi} \in S(\widehat{\mathbb{W}})$ defines a state by

$$\langle A\rangle_\psi := \psi^* A\psi \quad \text{for } A \in \mathbb{E}_o$$

upon noting that

$$\langle A\rangle_\psi = \langle A\rangle_{B\psi} \quad \text{for } A \in \mathbb{E}_o,\ B \in \mathbb{B}.$$

Thus B-transforms may be viewed as **gauge transformations** on the wave functions of infra Fock space.

In the applications, the group $\mathbb{B}$ of B-transforms may comprise one or more of the following:

- boson field shifts
- 1-particle operators mixing particles and antiparticles (as in the original work by BOGOLIUBOV on superconductivity)
- phase shifts in gauge theories
- spacetime symmetries (e.g., translations, rotations, boosts, dilatations, diffeomorphisms)
- permutation or braid group symmetries for identical particles
- (Stückelberg) renormalization group transformations

Thus B-transforms unify a number of previously independent phenomena in quantum field theories.

It is then natural to ask if there is a role to be played by the diffeomorphism group and its representations in relativistic quantum field theories.

Goldin and Sharp, 2019

The new setting gives an easy access
to **diffeomorphism invariance**.

Goldin, Menikoff, and Sharp pioneered in the 1970s the applied group theory approach to nonrelativistic quantum field theories via current algebra, and its close relations to classical nonrelativistic fluid mechanics.

There, the diffeomorphism group and its unitary representations play a prominent role.

Quantum fluids

Infra Fock spaces lead naturally to **quantum fluids**, which form a framework for diffeomorphism invariant quantum field theories.

Quantum fluid dynamics interpolates between the superselection sectors of a quantum field theory.

The remainder of the lecture (if there is still time left) will unfortunately have to be given on the blackboard, since I haven't manage to write the slides.

Conclusion

In these lectures

• I introduced a new way to think about quantization and quantum field theory.

• The new point of view simplifies the quantization process and allows one to apply geometric reasoning to the algebraic problems.

Though a lot is still to be done, this seems to me a very fruitful way to approach the unsolved construction problems in 4-dimensional relativistic quantum field theory.

Surely someday, we can believe, we will grasp the central idea of it all as so simple, so beautiful, so compelling that we will all say to each other, "Oh, how could it have been otherwise! How could we all have been so blind so long!"

John Archibald Wheeler, 1990

Then God saw everything that He had made, and indeed, it was very good.
(Genesis 1:31)

It is very pleasing to see that this also holds for the mathematical structure of physics – God's master plan when He designed the universe.

This is ongoing work in progress.

What I showed you is just the tip of a huge iceberg waiting to be charted and explored.

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For the discussion of questions concerning my coherent approach to quantum theory, please use the discussion forum
`https://www.physicsoverflow.org`