Logic in Context

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Abstract. Context logic formalizes the way mathematicians apply logic in their reasoning, shifting context as they find it useful. Context logic is defined by only three axioms – simple reflection laws for falsity, conjunction and equality. From these, both classical and intuitionistic reasoning elements arise in a natural way; classical reasoning being about consistency (nonrefutability), intuitionistic reasoning about verifiability as the criterion for truth. In the class of categorical contexts, these criteria coincide, and classical reasoning is valid when verifying statements. We discuss tautologies in context logic, and give a semantic analysis of the relative expressiveness of classical logic and the or-free fragment of intuitionistic logic.

Contents

1 Changing contexts

Ye have heard that it hath been said, ...
But I say unto you, ...
Jesus, according to Matthew 5:38-39

This paper lies at the confluence of two major traditions in logic. The first is represented by the formalisation of context most evident in the logical approaches to Artificial Intelligence pioneered by McCarthy [25, 26, 27] and developed by numerous authors since. The second is represented by the equally rich field of Abstract Algebraic Logic [8], nowadays considered a part of the larger field of Universal Logic [3]. Our formalisation goals are similar in nature with those captured in the former tradition. At the same time, the formal tools we develop are similar in nature with those employed in the latter tradition. While the two traditions have allowed us to conceptually anchor our goals and our tools, respectively, in well-established logical traditions, our motivation for the logic developed in this paper and the treatment we give to it are largely independent.

In this paper, we give a rigorous algebraic account of *the logic of contextual mathematical reasoning* in terms familiar to mathematicians. We call this logic *context logic*. Except in Section 6, no background in mathematical logic is assumed. Everything is written to make the formal presentation easy to read for the mathematician, with hardly any notational overhead.

On the formal metalevel where the semantics is discussed, we assume that statements have a well-determined truth-value, and that every statement is either true or false, just like in the informal Platonic universe. Our formal background is classical logic and a set theory in which unions of countably many sets and quantification over subsets of a countably infinite set makes sense. (Quantification over all context logics is needed to define the meaning of a tautology.) In particular, a suitable metalevel would be constituted by a model of Zermelo-Fraenkel set theory without the axiom of choice.

On the object level, we do *not* make any such assumption, but consider reflection rules necessary to be able to reflect on the object level about the truth of statements made in certain formal contexts. This naturally leads to context logic, which turns out to be equivalent to the or-free fragment of intuitionistic logic.

Logical statements encountered in mathematical reasoning may have different degrees of validity, depending on the context in which they are stated. Some statements are unconditionally true, some unconditionally false. But many statements are in some cases true, in some cases false, and may be undecided or even undecidable in other cases – depending on a more or less formally specified context, though in any particular case of interest, it may be known that some of these are true and some are false. This knowledge is formalized by asserting a context, which, by making certain statements, singles them out as true temporarily, as long as the context is maintained. A context is therefore something that is "known" in the weak sense of being assumed to be true, often only temporarily for the sake of exploring its consequences.

Often, within a mathematical text, the context is augmented, reduced, or changed completely as needed, according to established informal principles. In particular, in indirect proofs and in arguments by cases, extra assumptions are introduced, to be removed again when a goal or a contradiction has been reached. A change of context may even alter the meaning of words and symbols. A change of context is indicated in mathematical arguments by phrases such as "let ...", "Case 1. ...", "Contradiction. Therefore ...", "We assume ...", "This concludes the proof of the lemma", "As a preparation, we consider ...", "In this section, we write ...", etc..

There is a useful, wider view of context changes. The explicitly assumed part of the context is just a finite family of sufficient conditions $c_1, \ldots, c_n$ on an otherwise arbitrary context in which the mathematical arguments in a particular part of a theory are valid. Thus the assumed part of the context just takes the form of the requirement that $c_1, \ldots, c_n$ belong to the actual context to which this part is (to be) applied. A perceived context change is then simply a change in the requirements on a fixed but unspecified context – the actual context of interest. Our formalization implements this view.

Given a context, the logical laws then force certain other statements to be true, namely those obtained by conjunction and implication from those already assumed to be true. The collection of statements true in some context (whether part of the context, enforced by the logic, or based on other, less immediate or even unspecified grounds) define the closure of this context.

In short, a context is just a collection of statements consisting of meaningful assertions in the form of texts. Its closure is the collection of all truths that hold in any situation where these statements are valid. In particular, since one can derive anything from a single false statement, the closure of a set containing a universally false statement is the set of all statements.

In Section 2, we shall make this intuition rigorous. We abstract from the detailed contents of texts and statements, and consider only the formal logical relations between statements. Results about their semantics are proved in Section 3. Section 4 then discusses the resulting logical tautologies, and Section 5 the relations to classical and intuitionistic logic. We shall see that this study can be viewed as a semantic analysis of the relative expressiveness of classical and intuitionistic logic. Section 6 relates context logic to a more traditional first-order predicate calculus. The final Section 7 shows how, via reflection rules, context logic accomodates natural deduction, the traditional method for handling context changes in formal logical reasoning.

Two companion papers expand on the present discussion by providing a context-free, purely syntactical view of context logic (Neumaier [29]) and a theory of models for context logic (Neumaier [30]).

Context in other formalizations. Context logic as defined here is a container for a multiplicity of concrete logics that are rich enough to do or verify significant pieces of mathematics in a cumulative way. It is not a logic *of* context, since, in its basic form, the statements *in* a context logic are not *about* contexts. The contexts only serve to delineate when statements are valid, so statements *about* contexts always live on the metalevel. In the subsequent paper NEUMAIER [29], however, we extend the context logic of this paper to theories containing objects other than statements, where statements are a particular kind of objects and contexts can be reflected. At its most general, context logic is thus a logic of objects and statements *in context*.

For the sake of completeness, we summarize here a number of other formalizations of context that appear in the literature. They have little in common with the much more specific notion of a (mathematical) context discussed in the present paper, since they generally try to capture the concept of context in natural language and applications of artificial intelligence (AI), which must cope with social and/or linguistic contexts of multiple, nonmonotonic, and potentially conflicting nature. In the terminology of HAASE et al. [20], a mathematical context as considered in this paper is extensive, knowledge-monotone, context-monotone, idempotent, and independently L-reducible, while typical contexts in social, linguistic, or AI applications usually fail to have one or more of these properties. As a result, the theories dealing with a logic *of* context in general are generally quite shallow from a mathematical point of view, and must be enriched with user-defined properties for each specific application. The context logic described in this paper is by no means the first or the only logic of context to be proposed. As mentioned in the introduction, formal theories of context have a long tradition in logic-based Artificial Intelligence (SERAFINI [35]), the problem of formalising context gaining momentum mainly through the work of McCarthy starting from the late '80s. McCarthy proposed a formalisation known as the *Propositional Logic of Context* (PLC) aiming to solve the problem of *generality*. Guha's thesis [19], Buvač [5] and Makarios [23, 24] developed and generalised McCarthy's approach to a quantified logic of contexts with a first-order semantics. A second attempt by GHIDINI & GIUNCHIGLIA [11] and GIUNCHIGLIA & BOUQUET [12], based on *Local Models Semantics* (LMS) and *MultiContext Logics* (MCL) is based on different conceptual premises and attempts to solve the *locality* problem instead. Many more references regarding this second approach, as well as additional authors, are available from GIUNCHIGLIA et al.[13]. Additional bibliography of work on context, in general, can be found in THOMASON [43]. Surveys covering some of the aforementioned approaches are BETTINI [2], BOLCHINI [4], SERAFINI & BOUQUET [35], STRANG & LINNHOFF-POPIEN [39]. More loosely related to the present work, the relationship between natural deduction and context as constructive modality is explored by DE PAIVA in [33]. Guha, McCool and Fikes investigate contexts for the Semantic Web in GUHA et al. [18]. Contexts are approached from a linguistic perspective in CONNOLLY [6]. Situation lattices to model and reason about context are described in YE et al. [47]. There is no relation to another kind of context logic discussed, e.g., in GARDNER & ZARFATY [10]. There have been a nuumber of international and interdisciplinary conferences on modeling and using context thus far, see, e.g., [37, 38].

2 Context logic

In the beginning was the word.
John 1:1

What is truth?
Pilate, according to John 18:38

Informally, a context logic consists of an algebra of statements together with a closure operation on sets of statements. The precise definition of the algebra of statements (however it is defined in each instance) constitutes the syntax of the logic. The precise definition of the closure operation (however it is defined in each instance) constitutes the semantics of the logic.

The concept of a context logic deliberately abstracts from the details of the syntax and the semantics of an individual logic, and only keeps the properties that a logic needs to satisfy in order to do meaningful deductions in the usual incremental manner. As long as the axioms given below are satisfied, the resulting logical structure is a context logic in the present sense.

Thus there are many possible context logics with different syntax and semantics; context logic studies their common structure. In particular, logical systems that differ only in their specification of the deduction calculus (Hilbert style, sequent, or natural deduction) will be isomorphic as context logics. We may even apply the results of context logic to incompletely specified logical systems such as those connected to statements in natural languages, whose syntax and semantics is difficult to make fully precise.

In context logic, the traditional free algebra of formulas (i.e. the syntactical logical language) is generalized to an algebra endowed with the most basic operations for mathematical reasoning. The traditional rules for manipulating these formulas is generalized to a closure operator, viewed as an abstract substitute for a consequence relation, that satisfies basic reasoning properties that we call reflection laws.

We shall indicate in Section 6 how a first-order predicate calculus fits into this scheme; many other logical calculi can be treated in a similar way. The higher level of abstraction therefore serves to be able to study a minimal logical basis needed for the formalisation of mathematical reasoning. It provides a top-down approach that starts with the closure operator and proceeds to define two simple operations on the object-level (conjunction and object equality) that *reflect* desirable metalevel properties of the closure operator. The actual syntax of a logical system for mathematical reasoning is hidden in the particular algebra of statements to which the general theory is applied. Similarly, the actual semantics of a logical system is hidden in the closure operation in each particular context logic.

We now formalize the above informal considerations. Let Σ be a fixed, countably infinite set whose elements are called **statements**. A **context** is a set of statements. We assume the existence of a closure operator that assigns to each context Γ another context $\overline{\Gamma}$, its

closure, such that, for all contexts Γ, Δ,

$$\Gamma \subseteq \overline{\Gamma} = \overline{\overline{\Gamma}}, \tag{1}$$

$$\Gamma \subseteq \Delta \quad \text{implies} \quad \overline{\Gamma} \subseteq \overline{\Delta}. \tag{2}$$

As already indicated in the introduction, the intended interpretation is that when all statements in some context Γ are assumed to hold then precisely the statements in $\overline{\Gamma}$ are guaranteed to be true, according to arbitrary, but fixed criteria of truth encoded in the closure relation. (This will be made formally precise below.) Note that closure operators on sets are often associated with *topologies*. However, in order to define a topology, a closure operator must satisfy two additional conditions that cannot be derived from the three already given, namely: $\overline{\emptyset} = \emptyset$ and $\overline{\Gamma \cup \Delta} = \overline{\Gamma} \cup \overline{\Delta}$. These two conditions are in fact not satisfied in many logics of interest, so we do not assume them.

We assume that there are a distinguished statement 0 and two binary operations that assign to any two statements x, y two further statements $x \wedge y$ and $x = y$. With these assumptions, we call Σ (together with the operations $\wedge$, $=$, and closure) a **context logic** if the three **reflection rules**

$$0 \in \overline{\Gamma} \quad \text{iff} \quad \overline{\Gamma} = \Sigma, \quad \textbf{(false reflection)} \tag{3}$$

$$x \wedge y \in \overline{\Gamma} \quad \text{iff} \quad x, y \in \overline{\Gamma}, \quad \textbf{(and reflection)} \tag{4}$$

$$(x = y) \in \overline{\Gamma} \quad \text{iff} \quad \overline{\Gamma \cup \{x\}} = \overline{\Gamma \cup \{y\}}, \quad \textbf{(equal reflection)} \tag{5}$$

hold for all contexts Γ. The intuitive meaning of 0, $\wedge$ and $=$ as **false**, **logical and**, and **logical equality** (or **equivalence**), respectively, are immediate consequences of these reflection rules.

Since inside Σ, the equality sign is used as a binary operation symbol rather than as a relation sign, we use the notation $x \equiv y$ to express that two statements x, y are **identical** as elements of Σ. We define another binary operation $\Rightarrow$ on Σ by

$$x \Rightarrow y \;\equiv\; (x \wedge y = x). \tag{6}$$

Clearly, $\Rightarrow$ has the meaning of logical **implication**. We also define the operators F, C, B on Σ and the distinguished statement 1 by

$$Fx \equiv (x \Rightarrow 0), \quad Cx \equiv FFx, \quad Bx \equiv (Cx \Rightarrow x), \quad 1 \equiv F0. \tag{7}$$

Using these, we define further binary operations $|$ and $\neq$ on Σ by

$$x|y \equiv (Fx \Rightarrow y) \wedge (Fy \Rightarrow x). \tag{8}$$

$$x \neq y \equiv F(x = y), \tag{9}$$

1 is interpreted as **true**, F and $|$ as weak forms of **not** and **or**, and $\neq$ is a weak form of **distinct**. We require that, in expressions, the operators F, C, B bind stronger than $\wedge$ and $|$, which bind stronger than $\in$, which binds stronger than $=$, which binds stronger than $\Rightarrow$, which binds stronger than $\equiv$. We take $x = y = z$ to mean $(x = y) \wedge (y = z)$, $x = y \Rightarrow z$ to mean $(x = y) \wedge (y \Rightarrow z)$, $x \Rightarrow y = z$ to mean $(x \Rightarrow y) \wedge (y = z)$, $x \Rightarrow y \Rightarrow z$ to mean $(x \Rightarrow y) \wedge (y \Rightarrow z)$, and similarly for longer chains of $=$ and $\Rightarrow$.

2.1 Proposition. *For all statements x, y,*

$$1 \in \overline{\Gamma}, \quad \textbf{(true reflection)} \tag{10}$$

$$(x \Rightarrow y) \in \overline{\Gamma} \quad \textit{iff} \quad y \in \overline{\Gamma \cup \{x\}}, \quad \textbf{(imply reflection)} \tag{11}$$

$$x, (x \Rightarrow y) \in \overline{\Gamma} \quad \textit{implies} \quad y \in \overline{\Gamma}. \quad \textbf{(modus ponens)} \tag{12}$$

Proof. Since $x \wedge y \in \overline{\Gamma \cup \{x \wedge y\}}$, and reflection gives $x, y \in \overline{\Gamma \cup \{x \wedge y\}}$, so that

$$\overline{\Gamma \cup \{x\}} \subseteq \overline{\Gamma \cup \{x \wedge y\}}. \tag{13}$$

Now $(x \Rightarrow y) \in \overline{\Gamma}$ iff $x \wedge y = x) \in \overline{\Gamma}$ iff $\overline{\Gamma \cup \{x \wedge y\}} = \overline{\Gamma \cup \{x\}}$ by equal reflection iff $\overline{\Gamma \cup \{x \wedge y\}} \subseteq \overline{\Gamma \cup \{x\}}$ by (13) iff $x \wedge y \in \overline{\Gamma \cup \{x\}}$ iff $x, y \in \overline{\Gamma \cup \{x\}}$ by and reflection iff $y \in \overline{\Gamma \cup \{x\}}$. This proves (11). Specializing x and y to 0 together with false reflection implies that $(0 \Rightarrow 0) \in \overline{\Gamma}$, and (10) follows by definition (7) of 1.

By and reflection (4), $x \in \overline{\Gamma}$, hence $\overline{\Gamma} = \overline{\Gamma \cup \{x\}}$. Again by and reflection, $(x \Rightarrow y) \in \overline{\Gamma}$. Hence by imply reflection, $y \in \overline{\Gamma \cup \{x\}} = \overline{\Gamma}$, which is (12). □

Alternatively, one can also start with $\Rightarrow$ in place of $=$ as undefined operation, and imply reflection (11) in place of equal reflection (5), and define the operation $=$ by

$$x = y \;\equiv\; (x \Rightarrow y) \wedge (y \Rightarrow x).$$

Then equal reflection can be proved as follows: By (1), imply reflection (11) can be written in the equivalent form $(x \Rightarrow y) \in \overline{\Gamma}$ iff $\overline{\Gamma \cup \{y\}} \subseteq \overline{\Gamma \cup \{x\}}$. (5) follows from this and the definition of equality.

Note that the Watson theorem prover [45] is based on a logical foundation resmbling ours, in that *"equality appears as a term-forming operation as well as in the role of a predicate"* [46]. However, Watson has expressions and abstraction, which makes it much more expressive than the propositional setting considered in this paper.

A context is called **closed** if it contains its closure, **paradoxical** if its closure contains 0, and **consistent** if it is not paradoxical. Clearly, the closure of any context is closed, and the intersection of closed contexts is closed. We say that a statement x

- **holds** (or is **true**, or is **valid**, or is a **truth**) in the context Γ if it belongs to $\overline{\Gamma}$,
- **fails** (or is **false**, or is **invalid**) in the context Γ if Fx holds in Γ,
- is a **contradiction** in the context Γ if $x = 0$ holds in Γ,
- is **consistent** (or **irrefutable**) in the context Γ if Cx holds in Γ,
- is **Boolean** (or **classical**) in the context Γ if Bx holds in Γ,
- is **ambiguous** in (or **independent** of) the context Γ if both $\Gamma \cup \{x\}$ and $\Gamma \cup \{Fx\}$ are consistent,

- is a **consequence** of the statement y if $x \in \overline{\{y\}}$,
- is **necessary** (or is a **fact**) if x holds in the empty context (and hence in every context).
- is **possible** if x holds in some consistent context.

That this suggestive terminology is appropriate will be established in Section 3. (Our term "ambiguous" is roughly synonymous with "undecidable", but to define the latter needs an algorithmic setting not required here.)

Clearly, the closure of a context consists of all statements that hold in this context. With this interpretation of closure, the reflection rules for and and imply are immediately meaningful. They are clearly necessary for any strong rigorous reasoning. The operations $\wedge$ and $=$ get their meaning **and** and **equal** from the corresponding reflection rules.

The derived equality operation $\Rightarrow$ therefore gets the meaning of **implies**. Note that "x implies y" has no causal connotation but simply means (by imply reflection) "y can be added to the stack of known truths if x is in it", which is precisely what mathematical reasoning needs.

The operation $|$, which we refer to as **weak or**, becomes a weak form of disjunction; weak, since $x|x = x$ is not necessarily a fact – see Theorem 4.2(iii) below. A strong or is briefly discussed in Section 5.

On the (informal) semantical level we shall use "not" as customary, so that double negation preserves the truth values. However, a context logic has no formal notion of "not"; in its place we have the formal operator F. Indeed, Fx ("x is false") is a weak form of "not x". If S is a true (false) informal statement then "not S" is false (true). However, if x is a formal statement ($x \in \Sigma$) then "not x" is interpreted as "not: $x \equiv 1$". In contrast to what holds in the assumed informal language, a formal statement need not be true or false in a general context, Thus "not true" is not necessarily the same as "false", and "not false" not necessarily the same as "true".

Cx is the statement $Fx \Rightarrow 0$, asserting that assuming the falsity of x implies a contradiction, giving Cx indeed the interpretation of Cx as irrefutability of x. Hence asserting Bx asserts that the **proof by contradiction**, $(Fx \Rightarrow 0) \Rightarrow x$, is valid for x. A context in which all statements are Boolean, i.e., where Bx holds for all x is called **classical**. A context logic is called **classical** if Bx is a fact for all x, so that the empty context and hence every context is classical. In a nonclassical context, we only have the weaker fact $(Fx \Rightarrow 0) \Rightarrow Cx$.

One could consider to require **not reflection**, namely the property $Fx \in \overline{\Gamma}$ iff $x \notin \overline{\Gamma}$. But this would imply that $0 \notin \overline{\Gamma}$ since $1 \in \overline{\Gamma}$ by true reflection (10). Thus not reflection would force consistency – an undesirable property since mathematicians often work long and persistently in inconsistent contexts to prove the context inconsistent. A famous example is the proof by Feit & Thompson [7] that there is no finite simple group of odd order. Thus not reflection is too strong to assume it in arbitrary mathematical contexts, and we shall not use it.

Note that $1 \neq 0$ is provable in every context logic. Indeed, if $1 \neq 0$ is false then $1 = 0$. Therefore every statement holds, and in particular $1 \neq 0$. Conversely, in a paradoxical context, and reflection implies $1 = 0$, i.e., truth is contradictory, there is no difference between true and false. The reflection rule for false expresses the idea that, if false becomes true (in a paradoxical context) then everything becomes true. This is reasonable since, in this case, even when something is false, it is true – assuming contradictory statements amounts to giving up the distinction between truth and falsity.

A context logic is called **paradoxical** if every statement holds in every context, and **consistent** otherwise.

2.2 Example. (Paradoxical logic)
On an arbitrary countable set Σ containing 0, with arbitrary operations $\wedge$ and $\Rightarrow$, we define the closure by $\overline{\Gamma} = \Sigma$ for all contexts Γ. The reflection rules hold trivially, no context is consistent, hence the logic is paradoxical. It is easily checked that every paradoxical logic has this form.

2.3 Example. (Ternary logic)
On the set $\{0, b, 1\}$, the operations $\wedge, =, \Rightarrow, \neq, |$, and the operators F, C, B are given by the truth tables

x	y	$x \wedge y$	$x = y$	$x \Rightarrow y$	$x \neq y$	$x\|y$
0	0	0	1	1	0	0
0	1	0	0	1	1	1
1	0	0	0	0	1	1
1	1	1	1	1	0	1
0	b	0	0	1	1	b
1	b	b	b	b	0	1
b	0	0	0	0	1	b
b	1	b	b	1	0	1
b	b	b	1	1	0	1

x	Fx	Cx	Bx
0	1	0	1
1	0	1	1
b	0	1	b

With the closure

$$\overline{\Gamma} := \begin{cases} \Sigma & \text{if } 0 \in \Gamma, \\ \Gamma \cup \{1\} & \text{otherwise,} \end{cases}$$

the reflection rules hold. Thus we get a consistent context logic with three distinct statements. It is the only context logic with exactly three statements (see the discussion in NEUMAIER [30]). It is essentially the ternary member in an infinite series of models for intuitionistic logic constructed by GÖDEL [15], without the (idempotent) strong intuitionistic or $\vee$. Note that $b|b = 1$, so that the weak or is not idempotent. This context logic is not classical since Bb does not hold in every context. In the empty context, b is not true. Thus, "b is not true" is true but $b \neq 1$ is false. This shows that $\neq$ signifies, in general, only a weak form of distinctness, and is not equivalent with "is not".

2.4 Example. (Binary logic)
Restricting the ternary context logic of Example 2.3 to the set $\{0, 1\}$ of classical truth values (i.e., using only the upper half of the tables), we recover the truth tables for the traditional binary logic with only two statements. These form a consistent, classical context logic. The closure simplifies to $\overline{\Gamma} = \Gamma \cup \{1\}$. The weak and the strong or coincide in this case.

3 Semantics

Our knowledge is patchwork, and our predictive power is limited. But when perfection comes, all patchwork will disappear.
St. Paul, in 1 Corinthians 13:9-10

In this section, we show that the definitions given essentially have the traditional informal meaning associated with the words used for our concepts.

In a context logic, the logical interpretation of statements depends on the context Γ in which they are considered.

3.1 Theorem. *In every context logic, we have:*

(i) Γ *is inconsistent iff every statement holds in* Γ.

(ii) $\Gamma \cup \{x\}$ *is consistent iff* $Fx \notin \overline{\Gamma}$.

(iii) In any context, a statement x *is a contradiction iff* Fx *holds.*

(iv) A statement x *is possible iff* Fx *holds in a paradoxical context only.*

(v) In a consistent context, a statement cannot be true and false.

(vi) If Γ *is consistent then for any statement* x*, at least one of* $\Gamma \cup \{x\}$ *and* $\Gamma \cup \{Fx\}$ *is consistent.*

(vii) If x *is consistent in the consistent context* Γ *then* $\Gamma \cup \{x\}$ *is consistent.*

(viii) A consistent statement is false only in a paradoxical context.

Proof. (i) holds by false reflection (3).

(ii) $\Gamma \cup \{x\}$ is inconsistent iff $0 \in \overline{\Gamma \cup \{x\}}$ iff (by imply reflection (11)) $(x \Rightarrow 0) \in \overline{\Gamma}$ iff $Fx \in \overline{\Gamma}$.

(iii) By equal reflection (5), $x = 0$ holds in Γ iff $\overline{\Gamma \cup \{x\}} = \overline{\Gamma \cup \{0\}}$, and by false reflection, this is the case iff $\Gamma \cup \{x\}$ is inconsistent. By (ii), this is the case iff Fx holds in Γ.

(iv) x is possible iff x holds in some consistent context Γ, and by (ii), iff $Fx \notin \overline{\Gamma}$ for some consistent context Γ. This holds iff $Fx \in \overline{\Gamma}$ only if Γ is inconsistent, which says that Fx holds only in a paradoxical context.

(v) If x is true and false in the context Γ then $x, Fx \in \overline{\Gamma}$. Since $Fx \equiv (x \Rightarrow 0)$, modus ponens (12) implies $0 \in \overline{\Gamma}$, contradiction.

(vi) For otherwise there is a statement x such that $\Gamma \cup \{x\}$ and $\Gamma \cup \{Fx\}$ are inconsistent. By (ii), both Fx and FFx are in $\overline{\Gamma}$, hence Fx is true and false in Γ, contradicting (v).

(vii) If not then by (ii), $Fx \in \overline{\Gamma}$. But $Cx (\equiv FFx)$ holds in Γ, hence Fx is true and false in Γ, contradicting (vi).

(viii) Let Γ be consistent. If x is false then $0 \in \overline{\Gamma \cup \{x\}}$, hence $\Gamma \cup \{x\}$ is inconsistent, contradicting by (vi) the consistency of x. □

A **realization** is a closed and consistent context Γ such that $\Gamma \cup \{x\}$ is inconsistent for every $x \notin \Gamma$. A context Γ in which every statement x is either true or false ($x \in \overline{\Gamma}$ or $Fx \in \overline{\Gamma}$) is called **categorical**. (This definition uses the classical or, which is allowed on the informal level.) A closed and categorical context is called **complete**. Two contexts Γ and Γ' are **compatible** if $\Gamma \cup \Gamma'$ is consistent.

The model theoretic notation $\Gamma \models x$ ("x holds in every model in which Γ holds") has approximately the same content as our assertion "x holds in Γ". The meaning becomes the same if we identify "model" with "realization", which is permitted if the context logic is a logical framework in which everything, including the informal discussion, happens. The models are then in fact the suitably defined quotients Σ/Γ_0, where Γ_0 is a realization. We prefer the set theoretic notation $x \in \overline{\Gamma}$ to the model theoretic notation $\Gamma \models x$, which is unfamiliar to the average mathematician and requires additional explanations of what it means to be a model.

If Cx, one is not able to add Fx without getting a contradiction. Indeed, from Cx and Fx one obtains the contradiction $0 = F1 = FFx = Cx = 1$. Thus consistency means irrefutability – nothing can disprove it, not even additional assumptions or information - unless these are already contradictory in the original context. Thus, if Cx then x can be added to a consistent context without violating consistency. This just diminishes the collection of contexts compatible with the current context. By a completion argument given in a moment, one can add statements until one has a realization (and, as we shall show in Section 5, a classical logic).

3.2 Theorem. *In every context logic, we have:*

(i) A statement consistent in some context holds in all realizations containing this context.

(ii) A context is consistent iff it is contained in a realization.

(iii) A statement is possible iff it holds in some realization.

(iv) A consistent context is complete iff it is a realization.

(v) A consistent context is in a unique realization iff it is categorical.

(vi) A context logic is consistent iff it has a realization.

Proof. (i) Let x be a consistent in the context Γ, so that $FFx \in \overline{\Gamma}$. Let Δ be a realization containing Γ. By the closure property (2), $\overline{\Delta} = \Delta$ contains $\overline{\Gamma}$ hence FFx, hence $\Delta \cup \{Fx\}$ is inconsistent by Theorem 3.1(ii), hence $Fx \notin \Delta$, hence $x \in \Delta$ since Δ is a realization.

(ii) Clearly, a context contained in a realization is consistent. Conversely, let Γ be a consistent context. Since Σ is countable, there is a sequence $x_1, x_2, \ldots$ enumerating the elements of Σ. Put $\Gamma_1 = \overline{\Gamma}$. For any k for which Γ_k is defined, closed, and contains Γ (in particular for $k = 1$), let Σ_k be the set of $x \notin \Gamma_k$ such that $\Gamma_k \cup \{x\}$ is consistent. If Σ_k is nonempty, we put $\Gamma_{k+1} := \overline{\Gamma_k \cup \{x_{l(k)}\}}$, where $l(k)$ is the least l such that $x_l \in \Sigma_k$. Clearly, $l(k) \geq k$. If the sequence of sets constructed this way ends since some Σ_k is empty, Γ_k is a realization containing Γ. If not, then the union Γ_∞ of all Γ_k is a realization. Indeed, it is closed and consistent. If $x \notin \Gamma_\infty$ but $\Gamma_\infty \cup \{x\}$ were consistent then $x = x_k$ for some k, and $\Gamma_{k+1} \cup \{x\}$ is consistent as a subset of $\Gamma_\infty \cup \{x\}$. But since $l(k+1) > k$, this contradicts minimality.

(iii) If x is possible then x is contained in a consistent context, hence by (ii) in a realization.

(iv) Let Γ be a consistent context. If Γ is complete then Γ is closed, hence $\overline{\Gamma} = \Gamma$. If $x \notin \Gamma$ then $Fx \in \Gamma$, hence $\Gamma \cup \{x\}$ is inconsistent by Theorem 3.1(ii). Thus Γ is a realization. Conversely, if Γ is a realization then $x \notin \Gamma$ implies that $\Gamma \cup \{x\}$ is inconsistent, hence $Fx \in \Gamma$ by Theorem 3.1(ii). Thus every statement is true or false in Γ, and since Γ is closed, it is categorical.

(v) follows directly from (ii) and (iv).

(vi) holds by (iv) since a context logic is consistent iff the empty set is consistent. □

An assertion like (ii) is usually referred to as a **Lindenbaum theorem**. (The first such result is credited to Lindenbaum by TARSKI [40, Theorem 12 and footnote p.38]; apparently it was never published by Lindenbaum himself.) As any Lindenbaum theorem, it is valid only if the informal metalevel (in which the assertion is proved) is classical, cf. SHAPIRO [36].

The context consisting of all facts is called **universal**. It is easy to see that the universal context is the closure of the empty context and therefore closed; that it is inconsistent iff there is only a single closed context (namely the context of all statements); that it is consistent iff there is a consistent context; and that it is consistent and categorical iff there is a unique realization (namely the universal context).

3.3 Proposition. *Let Γ_0 be a context in the context logic Σ. Then the context logic obtained from Σ by keeping the operations but redefining the closure of a context Γ to be the closure in Σ of $\Gamma \cup \Gamma_0$ defines another context logic, Σ/Γ_0, the* **restriction** *of Σ by Γ_0. In Σ/Γ_0, the context Γ_0 is universal.*

Proof. Every closed context in Σ/Γ_0 is closed in Σ. Now (1)–(11) in Σ/Γ_0 follow from the corresponding rules in Σ since $\Gamma \subseteq \Delta$ implies $\Gamma \cup \Gamma_0 \subseteq \Delta \cup \Gamma_0$ and $(\Gamma \cup \Gamma_0) \cup \{x\} =$

$(\Gamma \cup \{x\}) \cup \Gamma_0$. □

Restriction amounts to requiring the statements of Γ_0 as axioms. This shows that facts are relative, and that what is a fact in one context logic may be context-dependent in another context logic. We therefore turn now to statements whose truth status is independent of the context and even the context logic.

4 Tautologies in context logic

Of making many books
there is no end.
Eccl. 12:12

A **logical expression** $e(x_1, \ldots, x_n)$ in the variables $x_1, \ldots, x_n$ is a string built according to the traditional informal rules from $0, 1, x_1, \ldots, x_n$, the operators F, C, B, the binary operations $\wedge, \Rightarrow, =, \neq$, and $|$, and parentheses, where operators bind strongest, $\wedge$ and $|$ bind stronger than $=$ and $\neq$, and $\Rightarrow$ binds weakest.

A **law** is a logical expression $e(x_1, \ldots, x_n)$ such that all instantiations of the variables result in facts. A logical expression that is a law in every context logic is called a **tautology**.

Formulas such as Bx (the law justifying proof by contradiction) or $FFx = x$ (the double negation law) are not tautologies since they are not a fact in the ternary context logic of Example 2.3. However, as we shall see, many other familiar formulas are tautologies. The following result is basic for verifying such tautologies.

4.1 Proposition.
(i) $x \Rightarrow y$ is a fact iff $x \in \Gamma$ implies $y \in \Gamma$ for all closed contexts Γ.

(ii) $x = y$ is a fact iff $x \in \Gamma$ is equivalent to $y \in \Gamma$ for all closed contexts Γ.

Proof. (i) If $x \Rightarrow y$ is a fact then $x \Rightarrow y \in \Gamma$ for any context Γ. If Γ is closed then, by imply reflection (11), $x \in \Gamma$ implies $y \in \Gamma$. Conversely, if $x \in \Gamma$ implies $y \in \Gamma$ for all closed Γ then $x \in \overline{\Delta \cup \{x\}}$ for every context Δ, hence the assumption implies $y \in \overline{\Delta \cup \{x\}}$, and imply reflection implies $(x \Rightarrow y) \in \Delta$. Thus $x \Rightarrow y$ is a fact.

(ii) For a closed context Γ, we have $x \in \Gamma$ iff $x \in \overline{\Gamma}$ iff $\overline{\Gamma} = \overline{\Gamma \cup \{x\}}$, and $y \in \Gamma$ iff $y \in \overline{\Gamma}$ iff $\overline{\Gamma} = \overline{\Gamma \cup \{y\}}$. Thus $x \in \Gamma$ and $y \in \Gamma$ are equivalent iff $\overline{\Gamma \cup \{x\}} = \overline{\Gamma \cup \{y\}}$, and by equal reflection (5), this holds iff $(x = y) \in \overline{\Gamma}$. Therefore this holds for all closed contexts Γ iff $x = y$ is a fact. □

4.2 Theorem. *The following expressions are tautologies:*

T0:	*1*	*(truth)*
T1:	$x = x$	*(reflexivity)*
T2:	$(x = y) = (y = x)$	*(symmetry)*
T3:	$(x = y) \wedge (y = z) \Rightarrow (x = z)$	*(transitivity)*
T4:	$x = y \Rightarrow (x = z) = (y = z)$	*(equal substitution)*
T5:	$x = y \Rightarrow x \wedge z = y \wedge z$	*(and substitution)*
T6:	$(x = x) = 1$	*(true characterization)*
T7:	$(x = 1) = x$	*(statement characterization)*
T8:	$(x \Rightarrow y) = (x \wedge y = x)$	*(imply characterization)*
T9:	$x \wedge x = x$	*(idempotence)*
T10:	$1 \wedge x = x$	*(maximality)*
T11:	$0 \wedge x = 0$	*(minimality)*
T12:	$x \wedge y = y \wedge x$	*(commutativity)*
T13:	$(x \wedge y) \wedge z = x \wedge (y \wedge z)$	*(associativity)*

Proof. In the following, Γ is an arbitrary closed context.

T0 follows from true reflection (10), and T1 from equal reflection (5).

T2 follows from Proposition 4.1(ii) since the condition there is symmetric in x and y. To get T3, we observe that, for a closed context Γ, $(x = y) \wedge (y = z) \in \Gamma$ implies by and reflection (4) that $(x = y), (y = z) \in \Gamma$, hence by Proposition 4.1(ii) $x \in \Gamma$ iff $y \in \Gamma$ iff $z \in \Gamma$, hence again by Proposition 4.1(ii) $(x = z) \in \Gamma$. By Proposition 4.1(i), this gives $((x = y) \wedge (y = z) \Rightarrow x = z) \in \Gamma$, hence T3.

The other expressions are handled similarly, and we shall be briefer. For T4 we need to show that $(x = y) \in \Gamma$ implies that $(x = z) \in \Gamma$ iff $(y = z) \in \Gamma$. The assumption says that $x \in \Gamma$ iff $y \in \Gamma$ and gives T4 upon using Proposition 4.1(ii) on both sides of the claim.

T5 is proved as T4, using and reflection in place of Proposition 4.1(ii).

T7 holds since $(x = 1) \in \Gamma$ iff $x \in \Gamma$ and $1 \in \Gamma$ are equivalent, and the latter always holds. T6 follows from T7 and T1.

To show T8, we note that $x \wedge y \in \Gamma$ holds iff $x \in \Gamma$ always implies $y \in \Gamma$. This is the case iff $x, y \in \Gamma$ and $y \in \Gamma$ are equivalent, hence iff $x \wedge y \in \Gamma$ and $y \in \Gamma$ are equivalent. This is the case iff $(x \wedge y = x) \in \Gamma$.

T9, T10 and T12 follow directly from and reflection, and T11 by also using false reflection.

T13 is a direct consequence of and reflection, applied twice. □

As will be shown in the companion paper Neumaier [29], a complete analysis of tautologies can be done by purely syntactic (string-based) means. In particular, it is decidable whether a logical expression is a tautology. Moreover, it can be shown that the tautologies T0–T13 provide an axiomatic basis for a purely algebraic view of context logic, in the sense that every tautology can be proved from T0–T13.

By T1–T5 and reflection, the relation $\sim$ defined by $x \sim y$ iff $x = y$ is a fact is a congruence relation. This allows one to identify statements with the same logical content. We call a context logic **simple** if $x = y$ is a fact only when $x \equiv y$. In any context logic, we can define for every statement x (zero or more) new, mutually distinct texts x_l not yet in Σ by a **definition** $x_l := x$, which declares each $x_l = x$ a fact, declares operations with x_l by substitution of x for x_l, and adds the x_l precisely to the contexts containing x. This results in an **augmented** context logic that is equivalent to the original context logic. In any context logic, we may alter arbitrarily the result of operations by statements congruent to the original results, obtaining a **renamed** context logic. Clearly, every context logic can be viewed as a renamed augmented context logic of a corresponding simple **quotient logic**, obtained by keeping from each congruence class of statements only a single one. Therefore, from an algebraic point of view, one may restrict attention to simple context logics. This algebraic point of view will be systematically developed in the companion papers Neumaier [30, 29].

4.3 Theorem. *(cf.* Frink *[9])*
The following expressions are tautologies:

F0: $1 = F0$

F1: $x \wedge y = y \wedge x$

F2: $x \wedge (y \wedge z) = (x \wedge y) \wedge z$

F3: $x \wedge x = x$

F4: $(x \wedge y = 0) = (x \Rightarrow Fy) = (x \wedge Fy = x)$

F5: $x \wedge Fx = 0$

F6: $x \wedge 0 = 0$

F7: $x \wedge 1 = x$

F8: $x \Rightarrow Cx$

F9: $(x \Rightarrow y) \Rightarrow (Fy \Rightarrow Fx)$

F10: $(x \Rightarrow y) \Rightarrow (Cx \Rightarrow Cy)$

F11: $CFx = FCx = Fx$

F12: $CCx = Cx$

F13: $x = Fy \Rightarrow Cx = x$

F14: $C(Fx \wedge Fy) = Fx \wedge Fy$

F15: $C0 = 0, \quad F1 = 0$

F16: $(Fx \wedge Cy = 0) = (Fx \Rightarrow Fy)$

F17: $F(x \wedge y) = F(Cx \wedge Cy)$

F18: $C(x \wedge y) = Cx \wedge Cy$

Proof. In this proof, we will be using the tautologies $x \wedge y \Rightarrow x$, and that $x \Rightarrow y$ and $x \Rightarrow z$ imply $x \Rightarrow y \wedge z$, so we first prove these. The former one is rewritten as $(x \wedge y) \wedge x = x \wedge y$

according to the definition of $\Rightarrow$, which is immediately equivalent with $x \wedge y = x \wedge y$ by associativity, commutativity and idempotency of conjunction, according to T9, T12 and T13. The latter one follows by **imply reflection** and **and reflection**: $x \Rightarrow y \wedge z$ iff $y \wedge z \in \overline{\{x\}}$ iff $y \in \overline{\{x\}}$ and $z \in \overline{\{x\}}$ iff $x \Rightarrow y$ and $x \Rightarrow z$.

The rule F0 follows from the definition of 1 and **and reflection**. F1=T12, F2=T13, F3=T9.

If $x \Rightarrow Fy$ then $x \wedge y \Rightarrow Fy \wedge y = 0$, hence $x \wedge y = 0$; conversely, if $x \wedge y = 0$ then $x \wedge y \Rightarrow 0$, hence $x \Rightarrow (y \Rightarrow 0) = Fy$, hence $x \Rightarrow Fy$. Now the definition of $\Rightarrow$ gives $x \wedge y = 0$ iff $x \wedge Fy = x$ iff $x \Rightarrow Fy$, and equal reflection proves F4.

By imply reflection, $\Rightarrow$ is reflexive, antisymmetric, and transitive. This is used repeatedly below.

Putting $x = Fy$ in F4 gives $(Fy \wedge y = 0) = (Fy = Fy) = 1$, hence F5. F6 holds since $x \wedge 0 = x \wedge (x \wedge Fx) = (x \wedge x) \wedge Fx = x \wedge Fx = 0$ by F5. F7 follows from F6 using F4. F5 implies that $x \wedge FFx = x$ by F4, hence $x \Rightarrow FFx = Cx$, giving F8.

To prove F9, suppose that $x \Rightarrow y$. Then F8 gives $x \Rightarrow Cy = FFy$, hence F4 gives $x \wedge Fy = 0$, hence $Fy \wedge x = 0$, and applying again F4 we find $Fy \Rightarrow Fx$, hence with imply reflection F9. Applying F9 twice gives F10.

To prove F11 we note that F8 with x replaced by Fx gives $Fx \Rightarrow FCx$, while F9 applied to F8 gives $FCx \Rightarrow Fx$, hence $FCx = Fx$. Since $CFx = FFFx = FCx$, F11 follows. Applying F11 twice gives F12, and F13 also follows directly from F11.

To prove F14, we first note that $Fx \wedge Fy \Rightarrow Fx$ using the tautology $x \wedge y \Rightarrow x$ demonstrated at the beginning of the proof, hence $C(Fx \wedge Fy) \Rightarrow CFx = Fx$ by F10 and F11, and by symmetry also $C(Fx \wedge Fy) \Rightarrow Fy$. Therefore $C(Fx \wedge Fy) \Rightarrow Fx \wedge Fy$, using the second tautology from the beginning. The converse follows from F8, whence $C(Fx \wedge Fy) = Fx \wedge Fy$.

To prove F15 we put $x = F0$ into F5 to get $F0 \wedge C0 = 0$, hence $C0 = C(F0 \wedge C0) = F0 \wedge C0 = 0$ by F14. F16 follows by applying F11 to F4.

To get F17, we first evaluate $z := F(x \wedge y) \wedge (Cx \wedge Cy)$. We have $z \Rightarrow F(x \wedge y)$, hence F4 implies $z \wedge x \wedge y = 0$, hence $z \wedge z \Rightarrow Fy$ by F4. Similarly, $z \Rightarrow Cy$, hence $z \wedge x \Rightarrow Cy$. Combining these implications, we find that $z \wedge x \Rightarrow Cy \wedge Fy = 0$, hence $z \wedge x = 0$ and $z \Rightarrow Fx$ by F4. But since $z \Rightarrow Cx$, this implies $z \Rightarrow Fy \wedge Cx = 0$. Thus $z = 0$, and from the defining equation for z we find $F(x \wedge y) \Rightarrow F(Cx \wedge Cy)$. The reverse implication follows from F8 and F9, proving F17.

Finally, applying F to both sides of F17 and using F14 we get F18. □

Note that F4 says in conventional terminology that Fy is a complement of y; thus a simple context logic is a pseudo-complemented semilattice in the sense of Frink [9], and F5–F18 are essentially translations of the equations (5)–(18) there.

From Neumaier [29], we quote the following theorem:

4.4 Theorem. *For any statement x, the set consisting of the statements 0, 1, x, Fx, Cx, and Bx is closed under the logical operations, and we have the following operation tables. (Here, for easy checking, $u||v \equiv (Fu \Rightarrow v)$ defines the* **asymmetric or** *$||$, so that $u|v \equiv (u||v) \wedge (v||u)$.)*

$\wedge$	0	1	x	Fx	Bx	Cx
0	0	0	0	0	0	0
1	0	1	x	Fx	Bx	Cx
x	0	x	x	0	x	x
Fx	0	Fx	0	Fx	Fx	0
Bx	0	Bx	x	Fx	Bx	x
Cx	0	Cx	x	0	x	Cx

$=$	0	1	x	Fx	Bx	Cx
0	1	0	Fx	Cx	0	Fx
1	0	1	x	Fx	Bx	Cx
x	Fx	x	1	0	Cx	Bx
Fx	Cx	Fx	0	1	Fx	0
Bx	0	Bx	Cx	Fx	1	x
Cx	Fx	Cx	Bx	0	x	1

$\Rightarrow$	0	1	x	Fx	Bx	Cx
0	1	1	1	1	1	1
1	0	1	x	Fx	Bx	Cx
x	Fx	1	1	Fx	1	1
Fx	Cx	1	Cx	1	1	Cx
Bx	0	1	Cx	Fx	1	Cx
Cx	Fx	1	Bx	Fx	Bx	1

z	0	1	x	Fx	Bx	Cx
Fz	1	0	Fx	Cx	0	Fx
Cz	0	1	Cx	Fx	1	Cx
Bz	1	1	Bx	1	Bx	1

$\Vert$	0	1	x	Fx	Bx	Cx
0	0	1	x	Fx	Bx	Cx
1	1	1	1	1	1	1
x	Cx	1	Cx	1	1	Cx
Fx	Fx	1	Bx	Fx	Bx	1
Bx	1	1	1	1	1	1
Cx	Cx	1	Cx	1	1	Cx

$\vert$	0	1	x	Fx	Bx	Cx
0	0	1	x	Fx	Bx	Cx
1	1	1	1	1	1	1
x	x	1	Cx	Bx	1	Cx
Fx	Fx	1	Bx	Fx	Bx	1
Bx	Bx	1	1	Bx	1	1
Cx	Cx	1	Cx	1	1	Cx

$\neq$	0	1	x	Fx	Bx	Cx
0	0	1	Cx	Fx	1	Cx
1	1	0	Fx	Cx	0	Fx
x	Cx	Fx	0	1	Fx	0
Fx	Fx	Cx	1	0	Cx	1
Bx	1	0	Fx	Cx	0	Fx
Cx	Cx	Fx	0	1	Fx	0

From the tables, we may read off that under the assumption $Bx = 1$ we get $Cx = x$, hence a 4-valued Boolean logic. Under the assumption $Cx = 1$ we get the ternary logic from

Example 2.3. The assumptions $x = 1$ or $Fx = 0$ just fix the variable and leave the binary logic. Finally, setting $0 = 1$ reduces the logic to the paradoxical logic in which everything is true and false.

5 Classical and intuitionistic logic

But let your communication be, Yea, yea; Nay, nay: for whatsoever is more than these cometh of evil.
Jesus, according to Matthew 5:37

Context logic includes classical logic and intuitionistic logic as special cases, though the intuitionistic case needs additional structure. Classical logic was defined in Section 2 as a context logic in which Bx holds in every context for all statements x. **Intuitionistic logic** (Heyting [21]) requires the existence of an additional operation, the **strong or** $\vee$, characterized by the properties ($A \in \overline{\Gamma}$ or $B \in \overline{\Gamma}$ implies $A \vee B \in \overline{\Gamma}$) and ($A \vee B \in \overline{\Gamma}$, $C \in \overline{\Gamma \cup \{A\}} \cap \overline{\Gamma \cup \{B\}}$ imply $C \in \overline{\Gamma}$). $x \vee y \in \overline{\Gamma}$ iff $x \in \overline{\Gamma}$ or $y \in \overline{\Gamma}$. Since $x|Fx = Bx$, we have $Fx|Cx = BFx = 1$, while intuitionistically, $Cx \vee Fx = 1$ is not a tautology.

If $\Rightarrow$ is interpreted as an order relation, the intuitionistic disjunction $\vee$ (strong or) has the properties of a least upper bound: $(x \Rightarrow z) \wedge (y \Rightarrow z) \Rightarrow (x \vee y \Rightarrow z)$, whereas the statement $(x \Rightarrow z) \wedge (y \Rightarrow z) \Rightarrow (x|y \Rightarrow z)$ is not generally valid. A least upper bound $x \vee y$, if it exists, satisfies $x \vee y \Rightarrow x|y$, but the converse only holds in classical logic. In particular, we always have $x \vee x = x$ but $x|x = Cx$, hence $x|x = x \vee x$ iff Bx. Thus the formula defining weak disjunction $|$ is not valid for the $\vee$ (strong or) in place of $|$. In general, $x|y$ is an upper bound for x and y but not necessarily a least upper bound.

Thus context logic is more general than intuitionistic logic but (having no $\vee$) less expressive. Nevertheless, as shown in the companion papers Neumaier [29, 30], general context logic inherits most of the properties of intuitionistic logic, though sometimes in a slightly modified form.

In both classical and intuitionistic logic, F serves as negation. In general, the weak negation F may behave nonclassically. In particular, the proof by contradiction, $(Fx \Rightarrow 0) \Rightarrow x$, which is $FFx \Rightarrow x$, hence Bx, is not generally valid since it fails in the ternary logic of Example 2.3. But indirect proofs, characterized by $(x \Rightarrow 0) \Rightarrow \ (x = 0)$, are universally valid by Theorem 3.1(iii).

5.1 Theorem.
(i) Every categorical (and hence every complete) context is classical.

(ii) A context is classical iff $x|x = x$ holds for all statements x.

Note that $\{0, 1\}$ is a classical context that is not categorical in the context logic of Theorem 4.4.

Proof. (i) If Γ is categorical, any statement x satisfies $x \in \overline{\Gamma}$ or $Fx \in \overline{\Gamma}$. In both cases, $(FFx = x) \in \overline{\Gamma}$ by Theorem 4.4.

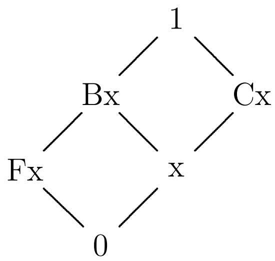
(ii) By F8, we have $x \Rightarrow Cx$, so that Bx, i.e., $Cx \Rightarrow x$, is equivalent to $Cx = x$, hence by Theorem 4.4, to $x|x = x$. □

Thus, for classical logic, and only then, $|$ serves as disjunction (strong or). The definition

$$x \sqcup y := F(Fx \wedge Fy) \tag{14}$$

also reduces in classical logic to disjunction (strong or). It is not difficult to prove that, in general, $x|y \Rightarrow x \sqcup y$. In the ternary logic of Example 2.3, $0|b = b$ but $0 \sqcup b = 1$, hence the disjunction $\sqcup$ is even weaker than $|$.

Simple intuitionistic context logics can be characterized algebraically via so-called Heyting algebras; something similar is possible for general context logics via bounded implicative semilattices; see the companion paper NEUMAIER [30]. Note that the free Heyting algebra in a single variable (the so-called Rieger-Nishimura lattice (NISHIMURA [32, 34]) is infinite. In contrast,Theorem 4.4 amounts to saying that the free context logic in a single variable comes from the Heyting algebra derived from the distributive lattice with the following Hasse diagram:



(We have $u \leq v$ iff either $u = v$ or u is joined in the Hasse diagram to v by an upwards going path.) Thus context logic is simpler than intuitionistic logic.

5.2 Theorem. *(*FRINK *[9])*

$$C\Sigma = F\Sigma = \Sigma_B$$

where $C\Sigma := \{Cx|x \in \Sigma\}$, $F\Sigma := \{Fx|x \in \Sigma\}$, $\Sigma_B := \{x \in \Sigma \mid Bx \in \overline{\emptyset}\}$, *is a Boolean algebra consisting of consistent elements only.*

Proof. By F11, $Fx = CFx$ for every $x \in \Sigma$ therefore $F\Sigma \subseteq C\Sigma$. By definition, $Cx = FFx$ for every $x \in \Sigma$ therefore $C\Sigma \subseteq F\Sigma$. Thus $C\Sigma = F\Sigma$. By F8, $x \Rightarrow Cx$ and this combined with $Bx \in \overline{\emptyset}$ i.e. $Cx \Rightarrow x$ gives $x = Cx$ for every $x \in \Sigma$ therefore $\Sigma_B \subseteq C\Sigma$. Conversely, by F12 $CCx = Cx$ therefore $CCx \Rightarrow Cx$ becomes $Cx \Rightarrow Cx$, obviously true. Thus also $C\Sigma \subseteq \Sigma_B$, which establishes $C\Sigma = \Sigma_B$. By F11, F12, F14 and F15. □

Relate this to GLIVENKO [14]; cf. FRINK [9]. Summarize general substitution results from the companion paper NEUMAIER [29].

The following result was proved by GÖDEL [16] in an intuitionistic setting; but the strong or is not needed.

5.3 Theorem. *In any context logic, the definition of implication $\to$ and or $\vee$ by*

$$x \to y \;\equiv\; F(x \wedge Fy), \quad x \sqcup y \;\equiv\; F(Fx \wedge Fy)$$

define a Boolean logic.

Proof. By Frink's formulas, Gödel's definitions are easily seen to be equivalent to

$$(x \to y) = Cx \Rightarrow Cy, \quad x \sqcup y = Cx|Cy,$$

and the equality relation $\sim$ defined by $x \sim y$ iff $x \to y$ and $y \to x$ reduces to $x \sim y$ iff $Cx = Cy$. Thus we simply get the Boolean quotient logic, which also is a sublogic. (Note that $B\Sigma$ is a filter, so that the quotient is well-defined.)

$(Cx \Rightarrow Cy) = (Cx \Rightarrow CCy) = C(x \Rightarrow Cy) = CF(x \wedge Cy) = F(x \wedge Cy) = x \to y.$

$Cx|Cy = (FCx \Rightarrow Cy) \wedge (FCy \Rightarrow Cx) = (Fx \Rightarrow Cy) \wedge (Fy \wedge Cx) = F(Fx \wedge Cy)$ if $F(Fx \wedge Fy) = (Fx \Rightarrow Cy)$. The forward implication for this follwos since Fx implies $F(Fx \wedge Fy) = FFy = Cy$, hence $Fx \wedge F(Fx \wedge Fy) \Rightarrow Cy$, and the reverse implication holds since $Fx \Rightarrow Cy$ implies $Fx \wedge Fy \Rightarrow Cy \wedge Fy = 0$, hence $F(Fx \wedge Fy)$. □

Declaring consistent statements true, i.e., assuming $Cx \Rightarrow x$, simplifies the logic a lot since it makes it classical, and many more rules hold. In a classical logic, the formula $e(0) \wedge e(1) \Rightarrow e(x)$ can be proved to be a law for every logical expression $e(x)$; see the companion paper NEUMAIER [29]. This implies that formulas such as Bx (the law justifying proof by contradiction) or $FFx = x$ (the double negation law) are laws in classical logic. In this sense, classical logic is a completion of the intuitionistic logic, obtained by declaring all consistent statements to be true.

6 Predicate calculus as a context logic

In this section we show that first-order predicate calculus with Tarski denotational semantics and closure under entailment is a context logic. A language for first-order predicate calculus is generally assumed to contain finite formulas built out of countably many variables, a finite number of constants and a finite number of connectives that behave truth-functionally from a semantic point of view. For further details, the reader is referred to the excellent monograph by NIENHUYS-CHENG & DE WOLF [31]; however, for our purposes it is sufficient to assume that the language is built using variables only and the two connectives relevant to context logic, conjunction and equality.

6.1 Theorem. *Let $\langle \mathcal{L}, \wedge, =, \models \rangle$ be a first-order predicate language with connectives $\wedge$ and $=$ interpreted as the binary Boolean "and" and "equal" respectively; and where $\models$ is defined* *in the usual way as logical consequence based on satisfaction in models. Then $\mathcal{L}$, equipped with the closure operator defined by*

$$x \in \overline{\Gamma} \quad \textit{iff} \quad \Gamma \models x,$$

is a context logic.

Proof. In the following, syntactical operators $\wedge$, $=$ and $\Rightarrow$ bind stronger than the semantical relations $\models$ and $\in$. We need to demonstrate that the context logic reflection laws hold in $\langle \mathcal{L}, \wedge, =, \models \rangle$, that is for all contexts $\Gamma \subseteq \mathcal{L}$:

$$0 \in \overline{\Gamma} \quad \text{iff} \quad \overline{\Gamma} = \mathcal{L}, \quad \textbf{(false reflection)} \tag{15}$$

$$x \wedge y \in \overline{\Gamma} \quad \text{iff} \quad x, y \in \overline{\Gamma}, \quad \textbf{(and reflection)} \tag{16}$$

$$(x = y) \in \overline{\Gamma} \quad \text{iff} \quad \overline{\Gamma \cup \{x\}} = \overline{\Gamma \cup \{y\}}, \quad \textbf{(equal reflection)} \tag{17}$$

We use the definition of the closure operator in $\langle \mathcal{L}, \wedge, =, \models \rangle$ to translate the three reflection laws to the following form:

$$\Gamma \models 0 \quad \text{iff} \quad \Gamma \models C \ (\forall C \in \mathcal{L}), \quad \textbf{(false reflection)} \tag{18}$$

$$\Gamma \models x \wedge y \quad \text{iff} \quad \Gamma \models x \text{ and } \Gamma \models y, \quad \textbf{(and reflection)} \tag{19}$$

$$\Gamma \models (x = y) \quad \text{iff} \quad \Gamma \cup \{x\} \models C \Leftrightarrow \Gamma \cup \{y\} \models C \ (\forall C \in \mathcal{L}), \quad \textbf{(equal reflection)} \tag{20}$$

'False reflection' states that a set of formulas has no model if and only if its models satisfy every formula. The necessity is trivial, while the sufficiency obtains by instantiating C to 0.

'And reflection' follows from the denotational semantics of $x \wedge y$, whose models are the elements of the intersection of the sets of models of x and y. Clearly, a model of Γ will be in this intersection if and only if it is in either set.

'Equal reflection' follows from the Deduction Theorem for first-order predicate calculus which allows us to rewrite 'equal reflection' as:

$$\Gamma \models (x = y) \quad \text{iff} \quad \Gamma \models (x \Rightarrow C) \Leftrightarrow \Gamma \models (y \Rightarrow C) \ (\forall C \in \mathcal{L}), \tag{21}$$

Instantiating $C = x$ and then $C = y$ on the right side, one obtains $\Gamma \cup \{x\} \models y$ and $\Gamma \cup \{y\} \models x$, and from the Deduction Theorem it follows that $\Gamma \models (x \Rightarrow y)$ and $\Gamma \models (y \Rightarrow x)$. □

7 Natural deduction and context logic

In this section we show how context logic accommodates natural deduction, the traditional method for handling context changes in logical reasoning. A good introduction to natural deduction approaches can be found in Pelletier [41, 42]. Natural deduction rules of inference are of two fundamental types, **introduction rules** and **elimination rules**. The following table gives the translation from the language used to classify inference rules in AI formalisations of context (McCarthy [25, 26, 27]) to the terminology used in natural deduction and, finally, context logic:

McCarthy	Natural Deduction	Context Logic
Discharge Rule	Introduction Rule	Downward Reflection
Importation Rule	Elimination Rule	Upward Reflection

The following is a dictionary showing how the main inference rules in natural deduction with contexts (Benzmüller [1]) generalise to context logic. Asserted contexts Γ in the form of finite sequences generalise to arbitrary contexts Γ in the form of sets. Finite deducibility of a formula A from a context Γ, denoted $\Gamma \vdash A$ in natural deduction with contexts, generalises to the requirement $A \in \overline{\Gamma}$ in context logic. In our natural deduction rules, we use the conventional notation $\top$ for true and $\bot$ for false.

The **and introduction** rule from natural deduction

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}(\wedge I) \tag{22}$$

and the left and right **and elimination** rules from natural deduction

$$\frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A}(\wedge E_l) \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B}(\wedge E_r) \tag{23}$$

combine to read in context logic as **and reflection**,

$$A \wedge B \in \overline{\Gamma} \quad \text{iff} \quad A, B \in \overline{\Gamma}. \tag{24}$$

The **imply introduction** rule from natural deduction

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash (A \Rightarrow B)}(\Rightarrow I) \tag{25}$$

and the **imply elimination** rule from natural deduction

$$\frac{\Gamma \vdash (A \Rightarrow B)}{\Gamma, A \vdash B}(\Rightarrow E) \tag{26}$$

combine to read in context logic as **imply reflection**,

$$(A \Rightarrow B) \in \overline{\Gamma} \quad \text{iff} \quad B \in \overline{\Gamma \cup \{A\}}. \tag{27}$$

The **false introduction** and **false elimination** rule from natural deduction

$$\frac{\Gamma \vdash C}{\Gamma \vdash \bot}(\bot I) \qquad \frac{\Gamma \vdash \bot}{\Gamma \vdash C}(\bot E) \tag{28}$$

combine to read in context logic as **false reflection**,

$$0 \in \overline{\Gamma} \quad \text{iff} \quad \overline{\Gamma} = \Sigma. \tag{29}$$

By comparison with the exposition in Section 2, we see that the validity of all natural deduction rules discussed so far constitutes precisely the condition for a logic described by natural deduction rules to be a context logic.

Some further natural deduction rules generalize directly to arbitrary context logics and are always valid there, as one easily verifies. The **true introduction** rule from natural deduction

$$\frac{}{\Gamma \vdash \top}(\top I) \tag{30}$$

reads in context logic as **true reflection**,

$$1 \in \overline{\Gamma}. \tag{31}$$

The **negation introduction** rule from natural deduction

$$\frac{\Gamma, A \vdash \bot}{\Gamma \vdash \neg A}(\neg I) \tag{32}$$

reads in context logic as **downward negation reflection**,

$$0 \in \overline{\Gamma \cup \{A\}} \quad \text{implies} \quad FA \in \overline{\Gamma}, \tag{33}$$

The **negation elimination** rule from natural deduction

$$\frac{\Gamma \vdash A \quad \Gamma \vdash \neg A}{\Gamma \vdash \bot}(\neg E) \tag{34}$$

reads in context logic as the **law of noncontradiction**,

$$A, FA \in \overline{\Gamma} \quad \text{implies} \quad 0 \in \overline{\Gamma}. \tag{35}$$

The **hypothesis introduction** rule from natural deduction

$$\frac{}{\Gamma, A, \Delta \vdash A}(HI) \tag{36}$$

reads in context logic as asserting a statement in any context that contains it,

$$A \in \overline{\Gamma \cup \{A\}}. \tag{37}$$

In classical logic, we furthermore have a **double negation elimination** rule

$$\frac{\Gamma \vdash \neg\neg A}{\Gamma \vdash A}(\neg\neg E) \tag{38}$$

which translates to context logic as:

$$FFA \in \overline{\Gamma} \quad \text{implies} \quad A \in \overline{\Gamma} \tag{39}$$

Relatedly, we have in classical logic the **proof by contradiction** rule

$$\frac{\Gamma, \neg A \vdash \bot}{\Gamma \vdash A}(\bot_c) \tag{40}$$

which translates to context logic as:

$$0 \in \overline{\Gamma \cup \{FA\}} \quad \text{implies} \quad A \in \overline{\Gamma} \tag{41}$$

Further natural deduction rules require the existence of additional structure in the context logic. If a binary operation $\vee$ (strong **or**) is defined on the set of statements, the left and right **or introduction** rules from natural deduction

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}(\vee I_l) \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}(\vee I_r) \tag{42}$$

combine to read in context logic as **downward or reflection**,

$$A \in \overline{\Gamma} \;\text{ or }\; B \in \overline{\Gamma} \quad \text{implies} \quad A \vee B \in \overline{\Gamma}. \tag{43}$$

The right **or elimination** rule from natural deduction

$$\frac{\Gamma \vdash A \vee B \quad \Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma \vdash C}(\vee E_r) \tag{44}$$

reads in context logic as a form of **upward or reflection**,

$$A \vee B \in \overline{\Gamma}, \;\; C \in \overline{\Gamma \cup \{A\}} \,\cap\, \overline{\Gamma \cup \{B\}} \quad \text{imply} \quad C \in \overline{\Gamma}, \tag{45}$$

which is implied by the converse of (43),

$$A \vee B \in \overline{\Gamma} \quad \text{implies} \quad A \in \overline{\Gamma} \;\text{ or }\; B \in \overline{\Gamma}. \tag{46}$$

Thus or introduction and or elimination together are implied by **strong or reflection**,

$$A \vee B \in \overline{\Gamma} \quad \text{iff} \quad A \in \overline{\Gamma} \;\text{ or }\; B \in \overline{\Gamma}. \tag{47}$$

Another class of natural deduction rules, namely the quantifier introduction and elimination rules require that the set of statements has a syntax in which variable substitution and quantification is possible according to standard practice. In the following, t is a term in the syntax of the set of statements (i.e. in the term algebra) and a^* is meant as a variable that is new in the context; the slash notation x/t or x/a^* indicates that the term t or the new-in-context variable a^*, respectively, are substituted for the variable x. The **universal quantifier introduction** rule and the **universal quantifier elimination** rule from natural deduction

$$\frac{\Gamma \vdash A(x/a^*)}{\Gamma \vdash (\forall x)A(x)}(\forall I) \qquad \frac{\Gamma \vdash (\forall x)A(x)}{\Gamma \vdash A(x/t)}(\forall E) \tag{48}$$

combine to read in context logic as **universal quantifier reflection**,

$$(A(x) \in \overline{\Gamma} \quad \text{for all } x) \quad \text{iff} \quad (\forall x)A(x) \in \overline{\Gamma}. \tag{49}$$

The **existential quantifier introduction** rule from natural deduction

$$\frac{\Gamma \vdash A(x/t)}{\Gamma \vdash (\exists x)A(x)}(\exists I) \tag{50}$$

reads in context logic as **downward existential quantifier reflection**,

$$(A(x) \in \overline{\Gamma} \quad \text{for some } x) \quad \text{implies} \quad (\exists x)A(x) \in \overline{\Gamma}. \tag{51}$$

The **existential quantifier elimination** rule from natural deduction

$$\frac{\Gamma \vdash (\exists x)A(x) \quad \Gamma, A(x/a*) \vdash C}{\Gamma \vdash C}(\exists E) \tag{52}$$

reads in context logic as a form of **upward existential quantifier reflection**,

$$(\mathbf{C} \in \overline{\Gamma \cup \{A(x)\}} \quad \text{for all } x), \quad (\exists x)A(x) \in \overline{\Gamma} \quad \text{imply} \quad C \in \overline{\Gamma}, \tag{53}$$

which is implied by the converse of (51),

$$(\exists x)A(x) \in \overline{\Gamma} \quad \text{implies} \quad (A(x) \in \overline{\Gamma} \quad \text{for some } x). \tag{54}$$

Thus existential quantifier introduction and existential quantifier elimination together are implied by **existential quantifier reflection**,

$$(\exists x)A(x) \in \overline{\Gamma} \quad \text{iff} \quad (A(x) \in \overline{\Gamma} \quad \text{for some } x). \tag{55}$$

We see that by abstracting from the syntactical and algorithmic content of natural deduction, the context logic formulation of the rules becomes more concise.

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