

Causal groups and local fields

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Abstract.

Whereas quantum mechanics quickly matured into a meaningful theory with solid mathematical foundations, the consolidation of quantum field theory took several decades and, as a matter of fact, has not yet come to a fully satisfactory end.

Buchholz & Fredenhagen, 2020 [12, p.1]

Beginning with the birth of modern quantum mechanics (1925), perturbative quantum mechanics and perturbative quantum field theory was **applied Lie algebras**. Beginning with WEYL [42] (1928), nonperturbative quantum mechanics is widely understood to be mostly **applied Lie groups**. But to regard nonperturbative quantum field theory as applied Lie groups is a much less travelled path, discovered by BOGOLIUBOV & SHIRKOV [4] (1955) in their causal S-matrix theory for nonperturbative relativistic quantum field theory, though the latter was later used primarily for causal perturbation theory. Independently, GOLDIN [19] (1971) introduced the group approach for the study of the current algebras arising in nonrelativistic quantum field theory.

In the present paper, this nonperturbative path will be enriched and better paved by introducing the concept of causal groups. It provides solid infrastructure for handling relativistic quantum field theory, and might be the path leading to its mathematically fully satisfactory consolidation.

Special causal groups defined by certain generators and relations arose in the recent (2020+) work by BUCHHOLZ & FREDENHAGEN [12, 13, 14] and BRUNETTI et al. [8, 9] as ingredients for the construction of dynamical C^* -algebras for quantum field theory over Minkowski spaces. The present generalization of their approach to arbitrary causal groups frees the causal S-matrix setting from its S-matrix connotations and extends it to arbitrary causal spaces.

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1 Causal structure

This section discusses causality in a general setting, covering discrete and continuous time, nonrelativistic and relativistic models of spacetime. We assume that the reader is familiar with the causal structure of Minkowski spacetime.

1.1 Causal spaces

As the survey in GARCÍA-PARRADO & SENOVILLA [17] shows, there are many notions of causal structure. All these exclude Galilei spacetime, which plays an important role in physics, hence none of them is general enough. Our concept of a causal space both covers the physical notions of causality that assume a manifold structure and generalizes the causal spaces of KRONHEIMER & PENROSE [18, 24] (our hyperbolic causal spaces) and the causal sets of BOMBELLI, LEE, MEYER & SORKIN [6] (our discrete hyperbolic causal spaces). One advantage of this generality is that one can get motivation and intuition from the consideration of much simpler toy problems.

A **causal space** is a Hausdorff space $\mathbb{M}$, whose elements are called **points**, with two transitive relations $\leq$ and $<$ on $\mathbb{M}$ such that there is a continuous **precedence function** $\tau : \mathbb{M} \times \mathbb{M} \rightarrow \mathbb{R}$ with

$$\begin{aligned}\tau(x, x) &= 0 \quad \text{for } x \in \mathbb{M}, \\ x \leq y &\Leftrightarrow \tau(x, y) \leq 0 \leq \tau(y, x). \\ x < y &\Leftrightarrow \tau(x, y) < 0 < \tau(y, x).\end{aligned}$$

The precedence function τ is not unique; the composition $\kappa \circ \tau$ with any strictly increasing continuous function $\kappa : \mathbb{R} \rightarrow \mathbb{R}$ with $\kappa(0) = 0$, is again a precedence function. Due to the properties of the precedence function, $\leq$ is a preorder (reflexive and transitive), the **causal preorder** of $\mathbb{M}$, and $<$ is a strict partial order (irreflexive, asymmetric, and transitive), the **strict causal order** of $\mathbb{M}$. Moreover,

$$\begin{aligned}x < y &\Rightarrow x \leq y, \\ x < y \leq z &\Rightarrow x < z, \\ x \leq y < z &\Rightarrow x < z, \\ x_\ell < y_\ell \ (\ell = 1, 2, 3 \dots) &\Rightarrow \lim_{\ell \rightarrow \infty} x_\ell \leq \lim_{\ell \rightarrow \infty} y_\ell\end{aligned}$$

if both limits exist. This implies that the **open intervals**

$$]x, y[:= \{z \in \mathbb{M} \mid x < z < y\}$$

are open sets and the **closed intervals**

$$[x, y] := \{z \in \mathbb{M} \mid x \leq z \leq y\}$$

are closed sets. As usual we write $x \leq y$ also as $y \geq x$ and $x < y$ also as $y > x$. The inequality $x < y$ is interpreted informally as the

strict law of causality: $x < y$ iff a change of forces (*cause*) at x can possibly influence a response (*effect*) at y .

1.2 Past, present, and future

Let S be a subset of the causal space $\mathbb{M}$. We write $\overline{S}$ for the closure of S and ∂S for its boundary. The **past** of S is the open set

$$\text{Past } S := \{y \in \mathbb{M} \mid y < x \text{ for some } x \in S\}, \quad (1)$$

and the **closed past** of S is the closed set

$$\text{cPast } S := \{y \in \mathbb{M} \mid y \leq x \text{ for some } x \in \overline{S}\}.$$

The **present** of S is the closed set

$$\text{Present } S := \text{cPast } S \setminus \text{Past } S.$$

The **future** of S is the open set

$$\text{Future } S := \{y \in \mathbb{M} \mid y > x \text{ for some } x \in S\}, \quad (2)$$

and the **closed future** of S is the closed set

$$\text{cFuture } S := \{y \in \mathbb{M} \mid y \geq x \text{ for some } x \in \overline{S}\}.$$

We have

$$S \subseteq \text{cPast } S = \text{Past } S \cup \text{Present } S, \quad S \cup \text{Future } S \subseteq \text{cFuture } S.$$

A subset S of $\mathbb{M}$ is called **convex** if

$$x, z \in S, \quad x \leq y \leq z \quad \Rightarrow \quad y \in S.$$

The **past** (or **past cone**), **closed past**, **present** (or **null cone**), **future** (or **future cone**), and **closed future** of a point $x \in \mathbb{M}$ are defined as

$$\text{Past}\{x\} = \{y \in \mathbb{M} \mid y < x\}, \quad \text{cPast}\{x\} = \{y \in \mathbb{M} \mid y \leq x\},$$

$$\text{Present}\{x\} = \text{cPast}\{x\} \setminus \text{Past}\{x\},$$

$$\text{Future}\{x\} = \{y \in \mathbb{M} \mid y > x\}, \quad \text{cFuture}\{x\} = \{y \in \mathbb{M} \mid y \geq x\}.$$

respectively. All these are convex sets. $x < y$ and $y > x$ say that x is in the past cone of y , equivalently that y is in the future cone of x . The open intervals $]x, y[$ are the intersections of $\text{Future}\{x\}$ and $\text{Past}\{y\}$. We have

$$x \in \text{Present}\{x\}, \quad \text{cPast}\{x\} \cap \text{cFuture}\{x\} = \{x\},$$

Informally, the present of x is the set of points that become the past once x moves into its future.

We say that $x \in \mathbb{M}$ is **prior** to $y \in \mathbb{M}$ and write $x \wedge \vee y$ if $x \wedge \vee y$ iff the closed past of x is disjoint from the closed future of y ; the relation symbol $\wedge \vee$ is suggestive of this property. We have $x \wedge \vee y$ iff $x \geq y$ does not hold. In general, being prior is neither transitive nor symmetric nor antisymmetric. (This is why we do not use the more traditional notation $x \lesssim y$ for $x \wedge \vee y$ introduced by BOGOLJUBOV & SCHIRKOW [4, p.468].) We call $x, y \in \mathbb{M}$ **causally unrelated** and write $x \times y$ if $x \wedge \vee y$ and $y \wedge \vee x$.

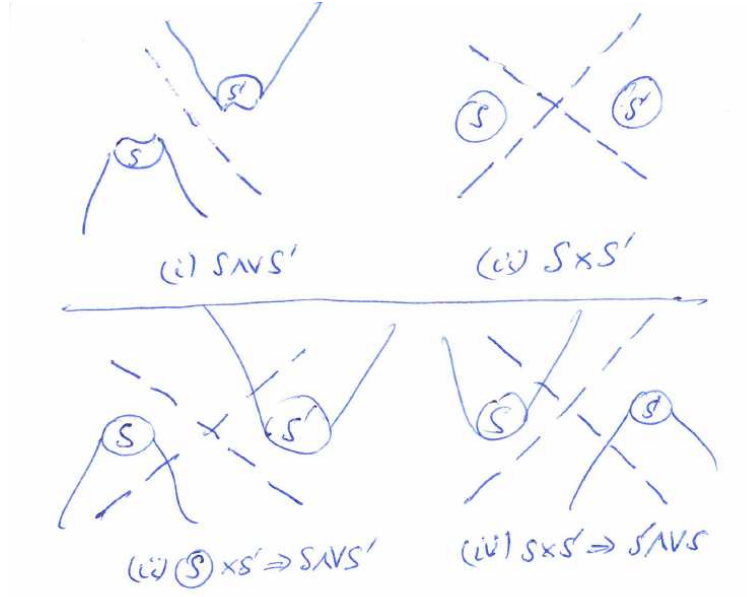


Figure 1: Causal prior and independence relations

If S, S' are subsets of $\mathbb{M}$ we say that S is **prior** to S' and write $S \wedge \vee S'$ if the closed past of S is disjoint from the closed future of S' . We call $S, S' \subseteq \mathbb{M}$ **causally unrelated** and write $S \times S'$ if $S \wedge \vee S'$ and $S' \wedge \vee S$. (There is no danger to misinterpret $S \wedge \vee S'$ as a Cartesian product since the latter is a set, not a statement.)

We use the parenthesis-saving convention that, in formulas, $<, >, \leq, \geq, \wedge$, and $\times$ bind weaker than all except logical operations (which bind weakest).

Associated with every causal space $\mathbb{M}$ is a **dual** causal space $\mathbb{M}_\downarrow$ with the same set of points as $\mathbb{M}$ but whose causal order $<_\downarrow$ is defined by

$$x <_\downarrow y \quad \Leftrightarrow \quad y < x.$$

Thus past and future are interchanged. In general, the present is not causal inversion invariant. The prior relation $\wedge$ is antidual in time, while causal independence is selfdual.

1.3 Classes of causal spaces

We call a causal space $\mathbb{M}$

- **linear** if the causal order is a strict linear order, i.e., for all $x, y \in \mathbb{M}$, exactly one of the relations $x = y$, $x < y$, and $x > y$ holds.
- **elliptic** if $x \leq y$ implies $x = y$ (so that $x < y$ never holds),
- **parabolic** if there is a **parabolic time map**, a nonconstant function $t : \mathbb{M} \rightarrow \mathbb{R}$ such that $\tau(x, y) := t(x) - t(y)$ is a precedence function,
- **hyperbolic** if $\leq$ is antisymmetric and hence a partial order,
- **locally compact** if every interval in $\mathbb{M}$ is a compact set,
- **locally finite** if every interval in $\mathbb{M}$ is a finite set,
- **discrete** if it is countable, locally finite, and its topology is discrete.
- **finite** if $\mathbb{M}$ is a finite set,

For an elliptic causal space, two points are causally unrelated iff they are distinct, and the present of any point is the whole spacetime. For a parabolic causal space, two points are causally unrelated iff they have the same time, and the present of a point x consists of all points with the same time as x .

In a parabolic causal space, the past, present, and future of a point $x \in \mathbb{M}$ form a partition of $\mathbb{M}$. Our notion of present agrees with tradition only if the causal set is parabolic, In all other cases, the present of x is in our terminology *not* – as tradition wants to have it – the set of points $y \in \mathbb{M}$ with time coordinate $xy_0 = x_0$, a set without any physical meaning. In Minkowski space, the present (null cone) of x is the boundary of its past cone, often called the **past light cone** of x .

A **timeline** is a linear causal space $\mathbb{M}$; then any $x \in \mathbb{M}$ is called a **time**. Simple examples of parabolic timelines (with the identity as time map) are the finite timeline $\mathbb{M} = [1 : T] := \{1, \dots, T\}$, the discrete timeline $\mathbb{M} = \mathbb{Z}$, and the **real timeline** $\mathbb{M} = \mathbb{R}$, with their natural linear orders.

A discrete (resp. finite) model of parabolic spacetime is the product $\mathbb{Z} \times S$ (resp. $[1 : T] \times S$), where S is a finite set, and $(t, s) < (t', s')$ iff $t < t'$ and no restrictions on s, s' . In contrast to [6], we do not attach any fundamental physical significance to discrete causal spaces, and use them only as toy models for gaining insight into the causal structure. They are also useful for approximate numerical computations (**discrete mechanics**, see, e.g., MARS DEN & WEST [26]), but this aspect is not addressed here.

1.4 Causal symmetries

A **causality preserving map** of $\mathbb{M}$ is a continuous map $h : \mathbb{M} \rightarrow \mathbb{M}$ satisfying

$$x < y \quad \Rightarrow \quad h(x) < h(y).$$

By continuity of the precedence function, we then also have

$$x \leq y \quad \Rightarrow \quad h(x) \leq h(y),$$

$$x \wedge y \quad \Rightarrow \quad h(x) \wedge h(y),$$

$$x \times y \quad \Rightarrow \quad h(x) \times h(y).$$

A **causal symmetry** of $\mathbb{M}$ is an invertible causality preserving map from $\mathbb{M}$ to itself whose inverse is also causality preserving. The causal symmetries of $\mathbb{M}$ form a group, the **causal symmetry group** $\text{cSym}(\mathbb{M})$ of $\mathbb{M}$.

A **causality reversing map** of $\mathbb{M}$ is a continuous map $r : \mathbb{M} \rightarrow \mathbb{M}$ satisfying

$$x < y \quad \Rightarrow \quad r(y) < r(x)$$

and hence

$$x \leq y \quad \Rightarrow \quad r(y) \leq r(x),$$

$$x \wedge y \quad \Rightarrow \quad r(y) \wedge r(x),$$

$$x \times y \quad \Rightarrow \quad r(y) \times r(x), \quad r(x) \times r(y).$$

An **anticausal symmetry** of $\mathbb{M}$ is an invertible causality reversing map from $\mathbb{M}$ to itself whose inverse is also causality reversing. A **causal inversion map** is an anticausal symmetry T with $T^2 = 1$. A causal space is **selfdual** if it has an antianticausal symmetry.

A **symmetry** of a causal space $\mathbb{M}$ is a causal or anticausal symmetry of $\mathbb{M}$. The symmetries of $\mathbb{M}$ also form a group, the **extended causal symmetry group** $\text{cSym}_{\pm}(\mathbb{M})$ of $\mathbb{M}$. In an elliptic causal space, every bijection is a causal and anticausal symmetry. On the other hand, the majority of causal spaces has no symmetry apart from the identity.

A causal space $\mathbb{M}$ is called **translation invariant** if it is an additive abelian group and $x < y$ iff $y - x > 0$. In this case, each **translation** $\tau_z : \mathbb{M} \rightarrow \mathbb{M}$ by $z \in \mathbb{M}$, defined by

$$\tau_z(x) := x + z \quad \text{for } x \in \mathbb{M},$$

is a causal symmetry of $\mathbb{M}$. Translation invariant causal spaces are selfdual, with causal inversion maps $T_z : x \rightarrow z - x$ for any $z \in \mathbb{M}$.

In applications to physics, a **spacetime** is a causal space $\mathbb{M}$, usually with a manifold structure. Of main interest in physics are the translation invariant causal spaces $\mathbb{M} = \mathbb{R}^{d+1}$ (**Euclidean spacetime**, elliptic, modeling equilibrium physics), $\mathbb{M} = \mathbb{R} \times \mathbb{R}^d$ (**Galilei spacetime**, parabolic, modeling nonrelativistic physics), and $\mathbb{M} = \mathbb{R}^{1,d}$ (**Minkowski spacetime**, hyperbolic, modeling relativistic physics), each with its common special case $d = 0$, where $\mathbb{M}$ is the real timeline $\mathbb{R}$. In Galilei and Minkowski spacetime, points are denoted by $x = \begin{pmatrix} x_0 \\ \mathbf{x} \end{pmatrix}$ with $x_0 \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^d$, and the causal order is defined in Galilei spacetime by the precedence function

$$\tau(x, y) := x_0 - y_0$$

corresponding to the parabolic time map $t(x) := x_0$, and in Minkowski spacetime by the precedence function

$$\tau(x, y) := x_0 - y_0 + |\mathbf{x} - \mathbf{y}|,$$

where $|\mathbf{x}| = \sqrt{\mathbf{x}^T \mathbf{x}}$.

For $d > 1$, the causal symmetry group of Minkowski space $\mathbb{R}^{1,d}$ is the group generated by translations, Lorentz boosts, and dilatations. For $d = 1$, the causal symmetry group is much larger, isomorphic to the product of two group of orientation preserving homeomorphisms of the real line (corresponding to the two lines forming the 2-dimensional light cone).

In general relativity and quantum gravity, spacetime is a curved smooth orientable Lorentzian manifold. But when gravitation is dynamical, spacetime is not a causal space; the causal structure then resides instead in the tangent spaces, which have a pointed Minkowski structure. In an external gravitational field that does not create closed timelike curves, this induces on the curved spacetime the structure of a hyperbolic causal space (KRONHEIMER & PENROSE [24]).

1.5 Cauchy structures

Except for parabolic causal spaces, time plays no role at all in causality issues. In place of the Galilean time concept one needs for an observer-independent description of motion in

more general causal spaces the concept of a **many-fingered time**, pioneered by TOMONAGA [40] and SCHWINGER [35] in their covariant perturbative quantization of quantum electrodynamics, and used in classical general relativity since KUCHAR [25].

A **Cauchy structure** on a causal set $\mathbb{M}$ is a collection $\mathcal{S}$ of subsets of $\mathbb{M}$ called **Cauchy slices** such that, for all compact subsets A of $\mathbb{M}$ and all closed subsets P and F of S with $P \wedge \vee F$, there are Cauchy slices Σ_- , Σ_0 , and Σ_+ satisfying

$$\Sigma_- \subseteq \text{Past } S, \quad S \subseteq \text{Past } \Sigma_+, \quad (3)$$

$$P \subseteq \text{cPast } \Sigma_0, \quad \Sigma_0 \subseteq \text{Past } F. \quad (4)$$

The Cauchy structure is **complete** if for every Cauchy slice Σ , each $x \in \mathbb{M}$ is in exactly one of the sets $\text{Past } \Sigma$, Σ , and $\text{Future } \Sigma$.

In a parabolic causal space, the collection $\mathcal{S}$ of **time slices**

$$\Sigma_\tau := \{x \in \mathbb{M} \mid t(x) = \tau\}$$

defines a Cauchy structure iff the parabolic time map is unbounded from below and from above. In Minkowski space, the collection $\mathcal{S}$ of Cauchy surfaces defines a Cauchy structure. In general, a Cauchy structure can exist only on causal spaces with infinitely many points; but this is by far from sufficient for their existence.

1.6 Causal lattices

A **causal lattice** is a discrete, translation invariant causal space. $\mathbb{Z}$ denotes the ring of integers. The 2-dimensional **light cone lattice** is the causal lattice Λ_2 with points $x = (x_+, x_-) \in \mathbb{Z} \times \mathbb{Z}$ and hyperbolic inner product

$$(x, y) := x_+ y_- + x_- y_+ \in \mathbb{Z} \quad \text{for } x, y \in \Lambda_2,$$

whose causal order is defined by the precedence function

$$\tau(x, y) := \max(x_+ - y_+, x_- - y_-).$$

Its extended causal symmetry group is an extension of the group of lattice translations by a group of order 8 with elements

$$\mathbf{x} \rightarrow (\alpha x_+, \beta x_-),$$

$$\mathbf{x} \rightarrow (\alpha x_-, \beta x_+),$$

where $\alpha, \beta \in \{-1, 1\}$.

The 2-dimensional light cone lattice Λ_2 has a Cauchy structure given by the sets

$$\Sigma = \{x^k \mid k \in \mathbb{Z}\},$$

where

$$x^{k+1} - x^k \in \{(1, 0), (0, -1)\} \quad \text{for } k \in \mathbb{Z}$$

and

$$K := \{k \in \mathbb{Z} \mid x^{k+1} - x^{k-1} = (1, -1)\}$$

[*** lattice with diagonal null directions ***]

Figure 2: The 2-dimensional light cone lattice

is unbounded from above and from below, is a complete Cauchy slice. Drawn in the plane, the points of Σ lie on a zigzag curve with linear pieces in the direction of the null axes $x_{\pm} = 0$ and infinitely many bends in both spatial directions; the bends are at the point x^k with $k \in K$.

For $d > 2$, the d -dimensional **light cone lattice** is the hyperbolic causal lattice Λ_d with points $x = (x_d, \mathbf{x}) \in \Lambda_2 \times \mathbb{Z}^{d-2}$ and inner product

$$(x, y) := (x_d, y_d) - \mathbf{x}^T \mathbf{y} \in \mathbb{Z} \quad \text{for } x, y \in \Lambda_d,$$

whose causal order is given by

$$x < y \quad :\Leftrightarrow \quad \max(x_+ - y_+, x_- - y_-) < 0 < (x - y, x - y),$$

$$x \leq y \quad :\Leftrightarrow \quad \max(x_+ - y_+, x_- - y_-) \leq 0 \leq (x - y, x - y).$$

Λ_d has an infinite group of symmetries that includes besides the lattice translations all reflections R_z in the infinitely many points $z \in \Lambda_d$ with $(z, z) = \pm 1, \pm 2$, defined by

$$R_z x := x - \frac{2(x, z)}{(z, z)} z \quad \text{for } x \in \Lambda_d.$$

Even more generally, all **hyperbolic lattices** (see, e.g., NIKULIN [33], BELOLIPETSKY [3]) are also causal. Many of these have a large group of symmetries (see, e.g., NIKULIN [34], BORCHERDS [7]).

2 The causal law

In this section we look at the manifestation of the causal law, first in linear response theory, then in a class of nonlinear discrete dynamical systems.

2.1 Causal function spaces and linear response theory

The law of causality takes its simplest mathematical (and physically very useful) form in **linear response theory**, which describes how a change of forces applied to an arbitrary physical system changes its response after sufficiently short times.

A **causal function space** is a Euclidean vector space $\mathbb{H} \subseteq L^2(\mathbb{M}, \mathbb{F})$ of continuous, with respect to some Borel measure square integrable functions¹ from a causal space $\mathbb{M}$ to a complex Euclidean space $\mathbb{F}$, with Hermitian inner product²

$$f^*g := \int dx f(x)^* g(x) \quad \text{for } f, g \in \mathbb{H},$$

where w^*w' denotes the Hermitian inner product of $w, w' \in \mathbb{F}$. We call $\mathbb{M}$ the **base** of $\mathbb{H}$.

If $\mathbb{M}$ is translation invariant an additive abelian group, the measure is assumed to be translation invariant. If $\mathbb{M}$ is discrete, the measure dx is assumed to be the counting measure, so that integration amounts to taking an unweighted sum over the points of $\mathbb{M}$.

We call $f \in \mathbb{H}$ **prior** to $g \in \mathbb{H}$ and write $g \wedge \vee f$ if the closed past of the support of g and the closed future of the support of f are disjoint. We call two functions $f, g \in \mathbb{H}$ **causally unrelated** and write $f \times g$ if $f \wedge \vee g$ and $g \wedge \vee f$. In an elliptic causal space, $f \wedge \vee g$ iff $f \times g$ iff $g \wedge \vee f$ iff f and g have disjoint support, but in general, $\wedge \vee$ is not symmetric.

The causal properties of $\mathbb{H}$ inherited from the causal space become visible not in $\mathbb{H}$ itself but in the algebra $\text{Lin } \mathbb{H}$ of everywhere defined linear operators on $\mathbb{H}$: In **linear response theory**, we interpret function values $f(x)$ of a function $f \in \mathbb{H}$ as a **generalized force** applied to a physical system at the point x and assume that the response $q(x) \in \mathbb{F}$ of the system at the point $y \geq x$ to the force depends linearly on f . Linearity means that $q = Gf$ for some linear operator $G \in \text{Lin } \mathbb{H}$. The **causal law** expresses the observed fact that a change in the response cannot appear before the force changes. In mathematical terms, if $f, h \in \mathbb{H}$ and $z \in \mathbb{M}$ are such that

$$f(x) = g(x) \quad \text{for all } x \leq z$$

then

$$Gf(x) = Gg(x) \quad \text{for all } x \leq z.$$

Causality in the sense defined above is different from the **arrow of time** introduced into physics by EDDINGTON [15, pp. 68–71] in terms of statistical irreversibility. The latter depends on the presence of randomness. The statistical arrow of time can be explained

¹If $\mathbb{M}$ is a manifold, $\mathbb{H}$ could also be a space of sections of a Hermitian vector bundle with base $\mathbb{M}$ and standard fiber $\mathbb{F}$, with straightforward changes.

²In realistic physical spacetimes, the Borel measure is the Lebesgue measure, and we use the corresponding notation for the general case.

(and is indeed explained by Eddington) in terms of the present notion of causality by the impossibility to prepare forces that would undo the effects – perceived by us as random – introduced by chaotic motion. In particular, as shown by spin echos (HAHN [21]), we can reverse a seemingly irreversible process precisely to the extent that we can prepare these forces. On the other hand, the statistical arrow of time does not help the least to explain (without circular reasoning) the causal law.

The **support** $\text{Supp } f$ of $f \in \mathbb{H}$ is the closure of the set of all $x \in \mathbb{M}$ with $f(x) \neq 0$. By linearity and continuity, the causal law can be expressed equivalently as

$$\text{Supp } f \subseteq \text{Future}(z) \quad \Rightarrow \quad \text{Supp } Gf \subseteq \text{Future}(z). \quad (5)$$

In terms of the prior relation, the causal law takes the equivalent weak form

$$g \wedge \vee f \quad \Rightarrow \quad g^* G f = 0. \quad (6)$$

This weak form makes sense in the more general case where the responses Gf belong to the typically larger antidual space $\mathbb{H}^\times$, so that $G \in \text{Lin}^\times \mathbb{H} := \text{Lin}(\mathbb{H}, \mathbb{H}^\times)$. (This is a way of saying that G is a bilinear form on $\mathbb{H}$.)

We call $G \in \text{Lin}^\times \mathbb{H}$ a **causal map** if the **causality condition** (6) holds for all $f, g \in \mathbb{H}$. This is necessary and sufficient for the causal behavior of the response. A system with a linear response $q = Gf$ to the force f is called **causal** if G is causal.

Causal maps generalize **lower block triangular matrices** to the operator level. Indeed, it is easy to see that on a finite timeline $[1 : n]$, with functions f regarded as vectors $f \in \mathbb{R}^n$ with components $f_x = f(x)$, the causal maps are given by multiplication with lower triangular matrices. Similarly, on a finite parabolic causal set, the causal maps are given by multiplication with lower block triangular matrices.

If $\mathbb{M}$ is Minkowski space and the measure used to define $\mathbb{H}$ is the Lebesgue measure then the linear map G whose kernel is the retarded Green's function of a linear hyperbolic differential equation

$$D\psi = 0$$

is an important example of a causal map with a direct physical meaning. From such a G we may form the metric

$$\Pi := \frac{1}{2\pi i} (G - G^*) = \delta(D).$$

This implies that

$$\langle f, g \rangle := f^* \Pi g$$

is the natural inner product on the space of solutions of $D\psi = 0$.

2.2 Discrete model dynamics

To motivate the concept of a causal group we consider a class of nonlinear dynamical models simple enough that all functional analytic difficulties are absent. Our causal space is the parabolic space $\mathbb{M} = \mathbb{Z} \times \mathbb{S}$ with points $x = (t, \mathbf{x})$, where the parabolic time map projects to the first coordinate. Our **fiber of field values** is a vector space $\mathbb{V}$ of complex-valued

functions on $\mathbb{S}$, corresponding to a complex scalar field. We define the causal function space $\mathbb{H}$ consisting of all sequences f with values in $C^\infty(\mathbb{V}, \mathbb{V})$, indexed by $t \in \mathbb{Z}$ with bounded temporal support, i.e., $f_t = 0$ for sufficiently large $|t|$. We call the elements $f \in \mathbb{H}$ **acceleration sequences**. The correspondence with fields is given by

$$q(t, \mathbf{x}) = q_t(\mathbf{x}), \quad F(t, \mathbf{x}) = f_t(\mathbf{x}) \quad \text{for } q_t \in \mathbb{V}, \mathbf{x} \in \mathbb{S}. \quad (7)$$

We consider the discrete dynamical system

$$q_{t+1} - 2q_t + q_{t-1} = f_t(q_t), \quad (8)$$

with **discrete time** t , **dynamical variables** $q_t \in \mathbb{V}$, and a given acceleration sequence $f \in \mathbb{H}$. The **phase space** associated with (8) consists of the

$$\mathbf{z}_t := \begin{pmatrix} q_t \\ q_{t-1} \end{pmatrix} \in \mathbb{V}^2.$$

The **causal inversion map** C interchanges the two blocks of $\mathbf{z}_t$,

$$C\mathbf{z}_t := \begin{pmatrix} q_{t-1} \\ q_t \end{pmatrix}.$$

Given $\mathbf{z}_t$ for some t , the dynamics (8) can be uniquely solved for all t by recursion in both time directions,

$$\begin{aligned} \mathbf{z}_{t+1} &= A_t(f)(\mathbf{z}_t) := \begin{pmatrix} f_t(q_t) + 2q_t - q_{t-1} \\ q_t \end{pmatrix}, \\ \mathbf{z}_t &= A_t(f)^{-1}(\mathbf{z}_{t+1}) := \begin{pmatrix} q_t \\ f_t(q_t) + 2q_t - q_{t+1} \end{pmatrix}, \end{aligned}$$

The nonlinear **time step maps** $A_t(f)$ defined by this are invertible,

$$A_t(f)^{-1} = CA_t(f)C,$$

which expresses causal inversion invariance of the equations of motion (8).

Since $f \in \mathbb{H}$, the acceleration f_t vanishes for large $|t|$. Therefore the asymptotic dynamics at large negative and large positive times is linear in t . Thus it is possible to discuss asymptotic **scattering**. We parameterize the linear asymptotic solutions for large negative time by $\begin{pmatrix} u \\ v \end{pmatrix}$, where

$$q_t = u - tv \quad \text{for } t < \text{Supp } f, \quad (9)$$

so that

$$\mathbf{z}_t = \begin{pmatrix} q_t \\ q_{t-1} \end{pmatrix} = \Omega_t \begin{pmatrix} u \\ v \end{pmatrix} := \begin{pmatrix} u - tv \\ u - (t-1)v \end{pmatrix}. \quad (10)$$

Similarly, we parameterize the linear asymptotic solutions for large positive time by $\begin{pmatrix} u' \\ v' \end{pmatrix}$, where

$$q_t = u' - tv' \quad \text{for } t > \text{Supp } f. \quad (11)$$

The **classical S-matrix** $S(f)$ is not a matrix but a nonlinear map of the asymptotic phase space to itself, the classical analogue of the quantum S-matrix in quantum mechanical scattering.

$S(f)$ maps the phase space at sufficiently large negative time to the phase space at sufficiently large positive time by

$$S(f) \begin{pmatrix} u \\ v \end{pmatrix} := \Omega_T^{-1} \mathbf{z}_T = \begin{pmatrix} u' \\ v' \end{pmatrix},$$

where $T > \text{Supp } f$ and $\mathbf{z}_T$ is obtained by solving the dynamics (8) started with (9).

2.1 Theorem. *The transformations*

$$U(f, g) := S(f)S(g)^{-1} \quad \text{for } f, g \in \mathbb{H} \quad (12)$$

*satisfy the **transition relation***

$$U(f, g)U(g, h) = U(f, h) \quad \text{for } f, g, h \in \mathbb{H} \quad (13)$$

*and the **causality relation***

$$U(f + h, g + h) = U(f, g) \quad \text{if } h \wedge \vee g - f. \quad (14)$$

Proof. (13) follows from

$$U(f, g)U(g, h) = S(f)S(g)^{-1}S(g)S(h)^{-1} = S(f)S(h)^{-1} = U(f, h).$$

(ii) The prior relation $h \wedge \vee g - f$ implies for a parabolic causal set that there is a time τ such that

$$h_t = 0 \quad \text{for } t > \tau, \quad \text{and} \quad g_t = f_t \quad \text{for } t \leq \tau. \quad (15)$$

Since the supports of f and g are bounded, there is a $T > 0$ such that the solutions are linear in t for $|t| \geq T$. Thus

$$S(f) = \Omega_{T+1}^{-1} A_T(f_T) \cdots A_{-T}(f_{-T}) \Omega_{-T} = A_+(f) A_-(f), \quad (16)$$

where

$$A_+(f) := \Omega_{T+1}^{-1} A_T(f_T) \cdots A_{\tau+1}(f_{\tau+1}), \quad A_-(f) := A_\tau(f_\tau) \cdots A_{-T}(f_{-T}) \Omega_{-T},$$

with factors in decreasing order of $t \in [-T, T]$. Because of (15) we have

$$A_+(f + h) = A_+(f), \quad A_+(g + h) = A_+(g),$$

$$A_-(f + h) = A_-(g + h), \quad A_-(f) = A_-(g),$$

hence

$$\begin{aligned} U(f + h, g + h) &= S(f + h)S(g + h)^{-1} \\ &= A_+(f + h)A_-(f + h)A_-(g + h)^{-1}A_+(g + h)^{-1} \\ &= A_+(f + h)A_+(g + h)^{-1} = A_+(f)A_+(g)^{-1} \\ &= A_+(f)A_-(f)A_-(g)^{-1}A_+(g)^{-1} = S(f)S(g)^{-1} = U(f, g). \end{aligned}$$

Thus (12) holds. □

A factorization like (16) of a (quantum) S-matrix was first used in a causal context by BOGOLJUBOV & SCHIRKOW [4, p.467].

2.3 Symplectic structure and quantization

Let $\mathbb{V}$ be a real Euclidean vector space with bilinear inner product $x^T y$. Linear operators $A \in \text{Lin } \mathbb{V}^2$ are written as 2×2 block matrices

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \quad (17)$$

whose components $A_{jk} \in \text{Lin } \mathbb{V}$ are the operators defined by

$$A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} A_{11}x_1 + A_{12}x_2 \\ A_{21}x_1 + A_{22}x_2 \end{pmatrix} \quad \text{for } x_1, x_2 \in \mathbb{V}.$$

Examples of interest are the block matrices

$$C := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad K := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

They satisfy the relations

$$C^2 = 1, \quad C^{-1} = C = C^T, \quad CJC = -J,$$

$$J^T = -J, \quad J^2 = -1, \quad J^3 = -J,$$

$$K^T = K, \quad KJK = 0, \quad KJ^2 = J^2K.$$

$A \in \text{Lin } \mathbb{V}^2$ is called **symplectic** if $A^T J A = J$. For example, the matrices

$$A(c) := cK - J = \begin{pmatrix} c & -1 \\ 1 & 0 \end{pmatrix} \quad \text{for } c \in \text{Lin } \mathbb{V}$$

are symplectic since

$$A(c)^T J A(c) = (cK + J)J(cK - J) = c^2 KJK + c(KJ^2 - J^2K) - J^3 = J.$$

The set of all symplectic matrices $A \in \text{Lin } \mathbb{V}^2$ is a group, the **symplectic group** $Sp(\mathbb{V}, \mathbb{R})$. A **canonical transformation** (or **symplectomorphism**) is a map $\mathbf{Q} \in \text{Diff}(\mathbb{V}^2)$ such that

$$\frac{\partial \mathbf{Q}(\mathbf{z})}{\partial \mathbf{z}} \in Sp(\mathbb{V}, \mathbb{R}) \quad \text{for } \mathbf{z} \in \mathbb{V}^2.$$

2.2 Proposition. *The canonical transformations form a group $\text{Can}(\mathbb{V}, \mathbb{R})$.*

Proof. If $\mathbf{Q}, \mathbf{R} \in \text{Can}(\mathbb{V}, \mathbb{R})$ then

$$\frac{\partial (\mathbf{QR})(\mathbf{z})}{\partial \mathbf{z}} = \frac{\partial}{\partial \mathbf{z}} \mathbf{Q}(\mathbf{R}(\mathbf{z})) = \frac{\partial \mathbf{Q}}{\partial \mathbf{z}}(\mathbf{R}(\mathbf{z})) \frac{\partial \mathbf{R}(\mathbf{z})}{\partial \mathbf{z}} \in Sp(\mathbb{V}, \mathbb{R}),$$

hence $\mathbf{QR} \in \text{Can}(\mathbb{V}, \mathbb{R})$. If $\mathbf{R} = \mathbf{Q}^{-1}$ then $(\mathbf{QR})(\mathbf{z}) = \mathbf{z}$, hence $\frac{\partial (\mathbf{QR})(\mathbf{z})}{\partial \mathbf{z}} = 1$, so that

$$\frac{\partial \mathbf{R}(\mathbf{z})}{\partial \mathbf{z}} = \left(\frac{\partial \mathbf{Q}}{\partial \mathbf{z}}(\mathbf{R}(\mathbf{z})) \right)^{-1} \in Sp(\mathbb{V}, \mathbb{R}).$$

Hence $\mathbf{Q}^{-1} = \mathbb{R} \in \text{Can}(\mathbb{V}, \mathbb{R})$. □

Since

$$A_t(f)'(\mathbf{z}_t) = \begin{pmatrix} f'(q_t) + 2 & -1 \\ 1 & 0 \end{pmatrix} \in Sp(\mathbb{V}, \mathbb{R}),$$

the time step maps $A_t(f)$ are canonical transformations. Since $A_t(f)'(\mathbf{z}_t)$ has determinant 1, the time step maps are also volume preserving.

$$\Omega_t = \begin{pmatrix} 1 & -t \\ 1 & 1-t \end{pmatrix}.$$

satisfies $\Omega_t^T J \Omega_t = J$, hence is symplectic, a linear canonical transformation. As a product of canonical transformations, $S(f)$ is a canonical transformation.

In order to obtain physically relevant unitary representations via geometric quantization we need to lift these canonical transformations to quantomorphisms; see SOURIAU [38].

A **quantomorphisms** is a map in a symplectic line bundle with local coordinates (q, p, s) that leaves the 1-form $\theta := p^T dq - ds$ invariant. Their general form maps (q, p, s) to (Q, P, S) with

$$Q = Q(q, p), \quad P = P(q, p), \quad S = s + \pi(q, p),$$

where

$$\frac{d\pi}{dp} = P \frac{dQ}{dp}, \quad \frac{d\pi}{dq} = P \frac{dQ}{dq} - p. \tag{18}$$

In our discrete dynamical situation,

$$Q(q, p) := 2q - p - V'(q)/m, \quad P(q, p) := q, \quad \pi(q, p) := q^2 - pq + V(q)/m - qV'(q)/m.$$

produces the required quantomorphism if the force has the form

$$f(q) = -\nabla V(q)$$

for some scalar potential $V(q)$. Thus the dynamics can be quantized by geometric quantization.

3 Causal groups

In this section we generalize Theorem 2.1 in terms of the general notion of a causal group.

3.1 Causal groups and quantum field theory

Let $\mathbb{G}$ be a group and $\mathbb{H}$ be a causal function space with base $\mathbb{M}$. Motivated by Theorem 2.1, we call a map $U : \mathbb{H} \times \mathbb{H} \rightarrow \mathbb{G}$ satisfying the transition relation (13),

$$U(f, g)U(g, g') = U(f, g') \quad \text{for } f, g, g' \in \mathbb{H},$$

a **transition map** on $\mathbb{G}$ with **carrier space** $\mathbb{H}$. We call a transition map U **causal** if the causality relation (14),

$$U(f + e, g + e) = U(f, g) \quad \text{if } e \wedge \nabla g - f,$$

holds for all $f, g, e \in \mathbb{H}$ such that e and $g - f$ have compact support. Informally, the causality relation (14) says that the same additive changes e of the *common* past of f and g in a localized region do not affect the transition behavior restricted to compact regions.

We call $\mathbb{G}$ a **causal group** over $\mathbb{M}$ with **carrier space** $\mathbb{H}$ if it has a causal transition map. In this case we call the subgroup generated by all $U(f, g)$ the **causal transition group** with transition map U . Although it is the latter that is relevant in the applications, it does not really matter if the causal group is far bigger since the applications always involve explicitly the transition maps and the causal group only serves to define these.

The same causal group may be causal with respect to multiple causal transition maps. For example, we may define

$$U_{Q,e}(f, g) := QU(f + e, g + e)Q^{-1} \quad (19)$$

with invertible $Q \in \mathbb{G}$ and arbitrary $e \in \mathbb{H}$, and find that each $U_{Q,e}$ is also a causal transition map. Thus a causal group may contain many causal transition maps and hence many causal transition groups.

From (13) we find for $g = g' = f$ that

$$U(f, f) = 1, \quad (20)$$

and then for $g' = f$

$$U(f, g)^{-1} = U(g, f). \quad (21)$$

Given an arbitrary map $S : \mathbb{H} \rightarrow \mathbb{G}$, the map $U : \mathbb{H} \times \mathbb{H} \rightarrow \mathbb{G}$ defined by (12) satisfies (13). Conversely, given (13), the map $S_e : \mathbb{H} \rightarrow \mathbb{G}$ defined for any fixed element $e \in \mathbb{H}$ by

$$S_e(f) := U(f, e) \quad (22)$$

determines U through (12) with $S = S_e$ since (13) implies

$$S_e(f)S_e(g)^{-1} = U(f, e)U(g, e)^{-1} = U(f, g).$$

Restricted to an arbitrary path $f : \mathbb{R} \rightarrow \mathbb{H}$, the transition operators $U(f(s), f(t))$ define a time-dependent flow on $\mathbb{G}$.

Causal groups over d -dimensional elliptic, parabolic, and hyperbolic spacetimes with $d \leq 4$ are relevant for Euclidean, nonrelativistic, and relativistic (classical and quantum) field theories, respectively. A very detailed survey of algebraic quantum field theory in Minkowski space was given by STREATER [39]. A survey of algebraic quantum fields on globally hyperbolic manifolds was given by HOLLANDS & WALD [22]. The traditional perturbative point of view on quantum field theory is expounded in many books, perhaps explained most thoroughly in WEINBERG [41]. For an introduction to quantum field theory from a mathematician's point of view, see ZEIDLER [43]; this book draws connections to many topics of contemporary mathematics, and points out basic unresolved mathematical issues in quantum field theory. These sources together provide complementary perspectives on the subject.

For the purposes of this paper, we define a **local quantum field theory** to be a unitary highest weight representation of a causal group on a Euclidean vector space $\mathbb{Q}$. (Its completion is the Hilbert space of **physical state vectors** in the folium of the representation in the sense of von Neumann algebras.)

The causality condition (14) arose historically for the case where the causal space is Minkowski spacetime, in the context of the causal S-matrix theory for quantum field theory by BOGOLIUBOV & SHIRKOV [5]. It was first stated by BOGOLIUBOV & SHIRKOV [5, §17.5] (1959) in the slightly vague verbal form

"If there are two functions $g''(y)$ and $g'(y)$ which coincide with each other for y^0 smaller than a certain t , then the product $S(g'')S^\dagger(g')$ must not depend on the state of the system for $y^0 < t$."

To interpret this, we note that the unitarity of the $S(g)$ implies that

$$U(g'', g') := S(g'')S^\dagger(g') = S(g'')S(g')^{-1}$$

is a transition map, and we may take their statement to mean that if $g'' - g'$ vanishes for $y^0 < t$ then $U(g'', g')$ does not change when both g'' and g' are shifted by the same function with support in $y^0 < t$. This is the parabolic version of the causality relation (14).

Using (12) to express the transition map in terms of S , (14) may be viewed as a condition

$$S(f + e)S(g + e)^{-1} = S(f)S(g)^{-1} \quad \text{if } e \wedge \vee g - f \quad (23)$$

restricting the possibilities for S . Fully formal versions equivalent to (23) were given independently by EPSTEIN & GLASER [16, (C.C.)] (1973) and SLAVNOV [37, (6)–(7)] (1977). This relation is the basis of **causal perturbation theory** (see, e.g., [30]), now a quite extensive branch of algebraic quantum field theory.

3.2 Examples of causal groups

(i) On the parabolic space $\mathbb{M} = \mathbb{Z} \times S$ discussed in Subsection 2.2, the group $\text{Can}(\mathbb{H})$ of canonical transformations of the sequence space $\mathbb{H}$ is by Theorem 2.1 a causal group over $\mathbb{M}$ with carrier space $\mathbb{H}$.

(ii) Any smooth homomorphism S from the additive group of a causal function space $\mathbb{H}$ to a multiplicative abelian group $\mathbb{G}$ may be used to define a causal transition map by

$$U(f, g) := S(f - g);$$

condition (14) is automatically satisfied. In particular, any time-dependent flow on $\mathbb{G}$ defines a causal group with carrier space $\mathbb{R}$.

(iii) Let $\mathbb{M}$ be a causal space and $\mathbb{H}$ be a causal function space with base $\mathbb{M}$. Let ω be a real symplectic form on $\mathbb{H}$ such that

$$\omega(f, g) = 0 \quad \text{if } f \wedge \vee g. \quad (24)$$

(For example, this is always the case for $\omega(f, g) := \lambda \operatorname{Im} f^* g$ with fixed real λ .) Then the **Heisenberg group** $\mathbb{G}_\omega(\mathbb{H})$, consisting of all $W_\alpha(f)$ with $\alpha \in \mathbb{R}$, $f \in \mathbb{H}$ with product

$$W_\alpha(f)W_\beta(g) := W_{\alpha+\beta-\omega(f,g)}(f+g), \quad (25)$$

is a causal group with carrier space $\mathbb{H}$ and causal transition map

$$U(f, g) := W_{\omega(f,g)}(f - g).$$

The transition relation follows from the multiplication law (25) and the causality relation follows from (24).

(iv) The direct product of causal groups is causal.

(v) The **free causal group** on $\mathbb{H}$ is a group defined by generators and relations. The generators are the symbols $U(f, g)$ with $f, g \in \mathbb{H}$, and the defining relations are (13) and (14). Every causal group is a homomorphic image of the free causal group with the same carrier space.

(vi) Certain causal groups defined by generators and relations also arise via (23) as ingredients for the construction of dynamical C^* -algebras in the recent work by BUCHHOLZ & FREDENHAGEN [12, 13, 14] for bosonic fields. and BRUNETTI et al. [8, 9] for fermionic fields and fields in curved spacetimes. In their treatments, the free causal group is subject to additional relations related to a specific Lagrangian.

There are many more causal groups, some of which will probably be described in the final version of the paper.

3.3 Locality and quantum fields

3.1 Proposition.

$$U(f, g')U(f, g) = U(f, g' + g - f) \quad \text{if } g' - f \wedge \vee g - f, \quad (26)$$

$$[U(f, g'), U(f, g)] = 0 \quad \text{if } g' - f \times g - f. \quad (27)$$

Proof. Suppose that $g' - f \wedge \vee g - f$ and write $e := g' - f$. Then $g' = e + f$ and by (12) and (14),

$$U(f, g) = U(f + e, g + e) = U(g', g' + g - f) = S(g')S(g' + g - f)^{-1},$$

hence

$$U(f, g')U(f, g) = S(f)S(g')^{-1}U(f, g) = S(f)S(g' + g - f)^{-1} = U(f, g' + g - f).$$

This implies (26). If $g' - f \times g - f$ then (26) implies

$$U(f, g')U(f, g) = U(f, g' + g - f) = U(f, g + g' - f) = U(f, g)U(f, g'),$$

hence (27). $\square$

A **domain** of $\mathbb{M}$ is a nonempty, open subset $\mathcal{O}$ of $\mathbb{M}$ with compact closure. A **local net of groups** over a causal space $\mathbb{M}$ is a collection of subgroups $\mathbb{G}(\mathcal{O})$ of a group $\mathbb{G}$, one for each domain $\mathcal{O}$ of $\mathbb{M}$, such that

$$\mathbb{G}(\mathcal{O}) \subseteq \mathbb{G}(\mathcal{O}') \quad \text{if } \mathcal{O} \subseteq \mathcal{O}' \quad (28)$$

and $\mathbb{G}(\mathcal{O})$ and $\mathbb{G}(\mathcal{O}')$ commute whenever $\mathcal{O} \times \mathcal{O}'$.

3.2 Proposition. *Let $f \in \mathbb{H}$. Then the collection of groups $\mathbb{G}_f(\mathcal{O})$ generated by all $U(f, g)$ with $\text{Supp } g \subseteq \mathcal{O}$ is a local net of groups.*

Proof. (28) is obvious. Assuming that $\mathcal{O} \times \mathcal{O}'$, Proposition 3.1 implies that $\mathbb{G}_f(\mathcal{O})$ and $\mathbb{G}_f(\mathcal{O}')$ commute. $\square$

This generalizes the locality properties of the causality relation first investigated by SLAVNOV [37] and IL'IN & SLAVNOV [23].

There is a generic construction (detailed, e.g., in BUCHHOLZ et al. [11, p.273]) that associates to any group a C^* -algebra containing the given group as a group of operators of spectral norm 1. In particular, we may associate to each $\mathbb{G}_f(\mathcal{O})$ a C^* -algebra $\mathbb{A}_f(\mathcal{O})$ containing $\mathbb{G}_f(\mathcal{O})$ as a group of operators. For any group state (e.g., for the trivial unit state), one obtains a unitary representation of this C^* -algebra and from it by another standard construction a von Neumann algebra. These form a **local nets** of C^* -algebras or von Neumann algebras in the sense of HAAG & KASTLER [20].

The Haag–Kastler axioms are axioms on local nets of C^* -algebras or von Neumann algebras that codify conditions for an algebra of fields that their nice unitary representations provide examples of relativistic quantum field theories over Minkowski space. The additional conditions are related to Poincaré invariance, which are inadequate for quantum field theories over more general causal spaces.

If $\mathbb{M}$ is a curved hyperbolic spacetime, a **relative vacuum state** is a state satisfying the **microlocal spectrum condition** of BRUNETTI et al. [10]. For Minkowski space, this condition is equivalent to the ordinary spectral condition required by the Haag–Kastler axioms. These condition can be translated into a condition to be satisfied for a local net of groups so that the associated net of algebras satisfy the microlocal spectrum condition.

3.4 Physical states and Wightman functions

We now assume that the causal group $\mathbb{G}$ is a Lie group and the transition map $U(f, g)$ is continuously differentiable in g .

We call the algebra $U(\mathbb{G})$ of linear differential operators on $C^\infty(\mathbb{G})$ the **universal group algebra** of $\mathbb{G}$. (It is isomorphic to a semidirect product of $\mathbb{G}$ and the universal enveloping algebra of the Lie algebra of $\mathbb{G}$, but this is not relevant here.)

We may treat $U(f, g)$ as multiplication operator in the universal group algebra $U(\mathbb{G})$. For $f, g, e \in \mathbb{H}$ we define

$$A_f(e) := i\hbar \frac{d}{d\tau} U(f + \tau e, g) \Big|_{\tau=0} U(f, g)^{-1}; \quad (29)$$

The transition relations shows that this is indeed independent of g .

The $A_f(e)$ are linear operators on $C^\infty(\mathbb{G})$ generating the **field algebra** of a local quantum field theory. Using distribution-valued **quantum fields**

$$A_f(x) = i \frac{dU(f, g)}{df(x)} U(f, g)^{-1}, \quad (30)$$

familiar from causal perturbation theory, where the quotient on the right denotes functional differentiation we may rewrite (29) as

$$A_f(e) = \int_{\mathbb{M}} dx e(x) A_{f,g}(x).$$

In terms of (12), (30) reads

$$A_f(x) := i \frac{dS(f)}{df(x)} S(f)^{-1},$$

a formula going back to the early days of causal perturbation theory.

The standard causal commutation rules for the quantum fields (30) follow by differentiation from the commutation rule (27).

3.5 Tomonaga–Schwinger structures

A **Tomonaga–Schwinger structure** on $\mathbb{M}$ consists of

- a causal function space $\mathbb{H}$ with base $\mathbb{M}$,
- a Cauchy structure $\mathcal{S}$ on $\mathbb{M}$,
- sets $Z(\Sigma)$ ($\Sigma \in \mathcal{S}$) and Z_f ($f \in \mathbb{H}$), and
- for each $\Sigma \in \mathcal{S}$ and $f \in \mathbb{H}$ an invertible **Møller operator** $\Omega_f(\Sigma) : Z_f \rightarrow Z(\Sigma)$, such that

$$\Omega_0(\Sigma)^{-1} \Omega_f(\Sigma) = \begin{cases} \Omega_-(f) & \text{if } \Sigma \wedge \text{Supp } f, \\ \Omega_+(f) & \text{if } \text{Supp } f \wedge \Sigma. \end{cases} \quad (31)$$

The maps $\Omega_+(f), \Omega_-(f) : Z_f \rightarrow Z_0$ are uniquely determined by (31). $\mathbb{H}$ is called the **carrier space** of the Tomonaga–Schwinger structure. The **transition maps**

$$U_f(\Sigma', \Sigma) := \Omega_f(\Sigma') \Omega_f(\Sigma)^{-1} : Z(\Sigma) \rightarrow Z(\Sigma')$$

describe the **many-fingered time flow** from $Z(\Sigma)$ to $Z(\Sigma')$. Møller operators first appeared in MØLLER [27] (1945) for nonrelativistic quantum mechanics and in SCHWINGER [35] (1948) for relativistic quantum field theory.

3.3 Theorem. *Every Tomonaga–Schwinger structure with carrier space $\mathbb{H}$ determines on the group $\text{Sym}(Z_f)$ of bijections of Z_f a causal transition map whose carrier space is the subspace $\mathbb{H}_c$ consisting of all functions in $\mathbb{H}$ with compact support.*

Proof. As in the proof of Theorem 2.1 we use the factorization idea of BOGOLJUBOV & SCHIRKOW [4]. It is sufficient to give the proof for $f = 0$; the general case is obtained by shifting all arguments by f .

Suppose that e, f, g have compact support and the prior relation $e \wedge \vee g - f$ holds. Then $S := \text{Supp}(e) \cup \text{Supp}(f) \cup \text{Supp}(g)$ is compact, and

$$P := \text{Supp}(h), \quad F := \text{Supp}(g - f)$$

are closed subsets of S with $P \wedge \vee F$. By definition of a Cauchy structure, there are Cauchy slices Σ_-, Σ_0 , and Σ_+ such that (3) and (4) hold. As a product of bijections, the map

$$S(f) := \Omega_+(f)\Omega_-(f)^{-1} : Z_0 \rightarrow Z_0,$$

is a bijection of Z_0 , hence belongs to the group $\text{Sym}(Z_0)$. By (31),

$$\begin{aligned} S(f) &= \Omega_+(f)\Omega_-(f)^{-1} = \Omega_0(\Sigma_+)^{-1}\Omega_f(\Sigma_+)\Omega_f(\Sigma_-)^{-1}\Omega_0(\Sigma_-) \\ &= \Omega_0(\Sigma_+)^{-1}\Omega_f(\Sigma_+)\Omega_f(\Sigma_0)^{-1}\Omega_f(\Sigma_0)\Omega_f(\Sigma_-)^{-1}\Omega_0(\Sigma_-) \\ &= \Omega_0(\Sigma_+)^{-1}U_f(\Sigma_+, \Sigma_0)U_f(\Sigma_0, \Sigma_-)\Omega_0(\Sigma_-) \\ &= R_+(f)R_-(f), \end{aligned}$$

where

$$\begin{aligned} R_+(f) &:= \Omega_0(\Sigma_+)^{-1}U_f(\Sigma_+, \Sigma_0) : Z(\Sigma_0) \rightarrow Z_0, \\ R_-(f) &:= U_f(\Sigma_0, \Sigma_-)\Omega_0(\Sigma_-) : Z_0 \rightarrow Z(\Sigma_0). \end{aligned}$$

The same holds with g in place of f . Therefore

$$S(f) = R_+(f)R_-(f), \quad S(g) = R_+(g)R_-(g),$$

Now

$$e(x) = 0 \quad \text{if } x \notin P, \quad f(x) = g(x) \quad \text{if } x \notin F$$

by definition of P and F . Therefore

$$\begin{aligned} R_+(f + e) &= R_+(f), \quad R_+(g + e) = R_+(g), \\ R_-(f + e) &= R_-(g + e), \quad R_-(f) = R_-(g). \end{aligned}$$

Hence the transition maps defined by (12) satisfy

$$\begin{aligned} U(f + e, g + e) &= S(f + e)S(g + e)^{-1} \\ &= R_+(f + e)R_-(f + e)R_-(g + e)^{-1}R_+(g + e)^{-1} \\ &= R_+(f + e)R_+(g + e)^{-1} = R_+(f)R_+(g)^{-1} \\ &= R_+(f)R_-(f)R_-(g)^{-1}R_+(g)^{-1} = S(f)S(g)^{-1} = U(f, g). \end{aligned}$$

Thus the causality relation (14) holds. Therefore U is a causal transition map. $\square$

Tomonaga–Schwinger structures can be constructed from certain classes of ordinary or partial differential equations with uniquely solvable initial value problem.

We assume that $\mathcal{S}$ is a Cauchy structure on the causal space $\mathbb{M}$ and $\mathbb{H}$ is a causal function space with base $\mathbb{M}$. Suppose that $\mathbb{M}$ and the value spaces $\mathbb{F}, \mathbb{F}', \mathbb{F}_{\text{phys}}$ are smooth manifolds, $P : C^\infty(\mathbb{M}, \mathbb{F}) \rightarrow C^\infty(\mathbb{M}, \mathbb{F})$ and $F : C^\infty(\mathbb{M}, \mathbb{F}) \times \mathbb{H} \rightarrow C^\infty(\mathbb{M}, \mathbb{F}_{\text{phys}})$ are smooth local maps, i.e., $(Pu)(x)$ and $F(u, f)(x)$ depend only on $u(x)$, $f(x)$ and their derivatives. We only state the principle; to be really useful for 4-dimensional spacetimes, one probably needs to weaken the assumptions made.

3.4 Theorem. *Suppose that for every $\Sigma \in \mathcal{S}$, every $f \in \mathbb{H}$, and every $u_0 \in C^\infty(\Sigma, \mathbb{F})$, the equations*

$$F(u, f) = 0, \quad u|_\Sigma = u_0 \quad (32)$$

have a solution $u \in C^\infty(\mathbb{M}, \mathbb{F})$, and any two such solutions have the same Pu . Then

$$Z(\Sigma) := C^\infty(\Sigma, \mathbb{F}) \quad \text{for } \Sigma \in \mathcal{S},$$

$$Z_f := \{Pu \mid u \in C^\infty(\mathbb{M}, \mathbb{F}), F(u, f) = 0\} \quad \text{for } f \in \mathbb{H},$$

$$\Omega_f(\Sigma)[z] := z|_\Sigma \quad \text{for } z \in Z_f$$

defines a Tomonaga–Schwinger structure on $\mathbb{M}$ with carrier space $\mathbb{H}$.

Proof. This is straightforward. □

In parabolic spacetimes, existence theorems for ODEs in Banach spaces apply. Therefore ODEs and regular index 1 DAEs should produce in this way Tomonaga–Schwinger structures on parabolic spacetimes, thus recovering nonrelativistic field theories. In particular, we recover results by BUCHHOLZ & FREDENHAGEN [14] on nonrelativistic quantum mechanics, considered as $1 + 0$ -dimensional quantum field theories.

3.6 Causal groups on coherent spaces

The notion of a coherent space is a nonlinear version of the notion of a complex Euclidean space (i.e., a complex vector space with a definite Hermitian inner product): The vector space axioms are dropped while the notion of inner product is kept. Coherent spaces provide a setting for the study of geometry in a different direction than traditional metric, topological, and differential geometry. Just as it pays to study the properties of manifolds independent of their embedding into a Euclidean space, so it appears fruitful to study the properties of coherent spaces independent of their embedding into a Euclidean space.

We take the following definitions from NEUMAIER [31] and NEUMAIER & GHANI FARASHAHI [32], where further details and many examples can be found.

A **coherent product** on a set Z is a map $K : Z \times Z \rightarrow \mathbb{C}$ such that

$$(K1) \quad K(z, z')^* = K(z', z) \quad \text{for } z, z' \in Z;$$

$$(K2) \quad \text{For arbitrary } z_1, \dots, z_n \in Z, \text{ the **Gram matrix** } G \text{ with entries}$$

$$G_{jk} := K(z_j, z_k) \quad \text{for } j, k = 1, \dots, n$$

is positive semidefinite.

In particular, for an arbitrary family of vectors ψ_z ($z \in Z$) in a complex Euclidean space,

$$K(z, z') := \psi_z^* \psi_{z'}$$

defines a coherent product on Z . There are many other ways to construct coherent products from known coherent products. An example:

3.5 Lemma. (NEEB [29, p.48]) *Let Z_ℓ ($\ell \in L$) be a family of subsets of a set Z such that any finite subset of Z is contained in one of the Z_ℓ . Then $K : Z \times Z \rightarrow \mathbb{C}$ is a coherent product on Z iff each $K|_{Z_\ell \times Z_\ell}$ is a coherent product on Z_ℓ .* $\square$

A **coherent space** is a nonempty set Z with a distinguished coherent product K on Z . A **coherent map** of a coherent space Z is a map $A : Z \rightarrow Z$ for which there is a (not necessarily unique) **adjoint map** $A^* : Z \rightarrow Z$ such that

$$K(Az, z') = K(z, A^*z') \quad \text{for } z, z' \in Z. \quad (33)$$

A **symmetry** of Z is an invertible coherent map on Z with an invertible adjoint. A coherent map A is called **unitary** if it is invertible and $A^* = A^{-1}$.

A ***-semigroup** is a semigroup $\mathbb{G}$ with an **involution** $*$ satisfying

$$A^{**} = A, \quad (AB)^* = B^*A^* \quad \text{for } A, B \in \mathbb{G}.$$

A ***-representation** of a *-semigroup $\mathbb{G}$ is a representation of $\mathbb{G}$ on a Euclidean space $\widehat{\mathbb{H}}$ such that

$$\Gamma(A)^* = \Gamma(A)^* \quad \text{for } A \in \mathbb{G}.$$

Every group is a *-semigroup with the inverse as involution. Every unitary representation of the group is a *-representation for this involution.

3.6 Theorem. (Main theorem of coherent quantization)

(i) *Let Γ be a *-representation of a *-semigroup $\mathbb{G}$. Then every $\mathbb{G}$ -invariant subset Z of $\widehat{\mathbb{H}}$ is a coherent space with coherent product*

$$K(z, z') := z^* z',$$

and $\mathbb{G}$ is a semigroup of coherent maps of Z , with the involution as adjoint.

(ii) *Conversely, let Z be a coherent space Z . Then there is a **quantum space** of Z , a Euclidean space spanned (algebraically) by a distinguished set of vectors $|z\rangle$ ($z \in Z$) called **coherent states** satisfying*

$$\langle z|z'\rangle = K(z, z') \quad \text{for } z, z' \in Z, \quad (34)$$

where $\langle z| := |z\rangle^$. If $\mathbb{G}$ is a *-semigroup of coherent maps of Z , with the involution as adjoint then $\mathbb{G}$ has a *-representation Γ such that*

$$\Gamma(A)|z\rangle = |Az\rangle \quad \text{for } A \in \mathbb{G}, z \in Z.$$

Proof. The first part is straightforward, and the second part is a consequence of the Moore-Aronszajn theorem on reproducing kernels (ARONSZAJN [1, 2], attributed by him to MOORE [28]) and the coherent quantization results from NEUMAIER & GHANI FARASHAH [32, Section 3]. $\square$

Related to a quantum space $\mathbb{Q}(Z)$ is its **augmented quantum space** $\mathbb{Q}^\times(Z)$, the antidual of $\mathbb{Q}(Z)$, which contains the **completed quantum space** $\overline{\mathbb{Q}}(Z)$, the Hilbert space completion of $\mathbb{Q}(Z)$.

This theorem implies that if a causal group $\mathbb{G}$ over $\mathbb{M}$ is a group of symmetries of a coherent space then it is also a group of symmetries of the associated quantum space. If $\mathbb{M}$ is a hyperbolic spacetime then this quantum space carries a local quantum field theory in the sense of HAAG & KASTLER [20],

The point of the theorem is that constructing coherent spaces with interwining symmetry groups is much less demanding than constructing $*$ -representations.

3.7 Conjecture.

(i) For every quantum field theory $\mathcal{T}_q$ there is a coherent space Z with a causal group of symmetries whose classical limit is a classical field theory $\mathcal{T}_c$, and whose quantum space reproduces $\mathcal{T}_q$.

(ii) If $\mathcal{T}_q$ or $\mathcal{T}_c$ has an explicit construction then Z also has an explicit construction. $\square$

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