ROUBUST STABILITY ASSESSMENT VIA INNER BOUNDS AND EXACT SOLUTION FOR THE RANGE OF THE EIGENVALUES OF INTERVAL MATRICES

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Robust stability assessment of linear electrical circuits and control systems under interval parameters uncertainties can be equated to estimating the eigenvalues of interval matrices. This estimation can be made using inner and outer bounds or the exact solution for the range of the eigenvalues. In this paper, the problem of determining inner bounds and the exact solution for the eigenvalues ranges is considered. A method for computing such bounds is suggested. If certain monotinicity conditions are fulfilled, it can be extended to provide the exact solution for the eigenvalues range.

An example illustrating the applicability of the methods suggested is provided.

1. Introduction

The problem of estimating the range of the eigenvalues of matrices is closely related to stability analysis of linear control systems under interval uncertainties of the parameters. There are a lot of methods for its solution (see [3] – [7]) in the literature. Recently, a method has been proposed for computing outer bounds on the eigenvalues of matrices with interval components (the real and the complex case) and these estimations are relatively less conservative as compared to other methods [1, 2].

In this paper, two new methods for handling the interval eigenvalues problem is suggested. The first provides tight inner bounds. The second determines the exact range of the eigenvalues of the interval matrix considered. It is however applicable only if certain monotonicity conditions in interval form are fulfilled. The interval monotonicity conditions are checked using the outer bounds suggested in [1, 2].

The two new methods can be applied for both real and complex eigenvalues, but for simplicity we will discuss only the real eigenvalue case.

2. Problem statement

The problem statement is defined in [1]. For a clearer presentation of the new methods, the main points of the problem formulation will be briefly presented here again. Let A be a real n x n matrix, $\boldsymbol{A}$ - an interval matrix containing A, and A^-, A^+, A_0 and R_A – the left end, the right end, the center and the radius of $\boldsymbol{A}$, respectively. (Here and henceforth, ordinary font letters will denote real quantities while bold face letters will stand for their interval counterparts.) We consider the following "perturbed" eigenvalue problem:

(1) $$A.x = \lambda.x,\ A\in \boldsymbol{A} = [\ A^-,\ A^+] = A_0 + [\ -R(A),\ R(A)]$$

Note: Each matrix $A \in \boldsymbol{A}$, is non-singular.

Let $\lambda^*(A)$ denote such a real eigenvalue while $x^{(k)}(A) = (x_1^{(k)}(A),\ x_2^{(k)}(A),\ \dots ,\ x_n^{(k)}(A))$ be the corresponding (real) eigenvector, k = 1, ... , n', n' ≤ n. For simplicity of solving the problem we will make the following assumption (ensuring structural stability of the problem).

Assumption A_1: For any $k \in K$ = 1, ... , n', all $\lambda^{(k)}(A)$ and $x^{(k)}(A)$, corresponding to all $A\in \boldsymbol{A}$, remain real.

For simplicity, we shall henceforth drop the index k. On account of Assumption A_1, the range

(2) $$\boldsymbol{\lambda}^* = \{\ \lambda(A): A \in \boldsymbol{A}\}$$

is a real interval.

The interval hull of the interval vector of the eigenvalues of matrix $A \in \boldsymbol{A}$ will be note $\boldsymbol{\lambda}^*$ and $\boldsymbol{\lambda}^*$ will be called exact (interval hull) solution to the interval vector of the eigenvalues of (1). Any other interval eigenvalue vector $\boldsymbol{\lambda}_{out}$ such that $\boldsymbol{\lambda}^* \subset \boldsymbol{\lambda}_{out}$ will be referred to as an outer interval vector of the eigenvalues of (1). Similarly, an interval vector of eigenvalues of matrix $A \in \boldsymbol{A}$ - $\boldsymbol{\lambda}_{in}$ with the property $\boldsymbol{\lambda}_{in} \subseteq \boldsymbol{\lambda}^*$ will be referred to as inner solution to (1).

In this paper, we suggest two methods for finding an inner bound and exact solution of $\boldsymbol{\lambda}^*$.

To simplify presentation of the method (without any loss of generality), we need a second assumption. Let the pair (x^0, λ^0) be the solution of the (center, nominal) problem

(3) $$A_0.x^0 = \lambda^0.x^0$$

Assumption A_2: We assume that the absolute value of the n-th component $\left|x_n^0\right|$ of x^0 is the largest component of the other components, i.e.

(4) $$\left|x_n^0\right| \geq \left|x_i^0\right|,\ i \neq n$$

Note: If p-th component is the largest component, we need to interchange the places of the p-th and n-th row in A matrix as well as the position of the components x_p and x_n.

Now x^0 is normalized by letting

(5) $$\left|x_n^0\right| = 1$$

Further, we require that (5) be also valid for

(5') $$\left|x_n(A)\right| = 1,\ \forall A \in \boldsymbol{A}$$

Condition (5') simplifies the method which will be presented in the next section.

We introduce the n-dimensional real vector

(6) $$y = (y_1,\ y_2,\ \dots ,\ y_n)$$

with

(7) $$y_i = x_i(A),\ i = 1, \dots, (n\text{-}1)$$
$$y_n = \lambda(A)$$

Using (7), (1) is rewritten as

(8) $$\left|\begin{array}{l} a_{11}\cdot y_1 + a_{12}\cdot y_2 + \dots + a_{1,(n-1)}\cdot y_{n-1} - y_n\cdot y_1 + a_{1n} = 0 \\ a_{21}\cdot y_1 + a_{22}\cdot y_2 + \dots + a_{2,(n-1)}\cdot y_{n-1} - y_n\cdot y_2 + a_{2n} = 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{n1}\cdot y_1 + a_{n2}\cdot y_2 + \dots + a_{n,(n-1)}\cdot y_{n-1} - y_n\cdot 1 + a_{nn} = 0 \end{array}\right.$$

where

(9) $$a_{ij} \in \boldsymbol{A_{ij}} = [A_{ij}^{-}, A_{ij}^{+}]$$

System (8) is a non-linear (more precisely, a quadratic) interval system because of the products $y_n.y_i$ in the first (n-1) equations in (8). Let y_i^* denote the range of the i-th component y_i (A), $A \in \boldsymbol{A}$, of the solution to (8). Let Y^* be the vector made up of y_i^*. Consider the following problems:

Problem P_1 and P_2: Find an inner bound and exact solution to (8), for $\forall a_{ij} \in \boldsymbol{A}_{ij},\ i = 1,\dots,n;\ j = 1,\dots,n$.

3. Inner bounds

To obtain inner bounds of the eigenvalues we rewrite system (8) in the form:

(10a) $$f_i(a_{ij}, y_i) = \sum_{j=1}^{n-1} a_{ij}.y_j - y_n.y_i + a_{in} = 0,\ \ i = 1,\dots,(n-1)$$

(10b) $$f_i(a_{ij}, y_i) = \sum_{j=1}^{n-1} a_{ij}.y_j - y_n + a_{nn} = 0,\ \ i = n$$

We differentiate systems (10a) with respect to a_{lm} to get n^2 nonlinear systems:

(11) $$\frac{\partial f_i}{\partial a_{lm}} + \frac{\partial f_i}{\partial y_i}.\frac{\partial y_i}{\partial a_{lm}} = 0, i = 1,\dots,n$$

(11a) $$\delta_{il}\cdot\gamma_{im}\cdot y_m + \sum_{j=1}^{n-1} a_{ij}\cdot\frac{\partial y_j}{\partial a_{lm}} - y_i\frac{\partial y_n}{\partial a_{lm}} - y_n\cdot\frac{\partial y_i}{\partial a_{lm}} + \delta_{il}.\delta_{nm} = 0,\ i = 1,\dots,(n-1)$$

(11b) $$\delta_{nl}\cdot\gamma_{im}\cdot y_m + \sum_{j=1}^{n-1} a_{ij}\cdot\frac{\partial y_j}{\partial a_{lm}} - \frac{\partial y_n}{\partial a_{lm}} + \delta_{nl}.\delta_{nm} = 0, i = n$$

where δ_{il} is the Kronicker symbol i.e. $\delta_{il} = 1$ for $i = l$ and $\delta_{il} = 0$ otherwise while $\gamma_{im} = 1$ for $m < n$ and $\gamma_{im} = 0$ if $m = n$. Now we fix all a_{ij} and y_i , y_m , y_n in (11) at their centers and solve the resulting system

(12a) $$\sum_{j=1}^{n-1} a_{ij}^{0} \cdot \frac{\partial y_j}{\partial a_{lm}} + (a_{ii}^{0} - y_n^{0}) \cdot \frac{\partial y_i}{\partial a_{lm}} - y_i^{0} \cdot \frac{\partial y_n}{\partial a_{lm}} = -\delta_{il} \cdot \delta_{mn} - \delta_{il} \cdot \gamma_{im} \cdot y_m^{0}$$

$$j \neq i, \qquad i = 1,\ldots,(n-1)$$

(12b) $$\sum_{j=1}^{n-1} a_{nj}^{0} \cdot \frac{\partial y_j}{\partial a_{lm}} - \frac{\partial y_n}{\partial a_{lm}} = -\delta_{nl} \cdot \delta_{nm} - \delta_{nl} \cdot \gamma_{nm} \cdot y_m^{0}$$

for the derivative $\frac{\partial y_n}{\partial a_{lm}}$. We repeat this n^2 times for all l, m = 1,…,n. Afterwards we form two matrices A' and A" in the following way: Let d_{lm} denote $\frac{\partial y_n}{\partial a_{lm}}$. Then the elements a_{lm}' and a_{lm}'' of A' and A" are defined as follows: if $d_{lm} \geq 0$ then $a_{lm}' = a_{lm}^-$, $a_{lm}'' = a_{lm}^+$, otherwise $a_{lm}' = a_{lm}^+$, $a_{lm}'' = a_{lm}^-$. Finally we solve two eigenvalue problems:

(13a) $$A'.x = \lambda.x$$

(13b) $$A''.x = \lambda.x$$

The solution λ' of (13a) determines the lower endpoint and the solution λ" of (13b) gives the upper endpoint of the inner bound sought $\boldsymbol{\lambda}_{in}$.

4. Exact solution

In order to determine the exact solution for the range of the eigenvalues of (1) we need to find outer bounds on the derivative $\frac{\partial y_n}{\partial a_{lm}}$ when all a_{ij} and y_i , y_m , y_n in (11) are intervals. The intervals for a_{ij} are known since $\boldsymbol{A}_{ij}$ are given intervals. To find the intervals $\boldsymbol{y}_k$ corresponding to $\boldsymbol{y}_k$, k = 1,...,n, we have to find an outer solution to (1). We compute such a solution using the method from [1] (or [2] in the case of complex eigenvalue). Then we solve a linear interval system which results from (11) when aditionally

(11c) $$a_{ij} \in \boldsymbol{A}_{ij}, y_k \in \boldsymbol{y}_k$$

The system obtained is:

(14a)
$$\sum_{\substack{j=1\\ j\neq i}}^{n-1} \boldsymbol{a}_{ij}\cdot\frac{\partial y_j}{\partial a_{lm}} + (\boldsymbol{a}_{ii} - \boldsymbol{y}_n).\frac{\partial y_i}{\partial a_{lm}} - \boldsymbol{y}_i\cdot\frac{\partial y_n}{\partial a_{lm}} = -\delta_{il}.\delta_{mn} - \delta_{il}\cdot\gamma_{im}\cdot\boldsymbol{y}_m \qquad i = 1,\ldots,(n-1)$$

(14b)
$$\sum_{j=1}^{n-1} \boldsymbol{a}_{nj}\cdot\frac{\partial y_j}{\partial a_{lm}} - \frac{\partial y_n}{\partial a_{lm}} = -\delta_{nl}.\delta_{nm} - \delta_{nl}\cdot\gamma_{nm}\cdot\boldsymbol{y}_m$$

We solve system (13) using the linear version of the method from [1] (or [2]). Thus we obtain the intervals $\boldsymbol{D}_{lm}^{i}$, $i = 1,\ldots,n$. We are interested only in $\boldsymbol{D}_{lm}^{n}$ denoted for simplicity $\boldsymbol{D}_{lm}$. Since $\boldsymbol{D}_{lm}$ is obtained from (14) as an outer solution, it is guaranteed to contain the corresponding "point" derivative d_{lm} for any $a_{ij} \in \boldsymbol{A_{ij}}$, $y_k \in \boldsymbol{y_k}$, i.e.

(15)
$$d_{lm} \in \boldsymbol{D}_{lm} = [D_{lm}^{-}, D_{lm}^{+}]$$

On account of (15) if

(16a)
$$D_{lm}^{-} \geq 0$$

then the derivative d_{lm} is monotoniously increasing, also if

(16b)
$$D_{lm}^{+} \leq 0$$

the derivative d_{lm} is monotoniously decreasing. Now we need the following assumption:

Assumption 3: Either (16a) or (16b) is valid for all l, m = 1,...,n.

If Assumption 1 is true then we proceed in the same way as in the previous section. We form two matrices A' and A" whose elements are expressed by either A_{lm}^{-} or A_{lm}^{+} depending on whether (16a) or (16b) takes place for l, m combination. We solve the corresponding problems (13a) and (13b) to get two solutions λ' and λ". Finally, the exact range $\boldsymbol{\lambda}^{*}$ of the eigenvalue considered is determined as:

(17)
$$\boldsymbol{\lambda}^{*} = [\lambda', \lambda'']$$

5. Numerical example

The applicability of the methods will be illustrated by the following example with n = 2:

(18)
$$\left| \begin{array}{l} \boldsymbol{a}_{11}.\boldsymbol{x}_1 + \boldsymbol{a}_{12}.\boldsymbol{x}_2 - \boldsymbol{\lambda}.\boldsymbol{x}_1 = 0 \\ \boldsymbol{a}_{21}.\boldsymbol{x}_1 + \boldsymbol{a}_{22}.\boldsymbol{x}_2 - \boldsymbol{\lambda}.\boldsymbol{x}_2 = 0 \end{array} \right.$$

Here

(18a) $$\left| \begin{array}{ll} a_{11}^{0} = -3.8 & a_{12}^{0} = 1.6 \\ a_{21}^{0} = 0.6 & a_{22}^{0} = -4.2 \end{array} \right.$$

and

(18b) $$R(A) = [R_{ij}(A) = 0.1],\ i = 1, 2;\ j = 1, 2.$$

First, we determine the centre of the eigenvalues $\Lambda^{o} = [-3, -5]^{T}$ from (3) for centers of matrix A from the equation

(18c) $$\det(A^{0}.x - \lambda.x) = 0$$

and

(18d) $$\lambda_{\max} = \max_{i}\{\lambda^{(i)}(A_{0})\}$$

For this example, the index k corresponding to λ_{max} is $k = 1$ and for simplicity we'll drop it. The corresponding normalized vector of the centres of the variables from (4) and (5) is:

$$X^{0} = [1\ x_{2}^{0}]^{T}$$

And following the of Assumption 2 we change the places of p = 1 and n = 2 components of vector X^{0} so vector Y from (3) become in form:

(19) $$Y = \begin{bmatrix} y_{1} \\ y_{2} \end{bmatrix} = \begin{bmatrix} x_{2} \\ \lambda \end{bmatrix}$$

so for A^{0} vector Y^{0} is:

(20) $$Y^{0} = \begin{bmatrix} y_{1}^{0} \\ y_{2}^{0} \end{bmatrix} = \begin{bmatrix} x_{2}^{0} \\ \lambda^{0} \end{bmatrix} = \begin{bmatrix} 0,5 \\ -3 \end{bmatrix}$$

So the nonlinear system according to (8) (resp. (18)) is:

(21) $$\left| \begin{array}{l} \boldsymbol{a}_{22}.\boldsymbol{y}_{1} - \boldsymbol{y}_{2}.\boldsymbol{y}_{1} + \boldsymbol{a}_{21} = 0 \\ \boldsymbol{a}_{12}.\boldsymbol{y}_{1} - \boldsymbol{y}_{2}.1 + \boldsymbol{a}_{11} = 0 \end{array} \right.$$

We calculate the derivatives

(22) $$\frac{\partial y_{2}}{\partial a_{ij}},\ i = 1, 2;\ j = 1, 2 \ \text{ for } \ a_{ij}^{0} \ \text{ and } \ y^{0}$$

using system (12). It can be checked that the derivatives are positive so matrices A' and A" can be written in form:

(23) $$A' = A^0 - \mathrm{R(A)} = \begin{vmatrix} -3.9 & 1.5 \\ 0.5 & -4.3 \end{vmatrix}$$

(24) $$A'' = A^0 + \mathrm{R}(A) = \begin{vmatrix} -3{,}7 & 1{,}7 \\ 0{,}7 & -4{,}1 \end{vmatrix}$$

The eigenvalues of matrix A' and A" are:

$\lambda' = -3{,}21118$, $\lambda'' = -2{,}79$

So the inner bounds on the eigenvalue $\lambda^0 = -3$ are:

(25) $$\boldsymbol{\lambda}_{\text{in}} = [\lambda', \lambda''] = [-3{,}21118, \ -2{,}79]$$

We can obtain the exact solution of the system (15) if we estimate the sign of the interval derivatives $\boldsymbol{D}_{lm}$. These derivatives were computed as explained in Section 4 and they all turned out to be positive. Thus, for this example the exact range of the eigenvalue centred at $\lambda^0 = -3$ is also given by the interval (25).

We can compare these results from (25) with results of [1], which gives outer bounds on the range of the eigenvalue:

(26) $$\boldsymbol{\lambda}_{\text{out}} = [-3{,}259118, \ -2{,}74]$$

As expected, the range (25)is contained in the outer bound (26).

6. Conclusion

A method for computing an inner bound on the range of an eigenvalue of interval matrices has been suggested. It consist of setting up and solving the real system (12) n^2 times for $\frac{\partial y_n}{\partial a_{lm}} = d_{lm}^{n} = d_{lm}$.

Using the sign of d_{lm}, the two eigenvalue problems (13a) and (13b) are formed. The corresponding solutions λ' and λ " of the above problems give the inner bound sought.

A method for computing the exact range of the eigenvalue considered has also been proposed. It is based on the interval solutions $\mathbf{D}_{lm}$ of systems (14). The method is applicable if Assumption 3 is valid.

A two dimensional example (18) has been solved which illustrates the applicability of the above methods.

In order to improve the assessment of the robust stability, the approach herein suggested will be extended in a subsequent paper to the case where the elements a_{ii} in (1) are functions of independent parameters.

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Оценяване на робастна устойчивост посредством външни и точни оценки на областта на изменение на собствените стойности на интервални матрици

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Робастната устойчивост на линейни електрически вериги и системи за управление при интервална неопределеност на параметрите може да се определи чрез оценка на собствените стойности на матрици с интервални коефициенти. Тази оценка може да се намери чрез определяне на външни или вътрешни оценки както и на точната стойност на лявата и дясната гранични стойности на собствените числа на интервалните матрици. В настоящата статия са разгледани задачите за определяне на вътрешни оценки и оценки на точното решение за диапазона на изменение на собствените числа. Предложен е метод за определяне на вътрешните оценки. Той може да бъде приложен и за намиране на точното решение, ако е изпълнено условието за монотонност за целия интервал на изменение на параметрите.

Практическата приложимост на предложените методи е илюстрирана чрез числов пример.

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