# Interval Frazer-Duncan criterion for stability analysis of linear systems with dependent coefficients in the characteristic polynomial

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*Abstract:* - The paper addresses the stability analysis of linear continuous systems under interval uncertainties. A new implementation of the interval Frazer-Duncan criterion is suggested to estimate the stability of the system considered. It is based on obtaining the interval extensions of the coefficients $a_0$ and $a_n$ in the characteristic polynomial as well as the determinant $\Delta_{n-1}$ from the Hurwitz matrix. In general, each of them is nonlinear function of independent system parameters. The interval extensions studied are determined by using modified affine arithmetic. Two sufficient conditions on stability and instability of the linear system considered are obtained. Numerical example illustrating the applicability of the method suggested is solved in the end of the paper.

*Key-Words:* - Robust stability analysis of linear systems, Interval Frazer-Duncan criterion, Interval extension, Affine arithmetic.

## 1 Introduction

It is well known that the linear system described by the characteristic polynomial

$$q(s) = a_0 s^n + a_1 s^{n-1} + ... + a_{n-1} s + a_n \qquad (1)$$

is stable if and only if the roots of the respective characteristic equation

$$q(s) = 0 \qquad (2)$$

have negative real part [1]. The necessary condition for system stability is to have positive coefficients in characteristic polynomial (1), i.e.

$$a_i > 0, \ i = 0,1,...,n. \qquad (3)$$

It is well known [1] that Hurwitz formulates necessary and sufficient conditions for the stability of linear systems described by characteristic polynomial (1). He defines the matrix

$$H(s) = \begin{bmatrix} a_1 & a_3 & a_5 & a_7 & ... & 0 & 0 \\ a_0 & a_2 & a_4 & a_6 & ... & 0 & 0 \\ 0 & a_1 & a_3 & a_5 & ... & 0 & 0 \\ 0 & a_0 & a_2 & a_4 & ... & 0 & 0 \\ ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & ... & ... & ... & a_{n-1} & 0 \\ 0 & 0 & ... & ... & ... & a_{n-2} & a_n \end{bmatrix}. \qquad (4)$$

Based on (4), he introduces $\Delta_h(s)$ which is the determinant of the h^th^ minor on the main diagonal of the Hurwitz matrix $H(s)$.

*Remark 1:* If $h = n$, then $\Delta_n(s)$ is the determinant of the Hurwitz matrix $H(s)$ and $\Delta_{n-1}(s)$ denote the determinant derived from $H(s)$ by deleting the last row and column of $H(s)$ (the so-called Hurwitz determinants of order $n$ and $n$-1).

The Hurwitz criterion of stability is based on the following theorem.

*Theorem 1:* A necessary and sufficient condition for stability of the system described by the characteristic polynomial (1) is:

$$\forall \Delta_h(s) > 0, \ h = 1,2,...,n. \qquad (5)$$

In general, all the coefficients $a_i$, $i = 0,1,...,n$ in the characteristic polynomial (1) are nonlinear functions of independent parameters $p_j$, $j = 1,2,...,m$. Thus, if we evaluate the uncertainty in real systems, each of it takes their values in prescribed independent intervals $\boldsymbol{p}_j$, $j = 1,2,...,m$. Then the interval form of the Hurwitz criterion for stability requires the verification of all $n$ conditions (5) in interval form. A better possibility is formulated by Frazer and Duncan in the following theorem [2]:

*Theorem 2:* Necessary and sufficient conditions for stability of the system described by the characteristic polynomial (1) are:

1) there exists a $p = p^1 \in \boldsymbol{p}$ such that the characteristic polynomial

$$q(s,p^1)=a_0(p^1)s^n+a_1(p^1)s^{n-1}+...+a_{n-1}(p^1)s+a_n(p^1) \qquad (6a)$$

is stable and

2) the coefficients $a_0$, $a_n$ and the Hurwitz determinant of order $n$-1 are different from zero over the parameter box, that is

$$\left| \begin{array}{l} a_0(p)\neq 0 \\ a_n(p)\neq 0 \\ \Delta_{n-1}(p)\neq 0 \end{array} \right. , \quad p \in \boldsymbol{p}. \qquad (6b)$$

Based on the above theorem and some well-known facts related to the stability of the polynomials – positiveness of the polynomial coefficients (necessary condition, e.g. [1]), positiveness of all Hurwitz determinants (necessary and sufficient condition, e.g. [1]) – the following result is straightforward [2]:

*Theorem 3:* Necessary and sufficient conditions for stability of the system described by the characteristic polynomial (1) are:

1) the nominal system (1) (with $p^0$ being the centre of $\boldsymbol{p}$) is stable and

2) the coefficients $a_0$, $a_n$ and the Hurwitz determinant of order $n$-1 are all positive in $\boldsymbol{p}$, i.e.

$$\left| \begin{array}{l} a_0(p)> 0 \\ a_n(p)> 0 \\ \Delta_{n-1}(p)> 0 \end{array} \right. , \text{ за } \ p \in \boldsymbol{p}. \qquad (7)$$

Since the verification of Condition 1 of Theorem 2 presents no difficulties we shall henceforth assume that it is fulfilled and we shall concentrate on checking Condition 2.

## 2 Problem formulation

Let $f(p)$ denote any of the functions $a_0(p)$, $a_n(p)$ and $\Delta_{n-1}(p)$. Thus, we have to solve for each of these functions the following problem:

*Problem P1:* Check that

$$f(p)>0, \ p\in \boldsymbol{p}. \qquad (8)$$

There are various ways to verify (8). The simplest approach is to use some interval extension $\boldsymbol{F}(\boldsymbol{p}) = [\underline{F} \ \ \overline{F}]$ of the function $f(p)$ in $\boldsymbol{p}$. Theorem 1.1 [2] states that the interval extension $\boldsymbol{F}(\boldsymbol{p})$ always contains the range $f(\boldsymbol{p}) = [\underline{f}^*, \ \overline{f}^*]$ of the function $f(p)$

$$\boldsymbol{F}(\boldsymbol{p}) \supseteq f(\boldsymbol{p}). \qquad (9)$$

Hence (8) is satisfied if $\underline{F} > 0$.

Based on Theorem 2 and inclusion property (9), the following results are obvious:

*Corollary 1:* (Sufficient condition for stability) If for all end-points $\underline{F}_q > 0$ of the functions $f_q(p)$

$$\underline{F}_q > 0, \ q = 1,2,3,..., \qquad (10)$$

then the system considered is stable.

*Corollary 2:* (Sufficient condition for instability) If at least one of the endpoints

$$\overline{F}_q \leq 0, \ q = 1,2,3,..., \qquad (11)$$

then the system considered is not stable.

Various interval extensions can be used in implementing Corollaries 1 and 2: natural extension, mean-value form extension, extension using the global optimization methods [2]. The natural extensions are determined using the standard interval arithmetic [3]. Unfortunately this extension is the widest compared to the other types of extensions. The improved interval linearization [5] leads to shorter bounds of the considered extensions. Better results could be obtained if an affine arithmetic is applied to calculate the interval extensions $\boldsymbol{F}(\boldsymbol{p})$ [4]. The shortest interval extensions are obtained using the modified interval arithmetic which will be described briefly in the next section. This technique has been recently proposed [11] for the stability analysis of linear interval systems with generalization of the known Raus criterion.

The paper is organized as follows. The modified affine arithmetic is described in the next section. The method for obtaining the interval extensions of the functions $a_0(p)$, $a_n(p)$ and $\Delta_{n-1}(p)$ using G-intervals is presented in Section 4. Numerical example illustrating the applicability of the new technique for stability analysis of linear interval systems by Frazer-Duncan criterion is solved in Section 5. The paper ends up with concluding remarks in the last Section 6.

# 3 Modified affine arithmetic

Most often, the functions $f(p)$ are rational functions. Thus, we will define the main mathematical operations for these functions. To maintain completeness we start with the definition of the basic conception, the so-called *generalized interval*.

*Definition 1:* A generalized (G) interval $\widetilde{X}$ of length $k$ is defined as follows:

$$\widetilde{X} = x_0 + \sum_{i=1}^{k} x_i \boldsymbol{e}_i \qquad (12)$$

where $x_i$, $i = 0,1,...,k$, are real numbers while $\boldsymbol{e}_i$ are unit symmetrical intervals, i.e.

$$\boldsymbol{e}_i = [-1, 1]. \qquad (12a)$$

Let

$$\widetilde{Y} = y_0 + \sum_{i=1}^{k'} y_i \boldsymbol{e}_i \qquad (13)$$

be a G-interval of length $k'$. To simplify presentation, we assume that $k' = k$ where $k$ is the length of $\widetilde{X}$ (otherwise, we add zero components either in $\widetilde{X}$ or $\widetilde{Y}$ depending on whether $k$ is smaller or larger than $k'$).

In general, each of the rational functions can be composed of the simple mathematical operations as follows.

*Linear combination*. Let $\widetilde{X}$ and $\widetilde{Y}$ be two G-intervals of length $k$ given by (12) and (13). Also, let $\alpha, \beta \in R$. Then the linear combination of $\widetilde{X}$ and $\widetilde{Y}$, denoted $\alpha \widetilde{X} + \beta \widetilde{Y}$, is another G-interval $\widetilde{Z}$ of the same length $k$ if its elements $z_i$ are computed as follows:

$$z_i = \alpha x_i + \beta y_i, \quad i = 0,1,..., k. \qquad (14)$$

As a corollary we have the definitions of addition of two G-intervals ($\alpha = \beta = 1$) and subtraction of two G-intervals ($\alpha = 1, \beta = -1$).

Now we shall define the operations of multiplication and division of G-intervals. Unlike the linear combination, the operations of multiplication and division of G-intervals result in a G-interval of increased length.

*Multiplication*. The product $\widetilde{X}\widetilde{Y}$ of two G-intervals $\widetilde{X}$ and $\widetilde{Y}$ of length $k$ is a G-interval $\widetilde{Z}$ of length $k$ + 1 if the components $z_i$ of $\widetilde{Z}$ are computed as follows:

$$u = \sum_{i=1}^{m} |x_i|, v = \sum_{i=1}^{m} |y_i|, c = 0.5 \sum_{i=1}^{m} x_i y_i, \qquad (15a)$$

$$z_0 = x_0 y_0 + c, z_i = x_0 y_i + y_0 x_i, i = 1,..., k, \qquad (15b)$$

$$z_{m+1} = u\,v - |c|. \qquad (15c)$$

It has to be noted that the multiplication (15) leads to smaller overestimation as compared with the multiplication used the standard affine arithmetic in [6] because of the "correction" introduced by the additional term $c$.

To define the operation of division, we have to consider the operation reciprocal $1/\widetilde{Y}$ of a G-interval. To do this we need some definitions. The G-interval $\widetilde{X}$ is reduced to the corresponding (ordinary) interval $\boldsymbol{x} = [\underline{x}, \overline{x}]$ if the summation operations in (12) are carried out. By abuse of language, we shall also say that $\widetilde{X}$ does not contain zero (is positive or negative) if the corresponding reduced interval $\boldsymbol{x}$ does not contain zero (is positive or negative).

*Reciprocal*. Let $\widetilde{Y}$ be a G-interval of length $k$ that does not contain zero. Then the reciprocal $\widetilde{Z} = 1/\widetilde{Y}$ is another G-interval of length $k$+1 if its components $z_i$ are computed as follows:

$$s = -1/(\underline{y}\,\overline{y}), y_1 = -\sqrt{-1/s}, y_2 = -y_1, \qquad (16a)$$

$$y_s = \begin{cases} y_2, & if\ \underline{y} > 0 \\ y_1, & if\ \overline{y} < 0 \end{cases}, \qquad (16b)$$

$$\underline{f} = 1/y_s - s\,y_s, \quad \overline{f} = 1/\underline{y} - s\,\underline{y}, \qquad (16c)$$

$$f_0 = 0.5\left(\underline{f} + \overline{f}\right), \quad r_f = \overline{f} - f_0, \qquad (16d)$$

$$z_0 = s y_0 + f_0, \quad z_i = s\,y_i, i = 1,..., k, \qquad (16e)$$

$$z_{m+1} = r_f \qquad (16f)$$

when $\underline{y}$ and $\overline{y}$ are the endpoints of the reduced interval $\boldsymbol{y}$.

The above formulae follow directly from the general approach for enclosing univariate functions [7]-[10] by a linear interval form.

The division rule given below is based on the expression

$$\widetilde{X}/\widetilde{Y} = \frac{x_0}{y_0} + \frac{\sum_{i=1}^{m}(y_0 x_i - x_0 y_i)e_i}{y_0(y_0 + \sum_{i=1}^{m} y_j e_j)} = c + \frac{1}{\widetilde{Y}}\left[\sum_{i=1}^{m}(x_i - c\,y_i)e_i\right] \qquad (17)$$

if ($0 \notin \widetilde{Y}$).

*Division*. Let $\widetilde{X}$ and $\widetilde{Y}$ be G-intervals of length $k$ and $0 \notin \widetilde{Y}$. Then the division $\widetilde{X}/\widetilde{Y}$ is a G-interval $\widetilde{Z}$ of length $k$+2 whose components $z_i$ are computed as follows:

$$\widetilde{Q} = 1/\widetilde{Y}, \qquad (18a)$$

$$c = x_0 / y_0, \quad p_0 = 0, \quad p_i = x_i - c\,y_i, \quad i = 1,..., k, \qquad (18b)$$

$$\widetilde{P} = \sum_{i=1}^{m} p_i e_i, \qquad (18c)$$

$$\widetilde{V} = \widetilde{Q} \cdot \widetilde{P}, \qquad (18d)$$

$$z_0 = c + v_0, z_i = v_i, i = 1,..., k+2 \qquad (18e)$$

It is seen that the division increases the length of the resulting interval $\widetilde{Z}$ by two because of the reciprocal (18a) and multiplication (18d), each operation adding one more element to the initial $k$ elements of $\widetilde{X}$ or $\widetilde{Y}$.

## 4 Interval Frazer-Duncan criterion with G-intervals

In this section, we are interested in solving the Problem P1 for all the functions $a_0(p)$, $a_n(p)$ and $\Delta_{n-1}(p)$. A method capable of finding the interval extensions $\boldsymbol{F}(\boldsymbol{p})$ of functions $f(p),\ p \in \boldsymbol{p}$ that uses affine arithmetic will be suggested here. This method consists of the following:

1) The nonlinear functions $a_0(p)$ and $a_n(p)$ are given in explicit form of the vector of system parameters $p$.

2) The nonlinear function $\Delta_{n-1}(p)$ is dependent on the vector of system parameters $p$ in implicit form. For this reason, we work out the determinant $\Delta_{n-1}$ and get the expression of the respective nonlinear function $\Delta_{n-1}(p)$ in explicit form of the independent parameters $p_j,\ j = 1,2,...,m$.

To find the interval extensions considered we do the following: first, we present the components of parameter vector $p$ by generalized intervals

$$p_j = p_j^0 + \sum_{s=1}^{m} p_{js}\boldsymbol{e}_s,\ \boldsymbol{e}_s = [-1,\ \ 1]. \qquad (19)$$

Then we apply the necessary simple mathematical operations of modified affine arithmetic (described in previous Section 3) to make a linearization of the resulting functions $f(p)$. Thus, we get the interval extensions in the following form:

$$\boldsymbol{F}(\boldsymbol{p}) = f_0 + \sum_{j=1}^{n_f} f_j\boldsymbol{e}_j,\ \boldsymbol{e}_j = [-1,\ \ 1] \qquad (20)$$

where the lengths $n_f$ of the respective G-intervals depend on the type of nonlinearity of the functions $a_0(p)$, $a_n(p)$ and $\Delta_{n-1}(p)$ with respect to the independent parameters $p_j,\ j = 1,2,...,m$.

The G-intervals (20) reduce to the corresponding (ordinary) intervals

$$\boldsymbol{F}(\boldsymbol{p}) = f_0 + [-r_f,\ \ r_f] = [\underline{F}\ \ \ \overline{F}] \qquad (21)$$

where

$$r_f = \sum_{j=1}^{n_f} |f_j|, \qquad (21a)$$

$$\underline{F} = 0.5(f_0 - r_f), \qquad (21b)$$

$$\overline{F} = 0.5(f_0 + r_f) \qquad (21c)$$

if the operations in (20) are carried out.

At the end, we make the following conclusions based on the Theorem 2 Corollaries:

1) If all $\underline{F}_q > 0,\ q = 1,2,3$, then the system considered is stable.

2) If at least one of the endpoints $\overline{F}_q \le 0,\ q = 1,2,3$, then the system considered is not stable.

## 5 Numerical example

The applicability of the above technique will be illustrated by an example assessing the stability of the linear interval parameter system described by characteristic polynomial (1). In this example the order $n$ of the associate characteristic polynomial is $n = 5$, i.e.

$$q(s) = a_0 s^5 + a_1 s^4 + a_2 s^3 + a_3 s^2 + a_4 s + a_5 \qquad (22)$$

where

$$\begin{aligned}
&a_0(p) = 187\\
&a_1(p) = 80869 + 0.044p_3\\
&a_2(p) = 0.044p_1 + 2.556p_2 + 10.5p_3 + 4.064*10^7\\
&a_3(p) = 10.5p_1 + 2389.7p_2 + 340p_3 + 10^{-4}p_2p_3 +\\
&\quad + 3.638*10^9\\
&a_4(p) = 340p_1 + 2.139*10^5 p_2 + 10^{-4}p_1p_2 +\\
&\quad + 0.02p_2p_3 + 88.74*10^9\\
&a_5(p) = 0.02p_1p_2 + 5.218*10^6 p_2
\end{aligned} \qquad (22a)$$

It is seen from (22a) that the vector of parameters $p$ is 3-dimensional, i.e.

$$p = [p_1\ \ \ p_2\ \ \ p_3]^{\mathrm{T}}. \qquad (23)$$

The respective vectors of centers and radii are

$$p^0 = [14000\ \ \ 10000\ \ \ 480]^{\mathrm{T}} \qquad (23a)$$

and

$$R(p) = [3000\ \ \ 3000\ \ \ 100]^{\mathrm{T}}. \qquad (23b)$$

We formulate the Hurwitz determinant $\Delta_{n-1} = \Delta_4$ of order 4 and substitute (22a) in it. As a result we get $\Delta_4$ as explicit function of the system parameters $p_j,\ j = 1,2,3$. First, we determinate the values of the functions $a_0(p)$, $a_n(p)$ and $\Delta_{n-1}(p)$ for $p = p^0$. The results of the calculations are:

$$\left|\begin{aligned}
&a_0(p^0) = 187\\
&a_n(p^0) = a_5(p^0) = 5.2203*10^{10}\\
&\Delta_{n-1}(p^0) = \Delta_4(p^0) = 8.0770*10^{32}
\end{aligned}\right. . \qquad (24)$$

Then, we apply the simple mathematical operations “multiplication” and “linear combination" (in two cases – "addition" and "substraction") and get the following left bounds of interval extensions of the functions $a_0(p)$, $a_n(p)$ and $\Delta_{n-1}(p)$ when $p \in \boldsymbol{p}$:

$$\left| \begin{array}{l} \underline{a}_0 = 187 \\ \underline{a}_n = \underline{a}_5 = 3.6526*10^{10} \\ \underline{\Delta}_{n-1} = \underline{\Delta}_4 = 8.0234*10^{32} \end{array} \right. . \quad (25)$$

As it is seen from (24) the nominal ($p = p^0$ - the centre of $\boldsymbol{p}$) system described by the characteristic polynomial (22) is stable. It follows from (25) that the left bounds of all determined interval extensions are positive. Thus, based on Corollary 1 of Theorem 3 the system described by the characteristic polynomial (22) is stable.

## 6 Conclusion

A new interval technique for stability analysis of linear interval systems described by the characteristic polynomial (1) has been suggested. It is based on computing the interval extensions of the functions $a_0(p)$, $a_n(p)$ and $\Delta_{n-1}(p)$ when the coefficients $a_i$, $i = 0,1,...,n$ in the characteristic polynomial (1) are nonlinear functions of independent system parameters $p_j$, $j = 1,...,m$ which take their values in prescribed intervals $\boldsymbol{p}_j, j = 1,2,...,m$. The interval extensions considered are determined using modified affine arithmetic which provides the shortest outer bounds of the ranges studied. Two sufficient conditions for stability of the system considered are defined. A numerical example is solved at the end of the paper. In the example, the nominal system ($p = p^0$ - the centre of $\boldsymbol{p}$) is stable and all the interval extensions of the elements of the functions $a_0(p)$, $a_n(p)$ and $\Delta_{n-1}(p)$ are positive. Hence the system under consideration is stable.

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