Assessing the stability of linear time-invariant continuous interval dynamic systems

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***Abstract* – It is known that stability analysis of linear time-invariant dynamic systems under parameter uncertainties can be equated to estimating the range of the eigenvalues of matrices whose elements are intervals. In this paper, first the problem of finding tight outer bounds on the eigenvalue ranges is considered. A method for computing such bounds is suggested which consists, essentially, of setting up and solving a system of *n* mildly nonlinear algebraic equations, *n* being the size of the interval matrix investigated. The main result of the paper, however, is a method for determining the right end-point of the exact eigenvalue ranges. The latter makes use of the outer bounds. It is applicable if certain computationally verifiable monotonicity conditions are fulfilled. The methods suggested can be applied for robust stability analysis of both continuous- and discrete-time systems. Numerical examples illustrating the applicability of the new methods are also provided.**

***Index Terms* – Robust stability analysis, eigenvalues of interval matrices, outer bounds, right end-point of the eigenvalue range.**

I. INTRODUCTION

It is well known that stability analysis of linear time-invariant systems under parameter uncertainties can be formulated as the problem of estimating the range of the eigenvalues of interval matrices (matrices whose elements are independent intervals). The known methods for solving this problem (cf. e.g. [1] – [5]) only provide estimates that are outer bounds on the exact eigenvalue ranges. In some cases, these estimates may be rather conservative (they overestimate the range considerably) and lead to inconclusive stability analysis results.

In this paper, a method for determining the right end-point of the eigenvalue ranges of real interval matrices is suggested. This method (called for brevity exact method) is applicable if certain computationally verifiable monotonicity conditions are fulfilled. It is based on the use of an approximate method, which provides tight outer bounds on the eigenvalue ranges.

The paper is organized as follows. The problem statement is given in Section II. The method for obtaining an outer bound on the range of a given eigenvalue is presented in Section III. In the next section, the exact method is suggested. Numerical examples illustrating the applicability of the new methods are solved in Section V. The paper ends up with concluding remarks in Section VI.

II. PROBLEM STATEMENT

Let A be a real n x n matrix, $\boldsymbol{A}$ - an interval matrix containing A, and A^{-}, A^{+}, A^{0} and R – the left end, the right end, the center and the radius of $\boldsymbol{A}$, respectively (throughout the paper, bold face letters will be used to denote interval quantities while ordinary letters will stand for their non-interval counterparts). We are interested in the stability analysis of continuous interval dynamic systems described by

$$\dot{x}(t) = Ax(t), \; A \in \boldsymbol{A}, \; t \geq 0 . \tag{1}$$

We make the natural assumption that the nominal system corresponding to $A = A^0$ is (asymptotically) stable, i.e. all the eigenvalues $\lambda_k^0 = \lambda_k(A^0)$, $k = 1,...,n$, have negative real parts $\text{Re}\left[\lambda_k^0\right]$ and that λ_k^0 have been ordered in decreasing value of $\text{Re}\left[\lambda_k^0\right]$ (i.e., $\text{Re}\left[\lambda_1^0\right] \geq \text{Re}\left[\lambda_2^0\right] \geq ...$).

In accordance with the basic approach to investigating the (asymptotic) stability of (1) adopted in this paper, we consider the following "perturbed" eigenvalue problem

$$A\,x = \lambda\,x,\;\; A \in \boldsymbol{A} = [A^{-}, A^{+}] = A^{0} + [-R, R]\,. \tag{2}$$

Let λ_k denote an eigenvalue of A. As is seen from (2) each λ_k is a function of A, i.e. $\lambda_k = \lambda_k(A)$. Thus, we are led to consider the sets

$$S_k = \{\lambda_k(A) : A \in \boldsymbol{A}\},\;\; k = 1,\ldots,n \tag{3}$$

(where the number index k corresponds to the ordering of λ_k^0). In general, S_k may be a set of complex eigenvalues $\lambda_k(A)$. In view of the stability analysis, we introduce the (real) intervals

$$\boldsymbol{I}_k^* = \mathrm{Re}[S_k] = \{\mathrm{Re}[\lambda_k]:\;\lambda_k = \lambda_k(A),\;\; A \in \boldsymbol{A}\},\;\; k = 1,\ldots,n\,. \tag{4}$$

Let $\boldsymbol{I}_k^* = \left[\left(I_k^*\right)^-,\;\; \left(I_k^*\right)^+\right]$. It is well known (e.g. [5]) that the continuous interval dynamic system (1) is stable iff (if and only if)

$$\left(I_k^*\right)^+ < 0,\;\; \forall k \in \{1,\ldots,n\}. \tag{5}$$

Thus, in order to establish the stability of (1), we need a method for computing the right end-point of each interval (4). Knowing these points, we can, in fact, determine the true margin of stability.

Remark 1: In practice, the stability margin of (1) can be determined with reasonable certainty if we confine ourselves to computing the first few values $\left(I_k^*\right)^+,\;\; k \in K = \{1,\ldots,n'\},\;\; n'$ being typically equal to 2.

In the next section, we shall first obtain outer bounds $\boldsymbol{I}_k$ on the interval $\boldsymbol{I}_k^*$. In Section IV, it will be shown how, using these bounds, the right end-point of the exact interval $\boldsymbol{I}_k^*$ can be determined, provided that certain monotonicity conditions are fulfilled.

III. OUTER BOUNDS ON THE EIGENVALUE RANGE

Consider again (2). It is seen from (2) that both λ and x are functions of A, i.e. $\lambda = \lambda(A)$ and $x = x(A)$. Let $x^{(k)}(A) = \left(x_1^{(k)}(A),\ x_2^{(k)}(A),\ \ldots\ ,\ x_n^{(k)}(A)\right)^{\mathrm{T}}$ be an eigenvector, corresponding to $\lambda_k(A)$ for a fixed $k \in K$ (T stands for transpose). Now let the pair $\left(\lambda^0,\ x^0\right)$ be the solution of the nominal (centre) problem

$$A^0 x = \lambda x. \tag{6}$$

To simplify the presentation of the method for obtaining outer bounds, we will first consider the case where the first *K* components λ_k of the eigenvalue vector λ^0 are real. We need the following assumption (ensuring structural stability of the problem).

Assumption A1: For $k \in K$ all $\lambda_k(A)$ and $x^{(k)}(A)$ remain real for all $A \in \boldsymbol{A}$.

On account of Assumption A1, the sets S_k from (3) for $k \in K$ will, in this case, be real intervals

$$\boldsymbol{I}_k^* = \{\lambda_k(A) : A \in \boldsymbol{A}\},\ \ k \in K. \tag{7}$$

Thus, $\boldsymbol{I}_k^*$ is, in this case, the range of $\lambda_k(A)$, when $A \in \boldsymbol{A}$.

For notational simplicity, we shall henceforth drop the index k. We are interested in finding an outer bound $\boldsymbol{I}$ on $\boldsymbol{I}^*$, i.e. an interval $\boldsymbol{I} = \left(I^-,\ \ I^+\right)$ with the property

$$\boldsymbol{I}^* \subset \boldsymbol{I}. \tag{8}$$

Thus, the problem at hand is the following

Problem P1: Find an outer bound $\boldsymbol{I}$ on $\boldsymbol{I}^*$, i.e. an estimation $\boldsymbol{I}$ having the inclusion property (8).

We now suggest a method for finding a "tight" outer bound $\boldsymbol{I}$ on $\boldsymbol{I}^*$, i.e. a bound with a small overestimation. To simplify the presentation of the method, we assume that the nth component x_n^0 of x^0 has the largest absolute value, i.e. $|x_n^0| \geq |x_i^0|$, $i = 1,...,n-1$ (this assumption is trivial and can be always achieved by reordering the components of x^0). We normalize the vector x^0 (dividing x^0 by x_n^0) to have

$$x_n^0 = 1\,. \tag{9a}$$

Assumption A2: We assume that (9a) is also valid for $x_n(A)$, i.e.

$$x_n(A) = 1,\ A \in \boldsymbol{A}\,. \tag{9b}$$

At this point, introduce the n-dimensional real vector

$$y = (y_1, y_2, ..., y_n)^{\mathrm{T}}, \quad y_i = x_i(A),\ i = 1,...,n-1,\ y_n = \lambda(A)\,. \tag{10}$$

Using (10) and (9), (2) is rewritten as

$$\sum_{j=1}^{n-1} a_{ij} y_j - y_n y_i + a_{in} = 0,\ i = 1,...,n-1, \quad \sum_{j=1}^{n-1} a_{nj} y_j - y_n + a_{nn} = 0\,, \tag{11a}$$

$$a_{ij} \in \boldsymbol{a}_{ij} = [a_{ij}^- \quad a_{ij}^+]\,, \tag{11b}$$

where a_{ij}^- and a_{ij}^+ are the elements of matrices A^- and A^+, respectively. System (11a) is a nonlinear (more precisely, an incomplete quadratic) system because of the products $y_n y_i$ in the first $n-1$ equations in it.

Let $\boldsymbol{y}_i^*$ denote the range of the ith component $y_i(A)$, $A \in \boldsymbol{A}$ of the solution y to (11). Let $\boldsymbol{y}^*$ be the vector made up of $\boldsymbol{y}_i^*$. Consider the following problem.

Problem P2: Find an outer solution $\boldsymbol{y}$ to (11), i.e. a solution enclosing the range vector $\boldsymbol{y}^*$:

$$\boldsymbol{y}^* \subset \boldsymbol{y}\,. \tag{12}$$

Obviously, the nth component of the solution $\boldsymbol{y}$ to Problem P2 is a solution to the original Problem P1.

We now proceed to solving Problem P2. The approach adopted is based on ideas suggested recently in [6], [7]. If $z = z_0 + u \in \boldsymbol{z}$ and $t = t_0 + v \in \boldsymbol{t}$, with $\boldsymbol{z}$ and $\boldsymbol{t}$ being intervals whose centers are z_0 and t_0, respectively, then

$$zt \in -z_0 t_0 + t_0 z + z_0 t + [-r_z r_t,\ r_z r_t] \tag{13}$$

where r_z and r_t are the respective radii. After letting

$$a_{ij} = a_{ij}^0 + u_{ij}\,,\ \ y_i = y_i^0 + v_i\,,\ i, j = 1,\ldots, n \tag{14}$$

where a_{ij}^0 are the elements of the centre matrix A^0 and y_i^0 are computed from (10) with $A = A^0$, we apply (13) to express the products in (11a). On substitution of (14) into (11a), having in mind that the centers a_{ij}^0 and y_i^0 satisfy system (11a) and following the techniques of [6], we get the system

$$a_{i,1}^0 v_1 + \ldots + (a_{i,i}^0 - y_n^0) v_i + \ldots + a_{i,n-1}^0 v_{n-1} - y_i^0 v_n = \boldsymbol{b}_i,\ i = 1,\ldots,n-1,\ \ a_{n1}^0 v_1 + a_{n2}^0 v_2 + \ldots + a_{n,n\text{-}1}^0 v_{n-1} - v_n = \boldsymbol{b}_n \tag{15a}$$

where $\boldsymbol{b}_i,\ i = 1,\ldots,n$ are intervals. These intervals are to be chosen in such a way that the solution set to (15a), displaced by y^0, should contain the solution set to the original system (11). It can be easily checked that the radii of $\boldsymbol{b}_i$ are

$$\begin{aligned} R(\boldsymbol{b}_i) &= \sum_{j=1}^{n-1} \left|y_j^0\right| R_{ij} + R_{in} + \sum_{j=1}^{n-1} R_{ij} r_j + r_n r_i,\ i = 1,2,\ldots,n-1\ , \\ R(\boldsymbol{b}_n) &= \sum_{j=1}^{n-1} \left|y_j^0\right| R_{nj} + R_{nn} + \sum_{j=1}^{n-1} R_{nj} r_j \end{aligned} \tag{15b}$$

where R_{ij} are the elements of R while $r_i = R(\boldsymbol{v}_i)$ is the radius of the unknown interval $\boldsymbol{v}_i$.

Now system (15) can be written in compact form

$$\widetilde{A}_0 v = b,\ \ b \in \boldsymbol{b} \tag{16}$$

where $\tilde{A}_0$ is the real coefficient matrix in (15a). Now we need the following assumption.

Assumption A3: The matrix $\tilde{A}_0$ is non-singular.

Let $C = \left|\tilde{A}_0^{-1}\right|$. If $r = (r_1, \quad r_2, \quad \ldots \quad , r_n)^{\mathrm{T}}$ and $r_{\boldsymbol{b}}$ denotes a column vector with components from (15b), then from (16)

$$r = Cr_{\boldsymbol{b}}\,. \tag{17}$$

Now we introduce the matrix $\breve{R}$ which is the same as R except for the last column whose elements are now zeros. Using (15b) and the new notation, (17) becomes

$$r = CR\left|x^0\right| + C\breve{R}r + Cg(r) \tag{18}$$

where x^0 is the normalized eigenvector and $g(r)$ is a nonlinear function with components $g_i(r) = r_i r_n$, $i = 1,2,\ldots,n-1$, $g_n(r) = 0$. Finally

$$r = d + Dr + Cg(r)\,, \tag{19a}$$

$$d = CR\left|x^0\right|, \quad D = C\breve{R}\,. \tag{19b}$$

The matrix equation (19a) is a nonlinear real-valued (non-interval) system of n equations in n unknowns r_i:

$$r_i = d_i + \sum_{j=1}^{n-1} d_{ij} r_j + r_n \sum_{j=1}^{n-1} c_{ij} r_j, \ i = 1,\ldots,n\,. \tag{20}$$

The smallest positive solutions r_i to (20) solve Problem P2. Indeed, if $r_i > 0$, we can introduce the intervals

$$\boldsymbol{y}_i = y_i^0 + [-r_i,\ r_i],\ i = 1,2,\ldots,n\,. \tag{21}$$

It can be proved that

$$\boldsymbol{y}_i^* \subset \boldsymbol{y}_i,\ i = 1,2,\ldots,n\,, \tag{22}$$

i.e. the intervals (21) are really outer bounds on the ranges $\boldsymbol{y}_i^*$ for all i. Hence

$$\boldsymbol{y}_n = y_n^0 + [-r_n,\ r_n] \tag{23}$$

is a solution to the original problem P1 since it is, in fact, a bound $\boldsymbol{I}$ on $\boldsymbol{I}^*$ satisfying the inclusion (8). More precisely, we have the following theorem.

Theorem 1: If the nonlinear system (20) has a positive solution $r = (r_1, \quad r_2, \quad \ldots \quad , r_n)^{\mathrm{T}}$, then the interval (23) is an outer bound on the range $\boldsymbol{I}^*$ of the real eigenvalue $\lambda_k(A)$ considered (for a given k from K).

The proof of Theorem 1 is based on the general approach of [7].

The present method for solving the original Problem P1 will be referred to as Method M1. As shown above, it comprises, essentially, the following computations. First, the "nominal" eigenvalue problem (6) is solved. Then, for each $k \in K$, the nonlinear system (20) is set up and solved. If all r_i found are positive, the outer bound $\boldsymbol{I}$ on the corresponding eigenvalue $\lambda_k(A),\ A \in \boldsymbol{A}$, is obtained by the interval (23). Since, in practice, the values of the R_{ij} are small relative to the magnitudes of the a_{ij}^0, system (20) is mildly nonlinear and its solution does not present any difficulties.

Remark 2: In our implementation, the simple iteration method, the method *fsolve* from MATLAB (using the initial vector $r^0 = 0$) as well as the method from [6] have been applied to solve (20).

Method M1 can be easily extended to the case of complex eigenvalues λ_k. Indeed, for a fix k let

$$\lambda = \lambda_{\mathrm{Re}} + j\lambda_{\mathrm{Im}}\ ,\quad x_i = x_{i,\mathrm{Re}} + jx_{i,\mathrm{Im}},\ i = 1,2,\ldots,n\,. \tag{24}$$

We introduce the $2n$-dimensional real vector y with components

$$y_i = x_{i,\mathrm{Re}}(A),\ y_n = \lambda_{\mathrm{Re}}(A)),\ y_{n+i} = x_{i,\mathrm{Im}}(A),\ y_{2n} = \lambda_{\mathrm{Im}}(A),\ i = 1,...,n-1. \quad (25)$$

We then apply the approach of Method M1 to obtain a nonlinear system of the type (19a) (the only difference is that now r has $2n$ components). If r is a positive solution the interval

$$\boldsymbol{y}_n' = y_n^0 + [-r_n,\ r_n] \quad (26)$$

determines an outer bound on the range of $\mathrm{Re}[\lambda_k(\boldsymbol{A})]$.

IV. Determination of the right end-point of the eigenvalue range

In this section, we are interested in determining the right end-points of the ranges $\boldsymbol{I}_k^*$ defined by (7) and (4), respectively. A method capable of finding these points will be suggested here. For the sake of simplicity, it will be presented only for the case of (7) (when the eigenvalues $\lambda_k(A)$ corresponding to all $A \in \boldsymbol{A}$ remain real).

A. Basic method. As in Section III, after dropping the index k and introducing vector (10), we rewrite system (2) in the form

$$\begin{aligned} f_i(a_{ij}, y_j) &= \sum_{j=1}^{n-1} a_{ij} y_j - y_n y_i + a_{in} = 0,\ i = 1,...,n-1 \\ f_n(a_{ij}, y_j) &= \sum_{j=1}^{n-1} a_{nj} y_j - y_n + a_{nn} = 0 \end{aligned} \quad ,\ a_{ij} \in \boldsymbol{a}_{ij}. \quad (27)$$

Let the derivative $\dfrac{\partial y_p}{\partial a_{lm}}$ $(p,l,m = 1,2,...,n)$ be denoted $d_{lm}^{(p)}$. We are interested only in $d_{lm}^{(n)}$ denoted for simplicity d_{lm}. From (27) d_{lm} depend on a_{ij}, i.e. $d_{lm} = d_{lm}(A)$. Suppose that an outer bound $\boldsymbol{d}_{lm}$ on d_{lm} is known when $A \in \boldsymbol{A}$, i.e.

$$d_{lm}(A) \in \boldsymbol{d}_{lm},\ \forall A \in \boldsymbol{A}. \quad (28)$$

At this point, we need the following assumption.

Assumption A4: The outer bound $\boldsymbol{d}_{lm} = [d_{lm}^-,\ d_{lm}^+]$ is either positive or negative, i.e.

$$d_{lm}^- \ge 0 \quad (29a)$$

or

$$d_{lm}^{+} \le 0 \qquad (29b)$$

for all $l, m = 1,...,n$.

On account of (29) and (28), if (29a) is valid, then y_n is monotonously increasing in a_{lm} for all possible $A \in \boldsymbol{A}$; also if (29b) holds, then y_n is monotonously decreasing in a_{lm} for all possible $A \in \boldsymbol{A}$. Therefore, the right end-point of range $\boldsymbol{y}_n^*$ can be determined in the following way. First, we form the matrix A' whose elements a'_{lm} are

$$a'_{lm} = \begin{cases} a_{lm}^{+}, \text{ if } d_{lm}^{-} \ge 0 \\ a_{lm}^{-}, \text{ if } d_{lm}^{+} \le 0 \end{cases}. \qquad (30)$$

We then solve the eigenvalue problem

$$A'x = \lambda x \qquad (31)$$

to obtain the eigenvalues λ' (corresponding to the kth real eigenvalue, $k \in K$). Finally, the right end-point of the range $\boldsymbol{I}^*$ considered is determined by the following theorem.

Theorem 2: If the monotonicity conditions (29) are valid, the elements of the matrix A' are defined as in (30) and the eigenvalues λ'_i, $i = 1,...,n$, are computed from (31), then the right end-point of the range $\boldsymbol{I}_k^*$ is given by the eigenvalue λ'_k.

The proof of Theorem 2 is straightforward and is based on the monotonicity conditions from Assumption A4.

To apply Theorem 2, we need the outer bounds $\boldsymbol{d}_{lm}$. They can be found in the following manner. We differentiate (27) with respect to a_{lm} to get n^2 systems

$$\frac{\partial f_i}{\partial a_{lm}} + \sum_{p=1}^{n} \frac{\partial f_i}{\partial y_p} \frac{\partial y_p}{\partial a_{lm}} = 0,\ i = 1,...,n,\ l, m = 1,...,n\ . \qquad (32)$$

For fixed indices l and m, (32) is written as

$$\sum_{j=1}^{n-1} a_{ij} \frac{\partial y_j}{\partial a_{lm}} - y_n \frac{\partial y_i}{\partial a_{lm}} - y_i \frac{\partial y_n}{\partial a_{lm}} = -\delta_{il}\delta_{nm} - \delta_{il}\gamma_{jm} y_m, \; i = 1,...,n-1, \tag{33a}$$

$$\sum_{j=1}^{n-1} a_{nj} \frac{\partial y_j}{\partial a_{lm}} - \frac{\partial y_n}{\partial a_{lm}} = -\delta_{nl}\delta_{nm} - \delta_{nl}\gamma_{nm} y_m \tag{33b}$$

where δ_{il} is the Kronecker symbol while γ_{im} = 1 for $m < n$ and γ_{im} = 0 if $m = n$. Consider the above system. To find an outer bound $\boldsymbol{d}_{lm}$ on the derivative $\frac{\partial y_n}{\partial a_{lm}}$, we first have to let a_{ij}, y_i, y_m and y_n vary within their respective intervals

$$a_{ij} \in \boldsymbol{a}_{ij}, \; y_p \in \boldsymbol{y}_p, \; i, j, p = 1,...,n. \tag{33c}$$

The intervals $\boldsymbol{y}_p$ in (33c) are computed using Method M1 from Section III. The resulting complete system (33a) to (33c) is, in fact, a linear interval system (see, e.g., [5] and the references cited therein) which is written as

$$\boldsymbol{B}z = \boldsymbol{b}. \tag{34}$$

We find the outer solution $\boldsymbol{z}$ to (34) using the linear version of Method M1 (with $g = 0$ in (18)). Finally, the outer bound $\boldsymbol{d}_{lm}$ sought (on the derivative $\frac{\partial y_n}{\partial a_{lm}}$) is determined as the nth component $\boldsymbol{z}_n$ of the outer solution $\boldsymbol{z}$ to the linear interval system (34). The above method for finding the right end-point of the range for the real eigenvalue will be referred to as Method M2 or exact method.

Remark 3: Method M2 is directly applicable to determine the right end-point of the range $\boldsymbol{I}^*_{\mathrm{Re}}$ for the real part of a complex eigenvalue. The only difference is that now the size of all systems involved is doubled.

B. Improved method. An improved version of Method M2 will be presented here which may find the end-point sought even in the case where some of the monotonicity conditions (29) are violated. The elements a_{lm}, for which conditions (29) hold, are fixed at their end-point values $a^{'}_{lm}$. The remaining elements are treated as entries of an interval matrix $\boldsymbol{A}_\rho$ of reduced size. Now Method M2 is applied to $\boldsymbol{A}_\rho$. If all monotonicity conditions (29), related to $\boldsymbol{A}_\rho$, are satisfied, the problem is solved. Otherwise, $\boldsymbol{A}_\rho$ is again reduced and a new iteration is initialized. This scheme of repeated application of Method M2 to reduced matrices of smaller and smaller size will be referred to as Method M3.

IV. Numerical examples

Example 1: In this example, $n = 2$ and the interval matrix $\boldsymbol{A}$ of the dynamic system studied is $\boldsymbol{A} = A^0 + [-R,\ R]$ with

$$a_{11}^0 = -3.8, \quad a_{12}^0 = 1.6, \quad a_{21}^0 = 0.6, \quad a_{22}^0 = -4.2\,, \tag{35a}$$

$$R_{ij}(\boldsymbol{A}) = 0.17,\ i, j = 1, 2\,. \tag{35b}$$

For this example $\lambda^0 = [-3,\ -5]^{\mathrm{T}}$ and we will confine ourselves to finding an outer bound $\boldsymbol{I}$ on the range $\boldsymbol{I}^*$ for the first eigenvalue $\lambda_1^0 = -3$ (since it is closer to zero). The eigenvector $x^0 = x^{0,(1)} = (x_1^0,\ x_2^0)$ associated with λ_1^0 is $x^0 = [0.894427,\ 0.447214]^{\mathrm{T}}$. Since $|x_1^0| > |x_2^0|$, the normalized vector is $x^0 = [1,\ 0.5]^{\mathrm{T}}$. Thus, the vector y^0 is $y^0 = [y_1^0,\ y_2^0]^{\mathrm{T}} = [\lambda^0,\ x_2^0]^{\mathrm{T}} = [-3,\ 0.5]^{\mathrm{T}}$ and $y = [y_1,\ y_2]^{\mathrm{T}} = [\lambda\ ,\ x_2]^{\mathrm{T}}$. So for this example system (11a) becomes

$$a_{11} + a_{12}y_2 - y_1 = 0, \quad a_{21} + a_{22}y_2 - y_1 y_2 = 0\,. \tag{36}$$

On account of (36), the system (20) now is

$$r_1 = 0.3570 + 0.2380 r_1 + 0.8000 r_1 r_2, \;\; r_2 = 0.1913 + 0.1275 r_1 + 0.5000 r_1 r_2 \,. \quad (37)$$

The solution to (37), obtained by the simple iteration method, has the components

$$r_1 = 0.3323, \quad r_2 = 0.5941 \,. \quad (38)$$

As both radii are positive, by Theorem 1 and (38) the outer bound $\boldsymbol{I}$ sought is

$$\boldsymbol{I} = \boldsymbol{y}_1 = y_1^0 + [-r_1,\; r_1] = [-3.3323,\; -2.6677] \,. \quad (39)$$

Thus, in view of (39), we conclude that the continuous dynamic system (1), whose interval matrix $\boldsymbol{A}$ is defined by (35), is stable. From (39) the stability margin M obtained by Method M1 is $M = -\left(I^+\right) = 2.6677$.

This example has been solved in [2] and [4], and the corresponding results for the margin M are $M = 2.5931$ and $M = 2.3098$, respectively. It is seen that the present paper's method M1 provides a more accurate estimate on the stability margin.

Example 2: We take up Example 1 where the components of the radius matrix R are however all equal to 0.1. We now want to find the right end-point of the exact range $\boldsymbol{I}^*$ associated with $\lambda_1(A)$. We apply Method M2 to solve the problem. The interval derivatives $\boldsymbol{d}_{lm}$, $l, m = 1, 2$ were computed as explained in Section IV and they all turned out to be positive. So the matrix $A^{'}$ can be written in the form $A^{'} = A^0 + R = \begin{bmatrix} -3.7 & 1.7 \\ 0.7 & -4.1 \end{bmatrix}$.

The eigenvalue $\lambda_k^{'}$ of matrix $A^{'}$, corresponding to $k = 1$, is $\lambda_1^{'} = -2.7909$. By Theorem 2, the right end-point of the exact range for $\lambda_1(A),\; A \in \boldsymbol{A}$, is $\left(I^*\right)^+ = -2.7909$. We also applied Method M1 to find the right end-point I^+ of the outer bound $\boldsymbol{I}$ on $\lambda_1(A),\; A \in \boldsymbol{A}$, which is $I^+ = -2.7761$. It is seen that $\left(I^*\right)^+ < I^+$; hence the exact (within the computation errors) stability margin is $M = -\left(I^*\right)^+ = 2.7909$.

Example 3: We consider a model of type (1) for a hard disk drive. It is 8-dimensional ($n = 8$) and the nonzero elements of the centre matrix A^0 are:

$$\begin{array}{llllll} a^0_{12} = 439.82 & a^0_{21} = -439.82 & a^0_{22} = -43.983 & a^0_{34} = 13823 & a^0_{43} = -13823 & \\ a^0_{44} = -138.23 & a^0_{56} = 25133 & a^0_{65} = -25133 & a^0_{66} = -2513.3 & a^0_{77} = -565.49 & \\ a^0_{81} = 0.00115 & a^0_{82} = -0.5750 & a^0_{84} = 2.3010 & a^0_{86} = 82.637 & a^0_{87} = 16.427 & a^0_{88} = -12.556 \end{array} \quad . \quad (40a)$$

The corresponding elements of the radius matrix R are:

$$\begin{array}{llllll} r_{12} = 21.991 & r_{21} = 21.991 & r_{22} = 6.8170 & r_{34} = 1382.3 & r_{43} = 1382.3 & \\ r_{44} = 74.644 & r_{56} = 1256.6 & r_{65} = 1256.6 & r_{66} = 1181.2 & r_{77} = 325.16 & \\ r_{81} = 2.449*10^{-4} & r_{82} = 0.0419 & r_{84} = 0.5680 & r_{86} = 15.716 & r_{87} = 3.1240 & r_{88} = 1.3530 \end{array} \quad . (40b)$$

For this example, the eigenvalues of the centre matrix are:

$$\begin{array}{l} \lambda^0_1 = -12.556\,,\ \lambda^0_2 = -21.991 + j439.27\,,\ \lambda^0_3 = -21.991 - j439.27\,, \\ \lambda^0_4 = -69.115 + j13823\,,\ \lambda^0_5 = -69.115 - j13823\,,\ \lambda^0_6 = -564.94\,, \\ \lambda^0_7 = -1256.6 + j25103\,,\ \lambda^0_8 = -1256.6 - j25103 \end{array} \qquad (41)$$

First, we will find an outer bound $\boldsymbol{I}$ on the range $\boldsymbol{I}^*$ for the first eigenvalue $\lambda^0_1 = -12.556$ (since it is the nearest to zero). The maximum absolute value component of the corresponding eigenvector x^0_1 is in the 8th position. Thus, the system (20) now is

$$\begin{aligned} r_1 &= 0.0501 r_1 + 0.0191 r_2 + 1.6280*10^{-4} r_1 r_8 + 22.783*10^{-4} r_2 r_8, \\ r_2 &= 0.0014 r_1 + 0.0506 r_2 + 22.783*10^{-4} r_1 r_8 + 0.6500*10^{-4} r_2 r_8, \\ r_3 &= 0.1000 r_3 + 0.00631 r_4 + 0.6580*10^{-6} r_3 r_8 + 0.7230*10^{-4} r_4 r_8, \\ r_4 &= 0.9083*10^{-4} r_3 + 0.1000 r_4 + 0.7230*10^{-4} r_3 r_8 + 0.6580*10^{-7} r_4 r_8, \\ r_5 &= 0.0500 r_5 + 0.0520 r_6 + 0.3960*10^{-5} r_5 r_8 + 0.3980*10^{-4} r_6 r_8, \\ r_6 &= 0.2500*10^{-4} r_5 + 0.0500 r_6 + 0.3980*10^{-4} r_5 r_8 + 0.1990*10^{-7} r_6 r_8, \\ r_7 &= 0.5880 r_7 + 0.0018 r_7 r_8, \\ r_8 &= 1.3530 + 0.0011 r_1 + 0.0710 r_2 + 2.0900*10^{-4} r_3 + 0.3380 r_4 + 20.643*10^{-4} r_5 + \\ &\quad + 11.582 r_6 + 12.784 r_7 + 0.0013 r_1 r_8 + 0.4000*10^{-4} r_2 r_8 + 1.6650*10^{-4} r_3 r_8 + \\ &\quad + 0.1510*10^{-6} r_4 r_8 + 0.0033 r_5 r_8 + 0.1640*10^{-5} r_6 r_8 + 0.0297 r_7 r_8. \end{aligned} \qquad (42)$$

The solution to (42) has been obtained by the simple iteration method. All radii are positive and $r_8 = 1.3827$. Therefore, by Theorem 1 the outer bound $\boldsymbol{I}$ sought is

$$\boldsymbol{I} = \left[I^-, \quad I^+\right] = \lambda_1^0 + [-r_8,\ r_8] = [-13.938,\ -11.173]. \tag{43}$$

It can be checked that the perturbations of the real part of $\lambda_2(A),\ A \in \boldsymbol{A}$ remain to the left of I^+. Thus, in view of (43), we conclude that the continuous dynamic system (1), whose interval matrix $\boldsymbol{A}$ is defined by (40), is stable. From (43) the stability margin M_1 obtained by Method M1 is thus $M_1 = 11.173$.

Next, we apply Method M2 to find the right end-point of the exact range $\boldsymbol{I}^*$ associated with $\lambda_1(A)$. The interval derivatives $\boldsymbol{d}_{lm},\ l,m = 1,...,8$ were computed as explained in Section IV and the nonzero values are:

$$\begin{array}{llllll} \boldsymbol{d}_{12} > 0 & \boldsymbol{d}_{21} < 0 & \boldsymbol{d}_{22} < 0 & \boldsymbol{d}_{34} > 0 & \boldsymbol{d}_{43} < 0 & \\ \boldsymbol{d}_{44} < 0 & \boldsymbol{d}_{56} > 0 & \boldsymbol{d}_{65} < 0 & \boldsymbol{d}_{66} < 0 & \boldsymbol{d}_{77} < 0 & \\ \boldsymbol{d}_{81} > 0 & \boldsymbol{d}_{82} < 0 & \boldsymbol{d}_{84} > 0 & \boldsymbol{d}_{86} > 0 & \boldsymbol{d}_{87} > 0 & \boldsymbol{d}_{88} < 0 \end{array} \tag{44}$$

So matrix A' can be formulated using conditions (29). The first component λ_1' of λ' is $\lambda_1' = -11.968$. By Theorem 2, the right end-point of the range of $\lambda_1(A),\ A \in \boldsymbol{A}$ is $\left(I^*\right)^+ = -11.968$. As expected, $\left(I^*\right)^+ < I^+$ and the true stability margin is $M_2 = -\left(I^*\right)^+ = 11.968 > M_1$.

VI. Conclusion

Two methods for stability analysis of linear time-invariant interval dynamic systems (1) have been suggested. They are based on computing estimates for the ranges (4) and (7) associated with the real parts of the eigenvalues of the system interval matrix $\boldsymbol{A}$. The first

Method M1 is an approximate method since it only provides outer bounds on the respective ranges. The second Method M2 is an exact method. If the monotonicity conditions (29) are satisfied, Method M2 yields the right end-point of the ranges. An improved version (method M3) has also been suggested which is capable of solving the problem even if not all monotonicity conditions are fulfilled. The applicability of the methods suggested has been illustrated by an 8-dimensional numerical example.

The present methods can be extended to encompass systems in which the elements a_{ij} of matrix A depend on a certain number of interval parameters $\boldsymbol{p}_i$, $i = 1,\ldots,q$. Such a generalization will be reported in a future publication.

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