Outer bounds on the real eingenvalues of interval matrices

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Abstract - **Stability analysis of linear circuits and systems under interval parameter uncertainties can be equated to estimating the eigenvalues of interval matrices. In this paper, the problem of determining outer bounds on the ranges of the real eigenvalues is considered. A method for computing such bounds is suggested. It consist of setting up and solving a system of n nonlinear equations, n being the size of the original square interval matrix. The latter system is only mildly nonlinear and its solution poses no numerical difficulties.**

An example illustrating the applicability of the method suggested is provided.

The approach adopted in bounding the real eigenvalues is rather general and can be extended to encompass the case of complex eigenvalues as well as to problems where the matrix elements are nonlinear functions of given interval parameters.

Index Terms - **Robust stability analysis, eigenvalues of interval matrices.**

I. INTRODUCTION

IT is well known that stability analysis of linear circuits and systems under parameter uncertainties can be formulated as the problem of estimating the range of the eigenvalues of interval matrices (see e.g. [1] – [5]). Such an approach is associated with an inherent difficulty which consists in the fact that the estimates thus obtained may be rather conservative. Therefore, there may be cases where the analysis results are inconclusive. Indeed, such a situation can arise when a small part of the estimation of the eigenvalue range lies in the right half of the complex plane.

In this paper, a new method for obtaining bounds on the real eigenvalues of interval matrices is suggested. It guaranties that the bounds are outer bounds, i.e. they contain the actual eigenvalue ranges. Furthermore, the method yields bounds that seem to be rather tight, i.e. with relatively small conservatism. From computational point of view, the method suggested reduces to solving an associated non-linear algebraic system than contains n equations of n unknowns, respectively, n being the size of the square interval matrix investigated.

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II. PROBLEM STATEMENT

Let A be a real n x n matrix, $\mathbf{A}$ - an interval matrix containing A, and A^-, A^+, A_0 and R_A – the left end, the right end, the center and the radius of $\mathbf{A}$, respectively. We consider the following "perturbed" eigenvalue problem:

$$A.x = \lambda.x,\ A\in \mathbf{A} = [\,A^-, A^+] = A_0 + [\,-R_A, R_A] \qquad (1)$$

In this paper, we are interested in the real eingenvalues of (1). Let $\lambda^*(\mathbf{A})$ denote such a real eigenvalue while $x^{(k)}(A) = (x_1^{(k)}(A), x_2^{(k)}(A), \dots, x_n^{(k)}(A))$ be the corresponding (real) eigenvector, k = 1, ... , n', n' ≤ n. We need the following assumption (ensuring structural stability of the problem).

Assumption A_1: For any $k\in K$ = 1, ... , n', all $\lambda^{(k)}(A)$ and $x^{(k)}(A)$, corresponding to all $A\in \mathbf{A}$, remain real.

For simplicity, we shall henceforth drop the index k. On account of Assumption A_1, the range

$$\boldsymbol{\lambda}^* = \{\,\lambda(A): A \in \mathbf{A}\} \qquad (2)$$

is a real interval. In checking the dynamic stability of linear systems, the ideal would be to determine $\boldsymbol{\lambda}^*$. Since presently this seems to be an intractable problem, we usually settle for an outer approximation $\boldsymbol{\lambda}$ of $\boldsymbol{\lambda}^*$, i.e. $\boldsymbol{\lambda}$ must include $\boldsymbol{\lambda}^*$:

$$\boldsymbol{\lambda}^* \subset \boldsymbol{\lambda} \qquad (3)$$

Thus, the problem at hand is the following:

Problem P_1: Find an outer bound $\boldsymbol{\lambda}$ on $\boldsymbol{\lambda}^*$, i.e. an estimation of $\boldsymbol{\lambda}^*$ having the inclusion property (3).

In this paper, we suggest a method for finding a "good" outer bound $\boldsymbol{\lambda}$ on $\boldsymbol{\lambda}^*$, that is a bound with a small overestimation.

To simplify presentation of the method (without any loss of generality), we need a second assumption. Let the pair (x^0, λ^0) be the solution of the (center, nominal) problem

$$A_0.x = \lambda.x \qquad (4)$$

Assumption A_2: We assume that the absolute value of the n-th component $\left|x_n^0\right|$ of x^0 is the largest component of the other components, i.e.

$$\left|x_n^0\right| \ge \left|x_i^0\right|,\ i \ne n \qquad (5)$$

Now x^0 is normalized by letting

$$\left|x_n^0\right| = 1 \qquad (6)$$

Further, we require that (6) be also valid for

$$\left| x_n(A) \right| = 1, \ \forall A \in \boldsymbol{A} \qquad (6')$$

Condition (6') simplifies the method which will be presented in the next section.

III. Problem Solution

We introduce the n-dimensional real vector

$Y = (y_1, y_2, \ldots, y_n)$

with

$$y_i = x_i(A), \ i = 1, \ldots, (n-1) \qquad (7a)$$

$$y_n = \lambda(A), \qquad (7b)$$

Using (7) and (6), (1) is rewritten as

$$\begin{cases} a_{11}.y_1 + a_{12}.y_2 + \ldots + a_{1,(n-1)}.y_{n-1} - y_n.y_1 + a_{1n} = 0 \\ a_{21}.y_1 + a_{22}.y_2 + \ldots + a_{2,(n-1)}.y_{n-1} - y_n.y_2 + a_{2n} = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n1}.y_1 + a_{n2}.y_2 + \ldots + a_{n,(n-1)}.y_{n-1} - y_n + a_{nn} = 0 \end{cases} \qquad (8a)$$

where

$$a_{ij} \in \boldsymbol{A}_{ij} = [A_{ij}^-, A_{ij}^+] \qquad (8b)$$

System (8) is a non-linear (more precisely, a quadratic) interval system because of the products $y_n.y_i$ in the first (n-1) equations in (8a).

Let y_i^* denote the range of the i-th component $y_i(A)$, $A \in \boldsymbol{A}$, of the solution to (8). Let Y^* be the vector made up of y_i^*. Consider the following problem:

Problem P_2: Find an outer solution $\boldsymbol{Y}$ to (8), i.e. a solution enclosing the range vector Y^*:

$$Y^* \subset Y \qquad (9)$$

Obviously, the n-th component of the solution $\boldsymbol{Y}$ to Problem P_2 is a solution to the original Problem P_1.

We now proceed to solving Problem P_2. The approach adopted is based on an idea suggested recently in [6]. If $z = z_0 + u \in Z$ and $t = t_0 + v \in T$, with Z and T being intervals whose centers are z_0 and t_0, respectively, then

$$u.v \in \ t_0.u + z_0.v + [-R_u.R_v, R_u.R_v] \qquad (10)$$

where R_u and R_v are the respective radiuses. We apply (10) to express the products in (8a) after letting

$$a_{ij} = a_{ij}^0 + u_{ij}, \ y_i = y_i^0 + v_i \qquad (11)$$

Having in mind that the centers a_{ij}^0 and y_i^0 satisfy system (8a) and following the techniques of [6] we get the system:

$$\begin{cases} (a_{11}^0 - y_n^0).v_1 + a_{12}^0.v_2 + \ldots + a_{1,(n-1)}^0.v_{n-1} - y_1^0.v_n = B_1 \\ a_{21}^0.v_1 + (a_{22}^0 - y_n^0).v_2 + \ldots + a_{2,(n-1)}^0.v_{n-1} - y_2^0.v_n = B_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{(n-1),1}^0.v_1 + a_{(n-1),2}^0.v_2 + \ldots + (a_{(n-1),(n-1)}^0 - y_n^0).v_{n-1} \\ \qquad - y_{(n-1)}^0.v_n = B_{n-1} \\ a_{n1}^0.v_1 + a_{n2}^0.v_2 + \ldots + a_{n,(n-1)}^0.v_{n-1} - 1.v_n = B_n \end{cases} \qquad (12a)$$

It can be easily checked that the radius of B_i is

$$R(B_i) = + \sum_{j=1}^{n-1} \left| y_j^0 \right| .R_{ij} + R_{in} + \sum_{j=1}^{n-1} R_{ij}.r_j + r_n.r_i \qquad (12b)$$

$$R(B_n) = + \sum_{j=1}^{n-1} \left| y_j^0 \right| .R_{nj} + \sum_{j=1}^{n-1} R_{nj}.r_j + R_{nn} \qquad (12c)$$

where R_{ij} is the radius if u_{ij} (equivalently of a_{ij}) and r_i is the radius of v_j (equivalently of y_j). Now system (12) can be written in a compact form

$$\widetilde{A}_0.v = B \qquad (13)$$

where $\widetilde{A}_0$ is the real coefficient matrix in (12a). Let $C = \left| \widetilde{A}_0^{-1} \right|$ and $r = (r_1, r_2, \ldots, r_n)$. From (13)

$$r = C.R(B) \qquad (14)$$

Now we introduce the following matrices: $R = \{R_{ij}\}$ is a (n x n) matrix, $\breve{R}$ is the same as R expect for the last column whose elements are now zero, $\breve{r}$ is a diagonal matrix with r_j along the diagonal except for the last element that is 1; we also introduce a n-dimensional vector $\breve{y}$ whose first (n-1) elements are y_j except for the last one that is 1. Using (12b), (12c) and the new notations, (14) can be written in the form:

$$r = C.R.\left| \breve{y} \right| + C.\breve{R}.r + C.r_n.\breve{r} \qquad (15)$$

or

$$r = d + D.r + C.r_n.\breve{r} \qquad (16a)$$

with

$$d = C.R.\left| \breve{y} \right|, \qquad D = C.\breve{R} \qquad (16b)$$

The matrix equation (16) is a non-linear real value (non-interval) system of n equations of n unknowns r_i:

$$r_i = d_i + \sum_{j=1}^{n-1} d_{ij}.r_j + r_n.\sum_{j=1}^{n-1} c_{ij}.r_j + c_{in}.r_n, \qquad i = 1, \ldots n \qquad (17)$$

The solution of (17) for positive r_i solves Problem P_2. Indeed

$$\boldsymbol{y}_i = y_i^0 + [-r_i, r_i], \ i = 1, 2, \ldots, n. \qquad (18)$$

It can be shown that

$$y_i^* \subset \boldsymbol{y}_i, \ i = 1, 2, \ldots, n, \qquad (19)$$

i.e. the intervals (18) are really outer bounds on the ranges y_i^* for all i. Hence, the interval

$$\boldsymbol{y}_n = y_n^0 + [-r_n, r_n] \qquad (20)$$

is the solution to the original problem P_1 and is, in fact, a bound $\boldsymbol{\lambda}$ on λ^* satisfying the inclusion (3). More specifically, we have the following theorem.

Theorem 3.1. If the nonlinear system (17) has a positive solution r which can be attained by the simple iteration method with initial vector $y^0 = 0$, then:

(i) the interval (20) is an outer bound on the range λ^* of the maximum eigenvalue $\lambda(A)$ of (1) when $A \in \boldsymbol{A}$;

(ii) each eigenvalue $\lambda(A)$ remains real for any $A \in \boldsymbol{A}$.

The proof of the theorem is given in the Appendix.

Thus, it has been shown that the original problem P_1 reduces to solving the non-linear (incomplete quadratic) system (17). Since R_{ij} are, most often, percents of a_{ij}^0 and (17) is only mildly non-linear, the solution of (17) does not present any problem.

IV. Numerical Example

The applicability of the method will be illustrated by the following example with n = 2:

$$\begin{vmatrix} a_{11}.x_1 + a_{12}.x_2 - \lambda.x_1 = 0 \\ a_{21}.x_1 + a_{22}.x_2 - \lambda.x_2 = 0 \end{vmatrix} \quad (21)$$

Here

$$\begin{vmatrix} a_{11}^0 = -3.8 & a_{12}^0 = 1.6 \\ a_{21}^0 = 0.6 & a_{22}^0 = -4.2 \end{vmatrix} \quad (22a)$$

and $R_{ij}(A) = 0.17$, $i = 1, 2; j = 1, 2.$ (22b)

First, we determine the centre of the eigenvalues

$$\Lambda^o = [-3, -5]^T \quad (23a)$$

and

$$\lambda_{max} = \max_i \{\lambda^{(i)}(A_0)\} \quad (23b)$$

For this example, the index k corresponding to λ_{max} is k = 1 and the corresponding normalized vector of the centres of the variables from (5), (6) and (6') is:

$$X^0 = [1\ x_2^0]^T \quad (24)$$

Note: Following (5) we normalized vector X^0 in relation to x_2^0 but indexes 1 and 2 can change to apply directly the describing algorithm we can written vector X^0 in form (24).

According to (7) vector Y is:

$$Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ \lambda \end{bmatrix} \quad (25)$$

so

$$Y^0 = \begin{bmatrix} y_1^0 \\ y_2^0 \end{bmatrix} = \begin{bmatrix} x_2^0 \\ \lambda^0 \end{bmatrix} = \begin{bmatrix} 0.5 \\ -3 \end{bmatrix} \quad (26)$$

So the system (8a) is:

$$\begin{vmatrix} a_{11}.1 + a_{12}.y_1 - y_2.1 = 0 \\ a_{21}.1 + a_{22}.y_1 - y_2.y_1 = 0 \end{vmatrix} \quad (27)$$

Applying (10) and simplifying the system we get the following non-linear system:

$$\begin{vmatrix} r_1 = 0.19125 + 0.1275.r_1 + 0.5r_1r_2 \\ r_2 = 0.357 + 0.238r_1 + 0.8r_1r_2 \end{vmatrix} \quad (28)$$

which can be written in the form:

$$\begin{vmatrix} -0.8725r_1 + 0.5r_1r_2 + 0.1912539 = 0 \\ 0.238r_1 - r_2 + 0.8r_1r_2 + 0.357 = 0 \end{vmatrix} \quad (29)$$

The solution of (29) is:

$$R(Y) = \begin{bmatrix} R(y_1) \\ R(y_2) \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \begin{bmatrix} 0.594102 \\ 0.332325 \end{bmatrix} \quad (30)$$

Finally, from (18) and (20):

$$\boldsymbol{Y} = \begin{bmatrix} \boldsymbol{y}_1 \\ \boldsymbol{y}_2 \end{bmatrix} = \begin{bmatrix} 0.5 \\ -3 \end{bmatrix} + \begin{bmatrix} [-0.594102, & 0.594102] \\ [-0.332325, & 0.332325] \end{bmatrix} = \begin{bmatrix} -0.094102, & 1.0941021 \\ -3.332325, & -2.667675 \end{bmatrix} \quad (31)$$

Thus, in view of (30), we conclude that the interval system (1) (resp. (21)) is stable with stability margin M = 2.667675 at $R(A) = 0.17$. This example has been solved by methods illustrated in [2] and [4] and the corresponding results at $R(A) = 0.17$ are: $M = 2.593129$ and $M = 2.3098$. A better result of stability margin could be found by using our simple approach.

The estimations of the real eigenvalues of interval matrix A (see (1)) at $R(A) = 0.17$ with our approach are outer bounds. We can demonstrate that with compare to results of "Monte Carlo" method, which gives inter bounds of the real eigenvalues are: λ_{max} = [-3.3.877, -2.7038].

V. Conclusion

The problem of bounding the real eigenvalues of interval matrices has been considered. It is related to the problem of assessing the robust stability of linear circuits or systems having interval parameters. A method for determining outer bounds on the eigenvalue ranges has been suggested. It requires the evaluation of the real eigenvalues and the corresponding eigenvectors from (4) for the center (nominal parameters) matrix A_0. The method essentially consists of setting up and solving the system of n non-linear equations (17) for the positive solutions r_i, i = 1, 2, … , n. The solution of the original problem is then found by the n-th radius r_n according to formula (20).

The approach herein suggested can be extended to treat also the case of complex eigenvalues. Another possible generalization is to encompass matrices whose elements are non-linear functions of a certain number of parameters.

Appendix

Consider system (16) written as

$$f(y) = c + By + Cg(y) - x \quad \text{(A1a)}$$

where

$$C > 0,\ B > 0,\ C > 0 \quad \text{(A1b)}$$

(the symbol < is meant componentwise) and

$$g(y) = \begin{pmatrix} y_1 y_n \\ y_2 y_n \\ . \\ . \\ y_{n-1} y_n \\ y_n \end{pmatrix} \quad \text{(A1c)}$$

The simple iteration method related to (A1) is:

$$y^{(\nu+1)} = c + B\, y^{(\nu)} + C\, g(y^{(\nu)}),\quad \nu \geq 0$$
$$y^{(0)} = 0 \quad \text{(A2)}$$

We need the following lemma.

Lemma. System (A1) has a positive solution $y^* > 0$ if and only if iteration (A2) is convergent.

(31)

Proof.

Sufficiency. Let (A2) be convergent and y^* be its limit. Because of (A1b) the limit y^* is positive. At the same time, y^* is the fixed point of

$$y = F(y) \quad \text{(A3a)}$$

with

$$F(y) = c + By + Cg(y) - x \quad \text{(A3b)}$$

and hence y^* is a positive solution to (A1).

Necessity. Let y^* be a positive solution to (1). Because of (A1b) each iteration $y^{(v+1)} > 0$ and $y^{(v+1)} > y^{(v)}$. On the other hand, $y^{(v+1)}$ is bounded by y^*, i.e. $y^{(v+1)} < y^*$. Hence iteration A_2 is convergent.

Now we can prove Theorem 3.1.

Proof of Theorem 3.1. The proof is based on the above Lemma and the general theory of [7].

Part (i). The convergence of (A2) corresponds to the convergence of the first stage (Procedure 3.2) of the method in [7]. Moreover, if the simple iteration process (A2) is convergent, then it follows from the Lemma that the Jacobian matrix $J(y^*)$ of $f(y)$ at y^* is not singular. Hence, $J(y)$ remains nonsingular in some neighbourhood of y^*. Thus, we can apply the second stage of the method in [7] (Procedure 3.3) starting with the vector

$$z = \lambda^* + (1+\varepsilon)[-y^*, y^*] \qquad \text{(A4)}$$

where $\varepsilon > 0$. Once again, the convergence of the simple iteration process (A2) with the new starting point guarantees that this second stage is also convergent and its limit is a vector whose components are given by (18). This proves the validity of (20).

Part (ii). Since y^* is finite, the solution set of (1) is also finite. This entails that each Jacobian related to (1) remains nonsingular in $\boldsymbol{A}$. Hence, problem (1) remains structurally stable, i.e. each $\lambda(A)$ remains real in $\boldsymbol{A}$.

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