Stability analysis of linear interval parameter systems via assessing the eigenvalues range

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Abstract – **The paper addresses the stability analysis of linear circuits and systems under interval parameter uncertainties. The problem is equivalent of estimating the eigenvalues of matrices, which elements are nonlinear functions of interval parameters. A method for obtaining the exact range of the eigenvalues is suggested. It can be applied if certain monotonicity conditions are fulfilled. A method for computing a tight outer bound on the eigenvalue range is also given. The outer bound is obtained as a solution of an algebraic nonlinear system. A procedure based on an iterative method for determining an inner bound on the eigenvalue range is also proposed. A numerical example, illustrating the applicability of the methods suggested, is solved at the end.**

Index Terms - **robust stability analysis, eigenvalues of interval matrices with dependent coefficients.**

I. INTRODUCTION

The dynamic behavior of lumped parameters time invariant linear systems can be described by the state space model. The stability of these systems is related to the stability analysis of the linear system

$$\dot{z}(t) = Az(t) + Lu(t) \quad (1)$$

where $z(t) = [z_i(t)],\ i = 1,...,n$ - vector of state space variables;

$u(t) = [u_j(t)],\ j = 1,...,m$ - vector of control signals.

The elements of matrix A are, in general, nonlinear functions of m parameters, which take on their values within prescribed intervals, i.e.

$$a_{ij}(p) = a_{ij}(p_1,..., p_m),\ i, j = 1,...,n \quad (2)$$

$$p_s \in \boldsymbol{p}_s, s = 1,...,m\ . \quad (2a)$$

System (1) is stable if and only if the eigenvalues of $A(p),\ p \in \boldsymbol{p}$, have negative real parts.

Ordinary letters will denote real quantities while bold face letters will stand for their interval counterparts. Thus, p and $\boldsymbol{p}$ are real and interval vectors of m parameters, respectively.

The problem statement for interval matrices with independent coefficients was defined in [3] (for real eigenvalues) and in [4] (for the complex case). The main points of the problem formulation for the case of dependent coefficients will be briefly presented here. We consider the following "perturbed" eigenvalue problem:

$$A(p)x = \lambda x,\ p \in \boldsymbol{p}\ . \quad (3)$$

Each matrix $A(p),\ p \in \boldsymbol{p}$, is assumed non-singular.

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It follows from (2) and (3) that each eigenvalue λ and its corresponding eigenvector x are implicit forms of p.

In this paper, we are interested in the intervals of the eigenvalues of (3).

We will estimate only the interval of the maximum eigenvalue obtained by the (center, nominal) problem

$$A(p^0)x = \lambda x \quad (4)$$

In general, the methods suggested later can be applied for any other real eigenvalues.

Let $\lambda^*(p^0) = \max\ \lambda_k(p^0),\ k = 1,...,n$ (5)

is a maximum eigenvalue while $x^* = [x_1, x_2,…, x_n]$ is the corresponding eigenvector. We make the following assumption (ensuring structural stability of the problem).

Assumption A1: Let $\lambda^*(p)$ and $x^*(p)$, corresponding to all $p \in \boldsymbol{p}$, remain real.

On account of *Assumption A1*, the range

$$\boldsymbol{\lambda}^* = \{\lambda(p): p \in \boldsymbol{p}\} \quad (6)$$

is a real interval.

Without any loss of generality we need a second assumption. If the pair (x^0, λ^0) is the solution of (4) then

Assumption A2: We assume that the absolute value of the n-th component $\left|x_n^0\right|$ of vector x^0 is the largest component of the other components, i.e.

$$\left|x_n^0\right| \geq \left|x_i^0\right|,\ i \neq n \quad (7)$$

If p-th component is the largest component, we need to interchange the places of the p-th and n-th row in A matrix as well as the position of the components x_p and x_n.

Now x^0 is normalized through

$$\left|x_n^0\right| = 1\ . \quad (8)$$

Further, we require that (8) be also valid for

$$\left|x_n(p)\right| = 1,\ p \in \boldsymbol{p}\ . \quad (8a)$$

We introduce the n-dimensional real vector

$$y = (y_1, y_2, \dots , y_n) \quad (9)$$

with

$$y_i = x_i(p),\ i = 1, \dots , (n-1)$$
$$y_n = \lambda(p) \quad (10)$$

Using (10), the eigenvalue problem (3) is rewritten as

$$\begin{cases} a_{11}y_1 + a_{12}y_2 + ... + a_{1(n-1)}y_{n-1} - y_n y_1 + a_{1n} = 0 \\ a_{21}y_1 + a_{22}y_2 + ... + a_{2(n-1)}y_{n-1} - y_n y_2 + a_{2n} = 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{n1}y_1 + a_{n2}y_2 + ... + a_{n(n-1)}y_{n-1} - y_n.1 + a_{nn} = 0 \end{cases} \quad (11)$$

where $a_{ij} = a_{ij}(p), p \in \boldsymbol{p}$. The system of interest (11) can be written in matrix form as:

$$\breve{A}(p)y - y_n y + A_n(p) = 0\ , \quad (11a)$$

where $\breve{A}(p)$ is the same as $A(p)$ except for the n-th column which is zero, $A_n(p)$ is the n-th column of $A(p)$.

It is seen that system (11) is nonlinear only because of the products $y_n y_i$, $i = 1,2,...,n$.

In previous papers of the authors, the problem of determining outer bound of the maximum eigenvalue of interval matrix with independent elements was solved ([3] – for the real and [4] – for the complex case). The inner bound and the exact range for the maximum eigenvalue of interval matrix with independent elements were obtained in [5]. This paper is extension of the previous results for the case where the elements a_{ij} and b_i are now dependent intervals (2).

The solution of (11) is the set

$$S(\boldsymbol{p}) = \{y : \breve{A}(p)y - y_n y + A_n(p) = 0, \quad p \in \boldsymbol{p}\} . (12)$$

The interval hull of $S(\boldsymbol{p})$ will be denoted $\boldsymbol{y}^*$ and $\boldsymbol{y}^*$ will be called exact range to (11). Any other $\boldsymbol{y}'$ such that $\boldsymbol{y}^* \subset \boldsymbol{y}'$ will be referred to as an outer bound to (11). Similarly, an interval vector $\boldsymbol{y}''$ with the property $\boldsymbol{y}'' \subseteq \boldsymbol{y}^*$ will be referred to as an inner bound to (11).

The present paper addresses the problem of determining the outer and inner bounds and the exact range of the solution of (11). First, a direct method for computing a tight and cheap outer bound $\boldsymbol{y}'$ is presented in Section II. It is based on the approach suggested in [3] and [4]. In Section III, the exact range $\boldsymbol{y}^*$ to (11) is determined for the case when certain monotonicity conditions, regarding the derivatives of y_i with respect to p_j, are fulfilled. The method suggested is an extension of the approach from [5] (applicable to linear interval systems with independent elements) to the general nonlinear case of dependencies (2). It is based on the use of the outer solution method from the previous section. An iterative method for obtaining the inner bound $\boldsymbol{y}''$ on $\boldsymbol{y}^*$ is presented in Section IV. It includes a procedure for obtaining the lower and the upper ends of the inner bound. An illustrative example is considered in Section IV.

II. OUTER SOLUTION

To apply the approach considered bellow we need the following preliminary facts [2]. The functions defined by (11) are

$$f_i(p,y) = \sum_{j=1}^{n-1} a_{ij}(p)y_j - y_n y_i - a_{in}(p), \; p \in \boldsymbol{p}, \; y \in \boldsymbol{y}$$
$$f_n(p,y) = \sum_{j=1}^{n-1} a_{nj}(p)y_j - y_n - a_{nn}(p), \; p \in \boldsymbol{p}, \; y \in \boldsymbol{y} \qquad (13)$$

The interval hull of $f_i(p,y)$ is $\boldsymbol{S}_{f_i}(p,y)$, $p \in \boldsymbol{p}$, $y \in \boldsymbol{y}$. On account of the inclusion property

$$f_i(p,y) \in \boldsymbol{S}_{f_i}(p,y), \; p \in \boldsymbol{p}, \; y \in \boldsymbol{y} \qquad (14)$$

the linear interval forms of $\boldsymbol{S}_{f_i}(p,y)$ are:

$$\boldsymbol{L}_{ij}(p) = \sum_{k=1}^{m} \alpha_{ijk} p_k + \boldsymbol{g}_{ij}, \quad p_k \in \boldsymbol{p}_k \qquad (15)$$

From (14) it follows

$$a_{ij}(p) \in \boldsymbol{L}_{ij}(p), \quad p \in \boldsymbol{p} .$$

To find the outer bound of $y_n = \lambda(p)$, $p \in \boldsymbol{p}$ we will appeal to the approach suggested recently in [3] for the case of independent a_{ij}. Let

$$a_{ij}(p) = \sum_{k=1}^{m} \alpha_{ijk} p_k + \boldsymbol{g}_{ij}, \quad p_k \in \boldsymbol{p}_k \qquad (16)$$

We apply (16) to (11) and we get the system:

$$\sum_{j=1}^{n-1}\sum_{k=1}^{m} [\alpha_{ijk} p_k + \boldsymbol{g}_{ij}] y_j - y_n y_i + \left(\sum_{k=1}^{m} \alpha_{ink} p_k + \boldsymbol{g}_{in} \right) = 0$$
$$\sum_{j=1}^{n-1}\sum_{k=1}^{m} [\alpha_{njk} p_k + \boldsymbol{g}_{nj}] y_j - y_n + \left(\sum_{k=1}^{m} \alpha_{nnk} p_k + \boldsymbol{g}_{nn} \right) = 0 \qquad (17)$$

We substitude the interval variables in (17) with

$$\boldsymbol{p}_k = p_k^0 + u_k, \; \boldsymbol{y}_j = y_j^0 + v_j, \; \boldsymbol{g}_{ij} = g_{ij}^0 + t_{ij} . \qquad (18)$$

On account of (18) we get the following system

$$\left| \begin{array}{l} (a_{11}^0 - y_n^0)v_1 + a_{12}^0 v_2 + ... + a_{1,(n-1)}^0 v_{n-1} - y_1^0 v_n = B_1 \\ a_{21}^0 v_1 + (a_{22}^0 - y_n^0)v_2 + ... + a_{2,(n-1)}^0 v_{n-1} - y_2^0 v_n = B_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{(n-1),1}^0 v_1 + a_{(n-1),2}^0 v_2 + ... + (a_{(n-1),(n-1)}^0 - y_n^0)v_{n-1} - \\ - y_{(n-1)}^0 . v_n = B_{n-1} \\ a_{n1}^0 v_1 + a_{n2}^0 v_2 + ... + a_{n,(n-1)}^0 v_{n-1} - v_n = B_n \end{array} \right. \qquad (19)$$

where y_j^0, $j = 1,...,n$ is the solution of the system (17) for the centers p^0 of the interval vector $\boldsymbol{p}$ while the meaning of the remaining symbols is similar to [3].

Now system (19) can be written in a compact form

$$\widetilde{A}_0 v = B \qquad (20)$$

where $\widetilde{A}_0$ is the real coefficient matrix in (17) for $p=p^0$.

Let $C = \widetilde{A}_0^{-1}$, thus, (20) can be written in the form:

$$v = CB = -\sum_{k=1}^{m} \left(C\breve{G}_k \right) u_k - CT\breve{y}^0 - \left[\sum_{k=1}^{m} \left(C\breve{H}_k \right) u_k \right] v - CTv +$$
$$+ Cv_n \breve{v} - \sum_{k=1}^{m} \left(CH_k^n \right) u_k - CT^n \qquad (21)$$

where $\breve{G}_k = \left[\breve{G}_{ik} \right]^T = \sum_{j=1}^{n-1} \alpha_{ijk} y_j^0$; $T = [t_{ij}]$; $H_k^n = \left[\alpha_{ink} \right]^T$;

$$\breve{H}_k = [\breve{H}_{ik}]^T = \left[\sum_{j=1}^{n-1} \alpha_{ijk} \right]^T ; \; i,j = 1,...,n; \; k = 1,...,m;$$

$\breve{y}^0$ and $\breve{v}$ are the same as vectors y^0 and v, respectively, expect for the last element which is now zero; T_n is the last column of matrix T. We note system (21) by the radii

$$r = d + Dr + |C| r_n \breve{r} , \qquad (22)$$

with

$$d = \left| - \sum_{k=1}^{m} \left(C\breve{G}_k \right) u_k - CT\breve{y}^0 - \sum_{k=1}^{m} \left(CH_k^n \right) u_k - CT^n \right| \qquad (22a)$$

$$D = \left| -\left[\sum_{k=1}^{m} (C\breve{H}_k) u_k \right] - CT \right| \qquad (22b)$$

The matrix equation (22) is a non-linear real value (non-interval) system of n equations of n unknowns r_i:

$$r_i = d_i + \sum_{j=1}^{n-1} D_{ij} r_j + r_n \sum_{j=1}^{n-1} \left| c_{ij} \right| r_j + \left| c_{in} \right| r_n,\ i = 1,..., n \qquad (23)$$

We solve system (23) for r_i and, based on the component r_n, the outer bounds of the maximum eigenvalue y_n' is:

$$y_n' = y_0^n + [-r_n,\ \ r_n]\ . \qquad (24)$$

The main result of this section is the following theorem.

THEOREM 2.1: *Assume the solution r of system (23) is positive. Then the outer bound on the range* λ^* *of the maximum eigenvalue λ(p) of (3) when* $p \in \boldsymbol{p}$ *is*

$$\boldsymbol{y}_n' = y_n^0 + \boldsymbol{r}_n',\ \text{ where } \boldsymbol{r}_n' = [-r_n, r_n] \qquad (25)$$

This theorem is valid for all the eigenvalues but to simplify the presentation we formulate the theorem only for the maximum eigenvalue.

The proof of the above theorem is similar to that of Theorem 3.1 in [3] and will therefore be omitted.

Thus, it has been shown that the problem of finding an outer bounds λ' on λ^* reduces to solving the non-linear (incomplete quadratic) system (23). Since system (23) is only mildly non-linear, because of the products $y_n y_i\ ,i = 1,...,n.$, its solution does not present any problem.

III. Exact solution

In this section, the outer bounds on the solution of system (11) will be applied in a method for computing the interval hull (exact range) $\boldsymbol{y}^*$. It is assumed that $a_{ij}(p)$ (see (2), (11)) are continuously differentiable functions in p. The method suggested is applicable only if certain monotonic conditions are fulfilled. It is based on the method suggested in [5], where the coefficients in the system (11) are independent.

We are interested in expressing the derivative of y_i with respect to p_l, i =1 ,…, n; l = 1, … , m. With this in mind, we differentiate (11) in p_l and on account of (2) we get:

$$\begin{aligned} &\sum_{j=1}^{n-1} a_{ij} \frac{\partial y_j}{\partial p_l} - \frac{\partial y_n}{\partial p_l} y_i - y_n \frac{\partial y_i}{\partial p_l} = -\sum_{j=1}^{n-1} \eta_{ijl}\, y_j + \eta_{inl} \\ &\sum_{j=1}^{n-1} a_{nj} \frac{\partial y_j}{\partial p_l} - \frac{\partial y_n}{\partial p_l} = -\sum_{j=1}^{n-1} \eta_{njl}\, y_j + \eta_{nnl}\ ,\ i = 1,..., (n\text{-}1) \end{aligned} \qquad (26)$$

$$\text{where} \quad \eta_{ijl} = \frac{\partial a_{ij}}{\partial p_l}, i = 1,..., n;\ j = 1,..., n;\ l = 1,..., m \qquad (27)$$

$$\eta_{njl} = \frac{\partial a_{nj}}{\partial p_l},\ j = 1,..., n;\ l = 1,..., m\ .$$

We solve the system (26) using the method proposed in Section II to determine the outer bounds of the derivatives

$$\frac{\partial y_i}{\partial p_l} = D_{il},\ p \in \boldsymbol{p} \qquad (28)$$

So we obtain the intervals $\boldsymbol{D}_{il}^h,\ h = 1,...,n$. With

$$\frac{\partial y_i(p)}{\partial p_l} \in \boldsymbol{D}_{il}, p \in \boldsymbol{p}\ . \qquad (29)$$

We will make the following assumption:

Assumption A3: We assume that each estimation $\boldsymbol{D}_{il}$, $l = 1,...,m$, satisfies either the condition

$$\boldsymbol{D}_{il} \geq 0 \ \text{ or } \ \boldsymbol{D}_{il} \leq 0\ . \qquad (30)$$

On account of inclusion (29) the fulfillment of *Assumption A3* guarantees that y_i is monotonic with respect to each p_l. Now we define two vectors as follows

$$\underline{p}_l^{(i)} = \begin{cases} \underline{p}_l^{(i)}, \boldsymbol{D}_{il} \geq 0, l = 1,..., m \\ \overline{p}_l^{(i)}, \boldsymbol{D}_{il} \leq 0, l = 1,..., m \end{cases},\quad i = 1,..., n \qquad (31a)$$

$$\overline{p}_l^{(i)} = \begin{cases} \overline{p}_l^{(i)}, \boldsymbol{D}_{il} \geq 0, l = 1,..., m \\ \underline{p}_l^{(i)}, \boldsymbol{D}_{il} \leq 0, l = 1,..., m \end{cases},\quad i = 1,..., n\ . \qquad (31b)$$

The exact range $\boldsymbol{y}^*$ of (11) can be found using the following theorem.

THEOREM 3.1*: If Assumption A3 holds for all i=1,…,n, then the n-th component* $\boldsymbol{y}_n^* = (\underline{y}_n^*, \overline{y}_n^*)$ *of the solution vector* $\boldsymbol{y}^*$ *is determined as follows:*

1) $\underline{y}_n^*$ is equal to the n-th component of the following system solution

$$\begin{aligned} &\sum_{j=1}^{n-1} a_{ij}(\underline{p}) y_j - y_n y_i + a_{in}(\underline{p}) = 0,\ i = 1,...,(n-1) \\ &\sum_{j=1}^{n-1} a_{nj}(\underline{p}) y_j - y_n + a_{nn}(\underline{p}) = 0. \end{aligned} \qquad (32a)$$

2) $\overline{y}_n^*$ is equal to the n-th component of the following system solution

$$\begin{aligned} &\sum_{j=1}^{n-1} a_{ij}(\overline{p}) y_j - y_n y_i + a_{in}(\overline{p}) = 0,\ i = 1,...,(n-1) \\ &\sum_{j=1}^{n-1} a_{nj}(\overline{p}) y_j - y_n + a_{in}(\overline{p}) = 0. \end{aligned} \qquad (32b)$$

IV. Inner solution

Here, a simple iterative method for computing an inner bounds $\boldsymbol{y}''$ on $\boldsymbol{y}^*$ will be presented. It is assumed that a_{ij} in (2) are continuously differentiable functions of p.

We assume that none of the certain monotonicity conditions (30) are fulfilled. In this case, we have the following procedure for finding the lower end-point $\underline{y}_i''$ of $\boldsymbol{y}_i''$, i = 1,…, n.

Procedure 4.1. For fixed $i = k$ we start by evaluating the derivative $d_{kl}(p) = dy_k(p)/dp_l$ for p=p^0. Let $d_{kl}^0 = d_{kl}(p^0)$, $A^0 = A(p^0)$, $\eta_l^0 = \eta(p^0)$ and $y^0 = y(p^0)$. We find the solution d_{kl}^0 of system (26) with respect to (28) and using (31) form the vector $p^1 = [p_l^1],\ \ l = 1,...,m,$ (where $\boldsymbol{D}_{kl}$ is replaced with d_{kl}^0). Then we solve the system (11) for the new vector p^1

$$\begin{vmatrix} \sum_{j=1}^{n-1} a_{ij}(p^1)y_j - y_n y_i + a_{in}(p^1) = 0 \\ \sum_{j=1}^{n-1} a_{nj}(p^1)y_j - y_n + a_{nn}(p^1) = 0 \end{vmatrix} \quad (33)$$

$$p^1 \in \boldsymbol{p},\ y \in \boldsymbol{y},\ i = 1,..., n$$

and find the solution y^1. If

$$y_k^1 \le y_k^0 \quad (34)$$

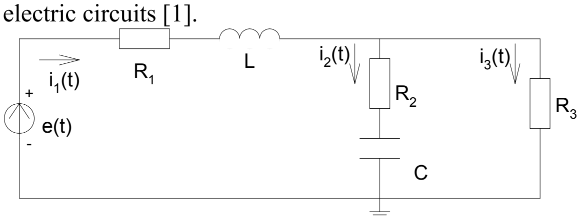
y_k^1 is renamed y_k^0, p^1 are renamed p^0 and the procedure is resumed from the start; otherwise it is terminate and the lower bound $\underline{y}_k^{"}$ is found.

We are interested only in $\underline{y}_n^{"}$.

A similar procedure 4.2. is valid for determining the upper end-point $\overline{y_k^{"}}$ of $\boldsymbol{y}_k^{"}$.

V. Numerical example

The circuit studied is shown in Fig. 5.1. Assuming that $R_i \in \boldsymbol{R}_i$, $L \in \boldsymbol{L}$, $C \in \boldsymbol{C}$, the vector of parameters is $p = (R_1, R_2, R_3, L, C)$ with $\boldsymbol{R}_1 = \boldsymbol{R}_3 = [99, 101]$, $\boldsymbol{R}_2 = [198, 202]$, $\boldsymbol{L} = [0.49, 0.51]$H, $\boldsymbol{C} = [240, 260]$μF. Such systems arise in tolerance analysis of linear AC electric circuits [1].



It is seen from the example that the expressions (2) are nonlinear functions of 5 parameters

$$A = \begin{bmatrix} -\dfrac{R_1 + R_2}{L.k} & -\dfrac{1}{L.k} \\ \dfrac{1}{C.k} & -\dfrac{1}{R_3 C.k} \end{bmatrix},\ k = 1 + \frac{R_2}{R_3} \quad M = \begin{bmatrix} \dfrac{1}{L} \\ 0 \end{bmatrix}. \quad (35)$$

The linearization of $a_{ij}(p)$ is made by the methods suggested in [2]. So we get the a_{ij} as the affine functions:

$$\begin{vmatrix} a_{11} = -0.67p_1 + 0.0001633p_2 - 1.334p_3 + 400.2p_4 - 200.087 + [-0.268\ \ 0.268] \\ a_{12} = 0.0022232p_2 - 0.004446p_3 + 1.3339p_4 - 1.3338 + 10^{-4}[-6.78\ \ 6.78] \\ a_{21} = -4.4484p_2 + 8.896p_3 - 5.34 * 10^6 p_5 + 2.669 * 10^3 + [-3.161\ \ 3.161] \\ a_{22} = 0.0445p_2 + 0.0445p_3 + 5.347 * 10^4 p_5 - 40.085 + [-0.02234\ \ 0.02234] \end{vmatrix}$$

According to (9) the vector y is:

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ \lambda \end{bmatrix} \quad (36)$$

resp. $$y^0 = \begin{bmatrix} y_1^0 \\ y_2^0 \end{bmatrix} = \begin{bmatrix} x_2^0 \\ \lambda^0 \end{bmatrix} = \begin{bmatrix} 0.00366795 \\ -18.244878 \end{bmatrix} \quad (37)$$

Applying (16) to (11) the nonlinear system (23) is:

$$\begin{vmatrix} -0.988r_1 - 0.0000207r_2 - 0.00565r_1r_2 + 0.0000644 = 0 \\ 45.05016r_1 - 2.02767r_2 - 7.54341r_1r_2 + 1.00325719 = 0 \end{vmatrix} \quad (38)$$

The solution of (39) is $r = [0.0000655\ \ 1.0067]^T$. Finally, from (24) we get the outer bound on the maximum eigenvalue

$$\boldsymbol{y}_2^{'} = [-19.2516\ \ -17.2382] \quad (39)$$

We calculate the derivatives $\partial y_i(p)/\partial p_l$, i=1,2; l=1,..,5 for $p \in \boldsymbol{p}$ using the system (26). It can be checked that the interval derivatives are positive except for the $\boldsymbol{D}_{24}$, so from (31) for vectors $\underline{p}$ and $\overline{p}$ we get

$$\underline{p} = [99\ \ 198\ \ 101\ \ 0.51\ \ 0.00024] \quad (40a)$$
$$\overline{p} = [101\ \ 202\ \ 101\ \ 0.49\ \ 0.00026] \quad (40b)$$

We are interested in the second component of the solutions of systems (11) for (40a) and (40b). The exact range is:

$$\boldsymbol{y}_2^* = [-19.21946\ \ -17.3127] \quad (41)$$

On account of (30) the sign of the derivatives $d_{kl}(p) = dy_k(p)/dp_l$, k=1,2; l=1,…,5 for $p = p^0$ is the same as those of $\boldsymbol{D}_{kl}$ respectively. So the inner bound of the solution of system (11) is the same as the exact solution obtained in (41).

VI. Conclusion

In this paper the problem of determining the outer and inner bounds and exact range of the solution of nonlinear system (11), where in general the coefficients are nonlinear functions (2) of system parameters, is discussed. A method for determining an outer solution $\boldsymbol{y}^{'}$ has been suggested in Section II. It is based on THEOREM 2.1 and it reduces to solving the incomplete quadratic system (23). The method is applicable if the solution r of system (23) is positive.

A version of this method for finding the outer solution can be used for determining the outer bounds of the derivatives $\boldsymbol{D}_{kl}$, $k = 1,...,n;\ l = 1,...,m$. If these bounds satisfied conditions (30) the method, proposed in Section III, can provide the exact solution for the eigenvalue range of the maximum eigenvalue.

In Section IV, two simple iterative procedures for determining a lower and upper bound of the inner solution $\boldsymbol{y}^{"}$ of (11) are suggested. The inner solution $\boldsymbol{y}^{"}$ is the same as the exact solution $\boldsymbol{y}^{*}$ because of the derivatives $\boldsymbol{D}_{kl}$ are strictly positive or strictly negative i.e. $0 \notin \boldsymbol{D}_{il}$. So the signs of the derivatives d_{kl}^0 are the same as respective signs of interval derivatives $\boldsymbol{D}_{kl}$.

A numerical example for analyzing the stability of electrical circuit has been solved in Section V. It illustrates the applicability of the above methods to determine the outer and inner bounds as well as the exact range of the eigenvalues of the system considered.

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