IEEE TRANSACTIONS ON

CIRCUITS AND SYSTEMS

I: FUNDAMENTAL THEORY AND APPLICATIONS

A PUBLICATION OF THE IEEE CIRCUITS AND SYSTEMS SOCIETY

WWW.IEEE-CAS.ORG

DECEMBER 2002 VOLUME 49 NUMBER 12 ITCAEX (ISSN 1057-7122)

PAPERS

TRANSACTIONS BRIEFS

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IEEE

Worst-Case Tolerance Analysis of Linear DC and AC Electric Circuits

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Abstract—**This paper addresses the problem of worst-case tolerance analysis of steady states in linear dc and ac electric circuits. The statement of the problem considered is in the form of linear algebraic equations whose elements are, in the general case, nonlinear functions of a given set of independent interval parameters. Three kinds of solutions are considered: 1) outer solution; 2) inner solution and 3) exact solution. A direct method for computing an outer solution and an iterative method for finding an inner solution are suggested. The inner and outer solutions thus found provide a tight two-sided bound on the exact solution of the tolerance problem investigated. The exact solution can be determined if certain monotonicity conditions are fulfilled. The verification of the conditions involves solving several associated outer solution problems. The computational efficiency of the methods suggested is demonstrated by a numerical example.**

Index Terms—**Interval analysis, interval methods, linear equations with dependent elements, worst-case tolerance analysis.**

I. Introduction

WORST-CASE tolerance analysis of linear circuits (systems) is a well-established research area in circuit theory. The traditional approach to handling this problem is to use the Monte Carlo method. Starting with the pioneer works [1] and [2], an alternative approach based on the application of interval analysis technique [3]–[5] has been in existence for several decades. The methods utilizing the latter approach are known as interval methods (cf. [6], [7] and the bibliography therein cited). Because of their high reliability the interest in these methods has lately considerably increased.

Most worst-case tolerance analysis problems for linear circuits can be formulated in the following ways:

1) in explicit form as corresponding global optimization problems [2], [6, Ch. 2], [8], [9];
2) in implicit form using a system of linear interval equations [1], [6, Ch. 3], [10]–[13].

This paper falls into the latter group of investigations. All known methods pertaining to this group are based, in one way or another, on the exact or approximate solution of a system of linear algebraic equations whose elements are either independent intervals (in the case of dc circuits [6], [10]) or are assumed to be independent intervals (in the case of ac circuits [11]–[13]). The requirement for independence of the elements imposes stringent restrictions on the form of the system used: tableau forms [6] or the so-called hybrid form [11]–[13]. The system must have independent interval elements since the interval methods for solving linear systems applied so far to the solution of tolerance problems are only capable of treating such systems.

A more general and more flexible approach is adopted in this paper. According to this approach, the worst-case tolerance problem considered is solved using a corresponding system of linear algebraic equations whose elements are now interdependent. More precisely, each element can be a linear or even nonlinear function of a given set of independent parameters. These take on their values within certain prescribed intervals. The interval solution of the linear system is then transformed in a nonlinear fashion to provide the solution of the tolerance problem on hand. The new approach covers all possible worst-case tolerance problems related to dc and ac linear circuits. For brevity, these problems will be referred to as tolerance analysis (TA) problems. The following three kinds of interval solutions to such problems will be considered:

1) outer solution;
2) inner solution;
3) exact solution.

The exact solution is the narrowest possible interval solution of the TA problem considered (the rigorous definition of the unique exact solution will be given in the next section). An outer solution is any interval solution that contains the exact solution. Similarly, an inner solution is any interval solution that is contained in the exact solution. Since the determination of the exact solution is not always possible with acceptable computational cost, cheap and tight inner and outer solutions will provide a good two-sided bound on the exact solution. Such outer and inner solutions are obtained in the present paper. After presenting the formulation of the TA problems in Section II, a simple direct method for computing an outer solution is suggested in Section III. It reduces essentially to inverting a real $n \times n$ matrix and solving a system of n real linear algebraic equations, n being the size of the original real linear interval system describing the TA problem considered. In Section IV, a simple iterative method for computing a tight inner solution is proposed. Finally, a more involved method for determining the exact solution is presented in Section V. The latter method is, however, applicable only if certain monotonicity conditions are satisfied. It requires solving several associated outer solution problems to computationally check the above monotonicity conditions. The efficiency of the methods presented is illustrated by way of a numerical example in Section VI. The paper ends up with several concluding remarks.

Manuscript received June 22, 2001; revised June 26, 2002. This paper was recommended by Associate Editor N. M. K. Rao.

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Digital Object Identifier 10.1109/TCSI.2002.805700

II. Formulation of the AT Problems

In this paper, an arbitrary TA problem can be formulated in the following manner. First, a real linear algebraic system of equations is set up

$$A(p)x = b(p) \tag{1a}$$

where p is an m-dimensional parameter vector, $A(p)$ and $b(p)$ are an $n \times n$ matrix and an n-dimensional vector, respectively. The elements of $A(p)$ and $b(p)$ are, in general, nonlinear functions of m parameters

$$a_{ij}(p) = a_{ij}(p_1, \ldots, p_m) \tag{1b}$$

$$b_i(p) = b_i(p_1, \ldots p_m) \tag{1c}$$

and the parameters take on their values within some prescribed intervals, i.e.,

$$p_k \in \boldsymbol{p}_k, \quad k = 1, \ldots, m \tag{1d}$$

Here and henceforth, ordinary font letters will denote real quantities while bold face letters will stand for their interval counterparts. Thus, $p = (p_1, \ldots, p_m)$ and $\boldsymbol{p} = (\boldsymbol{p}_1, \ldots, \boldsymbol{p}_m)$ will denote a real and an interval vector of m components, respectively.

The next step to formulating the TA problem is to define the relationship

$$u = g(x) \tag{2}$$

which specifies the vector of output variables u and where $g : R^n \to R^q$, $1 \leq q \leq n$. The pair (1a)–(1d) and (2) formulates the TA problem to be solved.

At this point, the following assumption is needed.

Assumption 1: Each matrix $A(p)$, $p \in \boldsymbol{p}$, is nonsingular.

Later, it will be shown that Assumption 1 can be easily verified numerically by a certain sufficient condition (Theorem 1).

The solution set of the pair (1a)–(1d) and (2) is the set

$$S(\boldsymbol{p}) := \left\{u : u = g(x), x = A^{-1}(p)b(p),\ p \in \boldsymbol{p}\right\}. \tag{3}$$

The interval hull of $S(\boldsymbol{p})$ will be denoted $\boldsymbol{u}^*$ and $\boldsymbol{u}^*$ will be called exact (interval hull) solution to problem (1a)–(1d), (2). Any other interval $\boldsymbol{u}'$ such that $\boldsymbol{u}^* \subset \boldsymbol{u}'$ will be referred to as an outer solution to (1a)–(1d), (2). Similarly, an interval vector $\boldsymbol{u}''$ with the property $\boldsymbol{u}'' \subseteq \boldsymbol{u}^*$ will be referred to as an inner solution to (1a)–(1d), (2).

The description (1a)–(1d), (2) is rather general and covers all possible dc and ac TA problems formulated in [6, Sec. 3, problems 3.1–3.10]. Typically, functions (1b) and (1c) involved in most TA problems are affine (linear) functions of the elements of $\boldsymbol{p}$

$$a_{ij}(p) = \alpha_{ij} + \sum_{k=1}^{m} \alpha_{ijk} p_k \tag{4a}$$

$$b_i(p) = \beta_i + \sum_{k=1}^{n} \beta_{ik} p_k. \tag{4b}$$

It is, however, known [6] that in the general case, some of the elements a_{ij} will be nonlinear functions if the circuit investigated involves dependent current sources with interval coefficients. Also, some of the elements b_j will be nonlinear functions if the magnitudes of some voltage or current sources are not known exactly (and are therefore given as intervals) and loop analysis is used to set up the TA system of equations. The nonlinear functions encountered in practice are continuously differentiable with respect to the parameters considered.

The specific TA problem considered is finally defined by choosing the function $g(x)$ in (2). In some cases, $g(x)$ is a linear function. Thus, if we are interested in the determination of the tolerances on all the components of x (as in [6, prob. 3.7, 3.8]), then

$$u = x \tag{5a}$$

so

$$g = E \tag{5b}$$

(where E denotes the identity matrix). In the overwhelming majority of applications, the problem is to estimate only the range of n' components $\boldsymbol{u}_i^*$ and $n' \ll n$ (typically $n' = 1$ in the case of direct current electrical circuits or $n' = 2$ in the case of alternative current electrical circuits). If we want to find the tolerance on one single component x_k of x, then (2) become

$$u_k = e_k^T \cdot x \tag{6}$$

where e_k^T is the transpose of the kth column of E. If the problem is to estimate the tolerance on n' output variables (with $n' > 1$), then the pair (1a)–(1d) and (6) is solved n' times.

In the general case [6, prob. 3.9, 3.10], $g(x)$ is a nonlinear function. For example, if we want to find the tolerance on the magnitude of a nodal voltage $\dot{V}_k$ [6, problem 3.9]), then

$$u = x_k^2 + x_{k+1}^2 \tag{7}$$

where x_k and x_{k+1} are the real and imaginary parts of the complex nodal voltage $\dot{V}_k$.

Once the TA problem is defined by fixing (1a)–(1d) and (2), we have finally to specify which of the solutions outer, inner, or exact is to be found.

III. Outer Solution

In this section, first a method for determining an outer solution $\boldsymbol{x}'$ to problem (1a)–(1d) and (5a)–(5b) is suggested. The solution $\boldsymbol{x}'$ is then used to find an outer solution $\boldsymbol{u}'$ to problem (1a)–(1d) and (2).

The derivation of the method is based on the general approach employed in [16], [17], on the one hand, and the result for the case of affine functions $a_{ij}(p_1, \ldots, p_m)$ [19], on the other.

To apply this approach, we need the following preliminary facts [14]–[16]. First, let $f : \mathbf{x} \subset R^n \to R^q$ be a continuously differentiable function. The function $f(x)$ can be enclosed in the interval vector $\boldsymbol{x}$ by the following linear interval form:

$$L_f(x) = Ax + a, \quad x \in \boldsymbol{x} \tag{8}$$

where A is a $q \times n$ real (noninterval) matrix while $\boldsymbol{a}$ is an interval vector. The form (8) can be determined in an automatic way using the algorithm of [16]. It has the inclusion property

$$f(x) \in \boldsymbol{L}_f(x), \quad x \in \boldsymbol{x}. \tag{9}$$

Secondly, consider the product

$$xy, \quad x \in \boldsymbol{x}, \quad y \in \boldsymbol{y} \tag{10}$$

where $\boldsymbol{x}$ and $\boldsymbol{y}$ are intervals. If x_0, y_0 and r_x, r_y are the respective centers and radii, then [14]

$$xy \in -x_0 y_0 + y_0 x + x_0 y + [-r_x r_y, r_x r_y]. \tag{11}$$

In accordance with (8), the corresponding linear interval forms of (4a) and (4b) are

$$\boldsymbol{L}_{ij}(p) = \sum_{k=1}^{m} \alpha_{ijk} p_k + \boldsymbol{a}_{ij}, \quad p \in \boldsymbol{p} \tag{12a}$$

$$\boldsymbol{l}_i(p) = \sum_{k=1}^{m} \beta_{ik} p_k + \boldsymbol{b}_i, \quad p \in \boldsymbol{p} \tag{12b}$$

and have the inclusion property

$$a_{ij}(p) \in \boldsymbol{L}_{ij}(p), \quad p \in \boldsymbol{p} \tag{13a}$$

$$b_i(p) \in \boldsymbol{l}_i(p), \quad p \in \boldsymbol{p}. \tag{13b}$$

The approach herein adopted to determining an outer solution to (1a)–(1d) is based on the use of (8)–(13) and the method for solving perturbed systems of equations in [17]. For this purpose, (1a)–(1d) is written in the form

$$f(x,p) = A(p)x - b(p) = 0, \quad p \in \boldsymbol{p}. \tag{14}$$

We temporarily assume $\boldsymbol{x}$ is a known interval vector. Then, $f(x,p)$ can be enclosed in $\boldsymbol{z} = (\boldsymbol{x}, \boldsymbol{p})$ by the linear interval form

$$\boldsymbol{L}_f(x,p) = A^x x + A^p p + \boldsymbol{v}, \quad x \in \boldsymbol{x}, \quad p \in \boldsymbol{p} \tag{15}$$

where A^x and A^p are $n \times n$ and $n \times m$ real matrices. On account of the inclusion property

$$f(x,p) \in \boldsymbol{L}_f(x,p), \quad x \in \boldsymbol{x}, \quad p \in \boldsymbol{p}. \tag{16}$$

Now, we shall obtain explicit expressions for A^x and A^p and $\boldsymbol{v}$. With this in mind, we first introduce the shorter notation $\boldsymbol{L}_{ij}$ for the intervals $\boldsymbol{L}_{ij}(\boldsymbol{p})$ and $\boldsymbol{l}_i$ for the intervals $\boldsymbol{l}_i(\boldsymbol{p})$. Let $\boldsymbol{L}$ denote the interval matrix whose elements are $\boldsymbol{L}_{ij}$ while L^0 denotes its center. Also, let x^0 be the center of vector $\boldsymbol{x}$. Then, on account of (11)

$$A(p)x \in L^0 x + \boldsymbol{L}x^0 + \boldsymbol{c}, \quad x \in \boldsymbol{x}, \quad L \in \boldsymbol{L} \tag{17a}$$

where

$$\boldsymbol{c} = -L^0 x^0 + [-r^c, r^c]. \tag{17b}$$

In (17b), the radius r^c of $\boldsymbol{c}$ is given by

$$r^c = R^L r^x \tag{17c}$$

where R^L is the radius of $\boldsymbol{L}$ and r^x is the radius of $\boldsymbol{x}$. It is seen from (12a) that the elements R^L_{ij} of R^L are computed as

$$R^L_{ij} = \sum_{k=1}^{m} |\alpha_{ijk}| r^p_k + R^a_{ij} \tag{17d}$$

where r^p_k is kth component of the radius r^p of $\boldsymbol{p}$ and R^a_{ij} is the radius of $\boldsymbol{a}_{ij}$. On account of (12a), the elements L^0_{ij} of L^0 are

$$L^0_{ij} = a^0_{ij} + \sum_{k=1}^{m} \alpha_{ijk} p^0_k. \tag{17e}$$

On the other hand, using (12a)

$$\left(\boldsymbol{L}x^0\right)_i = \sum_{j=1}^{n} \boldsymbol{L}_{ij} x^0_j = \sum_{k=1}^{m} \alpha_{ik} p_k + \boldsymbol{a}_i \tag{18a}$$

where

$$\alpha_{ik} = \sum_{j=1}^{n} x^0_j \alpha_{ijk} \tag{18b}$$

$$\boldsymbol{a}_i = \sum_{j=1}^{n} x^0_j \boldsymbol{a}_{ij}. \tag{18c}$$

In a similar way

$$l^0_i = b^0_i + \sum_{k=1}^{m} \beta_{ik} p^0_k \tag{19a}$$

$$\boldsymbol{l}_i = \boldsymbol{b}_i + \sum_{k=1}^{m} \beta_{ik} \boldsymbol{p}_k. \tag{19b}$$

Let $\boldsymbol{l}$ be the interval vector whose components are defined through (19b). Combining

$$b(p) \in \boldsymbol{l} \tag{20}$$

(17a), and (18a), we finally get the explicit expressions for A^x, A^p and $\boldsymbol{v}$ in (15)

$$A^x = L^0 \tag{21a}$$

$$A^p_{ik} = \sum_{j=1}^{n} \alpha_{ijk} x^0_j - \beta_{ik} \tag{21b}$$

$$\boldsymbol{v} = \boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}. \tag{21c}$$

Now consider the linear interval system related to (15)

$$L^0 x + A^p p + \boldsymbol{v} = 0, \quad p \in \boldsymbol{p} \tag{22}$$

Let $B = (L^0)^{-1}$ and $C = BA^p$. From (22), the solution for x denoted $\boldsymbol{h}$ is

$$\boldsymbol{h} = -Cp - B(\boldsymbol{a} + \boldsymbol{b}) - B\boldsymbol{c}. \tag{23}$$

So far, we have assumed that $\boldsymbol{x}$ is a known interval vector. Now, we proceed to determining $\boldsymbol{x}$ as an outer solution $\boldsymbol{x}'$.

First, we determine the center x^0 of $\boldsymbol{x}'$ as the solution of

$$L^0 x^0 = l^0 \tag{24}$$

where L^0 and l^0 are defined by (17e) and (19a), respectively.

We have next to determine the radius r of $\boldsymbol{x}'$. On account of (23), (17a)–(17e), and (21)

$$r^h = |C|r^p + |B|\left(r^a + r^b\right) + |B|Rr^s. \tag{25}$$

Let for simplicity of notation

$$c = |C|r^p + |B|\left(r^a + r^b\right) \tag{26a}$$

$$D = |B|R. \tag{26b}$$

Taking into account (25), a reasonable choice for r^h is to determine it as the solution y^* of the equation

$$y = c + Dy \tag{27}$$

or equivalently

$$(I \stackrel{\bullet}{-} \dot{D})y = c. \tag{28}$$

The main result of this section is the following theorem.

Theorem 1: Assume the solution y^* to system (28) is positive. Then

1) the interval vector

$$\boldsymbol{x} = x^0 + \boldsymbol{h}' \tag{29a}$$

where

$$\boldsymbol{h}' = [-y^*, y^*] \tag{29b}$$

is an outer solution to (1a)–(1d);

2) matrix $A(p)$ is nonsingular for each $p \in \boldsymbol{p}$.

The proof of the above theorem is similar to that of Theorem 2 in [19] and will therefore be omitted. The method used to solve (28) should however guarantee the nonsingularity of matrix $I - D$.

Based on Theorem 1, the present method for determining an outer solution to system (1a)–(1d) comprises the following computations. First, we evaluate matrix L^0 as well as matrix R using (17e) and (17d). Next, L^0 is inverted to get matrix B. Using (26), we set up system (28). If the solution of (28) is positive, then the outer solution is obtained from (29). If, on the other hand, system (28) does not have a positive solution, the method is not applicable.

If the TA problem is to find an outer solution $\boldsymbol{u}'$ to (1a)–(1d) and (2), $\boldsymbol{u}'$ can be computed as follows:

$$\boldsymbol{u} = g(x) \tag{30}$$

where $\boldsymbol{u}$ is the range of g in $\boldsymbol{x}$.

The above method for computing an outer approximation to the exact solution of the TA problem considered will be referred to as method M1.

IV. Inner Solution

In this section, first a simple iterative method for determining an inner solution $\boldsymbol{x}''$ to system (1a)–(1d) will be presented. In fact, we compute individually each component $\boldsymbol{x}_k''$ of $\boldsymbol{x}''$. As in the previous section, the method will then be extended to determining the component $\boldsymbol{u}_k''$ of the inner solution $\boldsymbol{u}''$ of problem (1a)–(1d) and (2).

The method is based on a local optimization technique and appeals to two procedures which determine separately the lower endpoint $\underline{x}_k''$ and the upper endpoint $\overline{x}_k''$ of $\boldsymbol{x}_k''$. Each procedure makes use of the derivatives of x_k with respect to p_l, $l = 1, \ldots, m$. These derivatives are computed in the following way.

System (1a) is written in detailed form as

$$\sum_{j=1}^{n} a_{ij}(p_1, \ldots m)\, x_j = b_i(p_1, \ldots m) = 0, \quad i = 1, \ldots, n. \tag{31}$$

We are interested in expressing the derivative of x_j with respect to p_l, $l = 1, \ldots m$. With this in mind, we differentiate (31) in p_l and on account of (1b) and (1c) we get

$$\sum_{j=1}^{n} a_{ij}(p)\frac{\partial x_j}{\partial p_l} = \gamma_{il}(p) - \sum_{j=1}^{n} \eta_{ijl}(p)x_j, \quad i = 1, \ldots, n \tag{32a}$$

where

$$\gamma_{il}(p) = \frac{\partial b_i(p)}{\partial p_l} \tag{32b}$$

$$\eta_{ijl}(p) = \frac{\partial a_{ij}(p)}{\partial p_l}. \tag{32c}$$

Systems (32a) and (32b)–(32c) will be rewritten as

$$A(p)\frac{dx}{dp_l} = \gamma_l(p) - \eta_l(p)x(p) \tag{32d}$$

where $\gamma_l(p)$ is a column vector and $\eta_l(p)$ is a matrix. Hence, if $B(p) = A^{-1}(p)$

$$\frac{dx}{dp_l}(p) = B(p)\left(\gamma_l(p) - \eta_l(p)x(p)\right), \quad p \in \boldsymbol{p}. \tag{33}$$

If we are interested in the kth component of dx/dp_l, then (33) becomes

$$\frac{dx_k}{dp_l}(p) = B_k(p)\left(\gamma_l(p) - \eta_l(p)x(p)\right), \quad p \in \boldsymbol{p} \tag{34}$$

where $B_k(p)$ is the kth row of $B(p)$. Now, as can be easily seen, $B_k(p)$ can be computed in a most efficient manner as follows. For a fixed p we solve the real system

$$A^T(p)y = e_k, \quad p \in \boldsymbol{p} \tag{35}$$

where A^T denotes the transpose of A and e^k is the kth column of the identity matrix. Finally, B_k is obtained as y^T.

We have the following procedure for finding the lower endpoint $\underline{x}_k''$ of $\boldsymbol{x}_k''$.

Procedure 1: For a fixed k we start by evaluating the derivative $d_{kl}(p) = dx_k/dp_l(p)$ for $p = p^0$. Let $d_{kl}^0 = d_{kl}(p^0)$, $A^0 = A(p^0)$, $B_k^0 = B_k(p^0)$, $\gamma_l^0 = \gamma_l(p^0)$, $\eta_l^0 = \eta(p^0)$ and $x^0 = x(p^0)$. On account of (34)

$$d_{kl}^0 = B_k^0\left(\gamma_l^0 - \eta_l^0 x^0\right) \tag{36}$$

where

$$B_k^0 = \left(y^0\right)^T \tag{37a}$$

and y^0 is the solution of the real system

$$\left(A^0\right)^T y = c_k. \tag{37b}$$

Now, we determine new values p_l^1, $l = 1, \ldots, m$, using the following formula

$$p_l^1 = \begin{cases} \underline{p}_l, & \text{if } d_{kl} \geq 0 \\ \overline{p}_l, & \text{if } d_{kl} < 0 \end{cases}, l = 1, \ldots, m \tag{38}$$

and form the vector $p^1 = (p_l^1, \ldots, p_m^1)$. We then solve the system

$$A\left(p^1\right) x = b\left(p^1\right) \tag{39}$$

to find the vector x^1. If

$$x_k^1 < x_k^0 \tag{40}$$

x_k^1 is renamed x_k^0, p_l^1 are renamed p_l^0, and the procedure is resumed from the start; otherwise, the procedure is terminated and the inner bound on $\underline{x}_k^*$ is given by the corresponding component x_k^0.

A similar procedure is valid for determining the upper endpoint $\overline{x}_k''$ of $\boldsymbol{x}_k''$.

Procedure 2: For a fixed k, we let $p = p^0$ and repeat the computations (36) and (37a)–(37b) of Procedure 1. Now, we determine a new vector y^1 with components x^1 using the formula

$$p_l^1 = \begin{cases} \overline{p}_l, & \text{if } d_{kl} \geq 0 \\ \underline{p}_l, & \text{if } d_{kl} < 0 \end{cases}, l = 1, \ldots, m. \tag{41}$$

Then, the corresponding system (39) is solved and a new vector x^1 is thus found. If

$$x_k^1 > x_k^0 \tag{42}$$

x_k^1 is renamed x_k^0, and p_l^1 are renamed p_l^0 and the procedure is resumed from the start; otherwise the procedure is stopped and the inner bound on $\overline{x}_k^*$ is given by the corresponding component x_k^0.

It is seen that Procedures 1 and 2 implement a strategy which is based on a local optimization scheme. If the actual dependencies of dx_k/dp_l on dp_l in $\boldsymbol{p}$ are monotone functions, these procedures will produce, in fact, the exact solution $\boldsymbol{x}_k^*$. In the general case, Procedures 1 and 2 will only provide inner bounds on $\boldsymbol{x}_k^*$.

Combined with the outer bounds $\boldsymbol{x}'$ the pair $(\boldsymbol{x}'', \boldsymbol{x}')$ provides a two-sided estimate of the exact solution $\boldsymbol{x}^*$ of system (1a)–(1d). The width of the interval vector $\boldsymbol{w} = \boldsymbol{x}' - \boldsymbol{x}''$ can serve as a measure of the accuracy of the approximations $\boldsymbol{x}'$ and $\boldsymbol{x}''$.

The above approach can also be applied in the case of handling the TA problem (1a)–(1d) and (2). The only difference is that now we compute the derivatives of u with respect to p_l. Thus, for u defined by (7)

$$\frac{du}{dp_l}(p) = 2x_k(p)\frac{dx_k}{dp_l}(p) + 2x_{k+1}(p)\frac{dx_{k+1}}{dp_l}(p), \quad p \in \boldsymbol{p}. \tag{43}$$

The inner solution $\boldsymbol{u}''$ is then found using Procedures 1 and 2 that have been modified accordingly.

V. Exact Solution

In this section, the exact solution to the TA problem (1a)–(1d) and (2), or problem (1a)–(1d) and (5b) will be sought. For simplicity of the presentation, first a method for computing the exact solution $\boldsymbol{x}_k^*$ of the simpler TA problem (1a)–(1d) and (5b) will be suggested. The method is applicable only if the derivatives dx_k/dp_l are guaranteed to be monotone in $\boldsymbol{p}$. These monotonicity conditions can be checked in the following way.

Let $\boldsymbol{x}'$ be the outer solution of (1a)–(1d) computed by the method presented in Section III. Similarly, let $\boldsymbol{B}_k$, $\boldsymbol{\gamma}$ and $\boldsymbol{\eta}$ denote the enclosure of the respective quantities for $p \in \boldsymbol{p}$. Then, we can define $\boldsymbol{D}_{kl}$ as follows:

$$\boldsymbol{D}_{kl} = \boldsymbol{B}_k\left(\boldsymbol{\gamma}_l - \boldsymbol{\eta}_l \boldsymbol{x}'\right) \tag{44}$$

and obviously

$$\frac{dx_k}{dp_l}(p) \in \boldsymbol{D}_{kl}, \quad p \in \boldsymbol{p}. \tag{45}$$

Hence, the derivative considered is guaranteed to be monotone in $\boldsymbol{p}$ if $0 \notin \boldsymbol{D}_{kl}$.

For the special case of linear functions $a_{ij}(p)$ and $b_i(p)$ the matrices γ and η are constant.

As in the previous section, $\boldsymbol{B}_k$ can be computed as an outer solution y of the following system:

$$A^T(p)y = e_k, \quad p \in \boldsymbol{p}. \tag{46}$$

To get a narrow interval vector $\boldsymbol{y}$, system (46) will be solved by the method M1 from Section III. Finally, $\boldsymbol{B}_k$ is obtained as $\boldsymbol{y}^T$.

Using (45), we determine the estimates $\boldsymbol{D}_{kl}$. Now we make the following assumption.

Assumption 2: We assume that each estimate $\boldsymbol{D}_{kl}$, $l = 1, \ldots, m$, satisfies either the condition

$$\boldsymbol{D}_{kl} \geq 0 \tag{47a}$$

or the condition

$$\boldsymbol{D}_{kl} \leq 0. \tag{47b}$$

On account of inclusion (45) the fulfillment of Assumption 2 guarantees that x_k is monotone with respect to each parameter p_l. Now, we define two vectors $\underline{p}^{(k)}$ and $\overline{p}^{(k)}$ as follows:

$$\underline{p}_l^{(k)} = \begin{cases} \underline{p}_l, & \text{if } \boldsymbol{D}_{kl} \geq 0 \\ \overline{p}_l, & \text{if } \boldsymbol{D}_{kl} \leq 0 \end{cases}, l = 1, \ldots, m \tag{48a}$$

$$\overline{p}_l^{(k)} = \begin{cases} \overline{p}_l, & \text{if } \boldsymbol{D}_{kl} \geq 0 \\ \underline{p}_l, & \text{if } \boldsymbol{D}_{kl} \leq 0 \end{cases}, l = 1, \ldots, m. \tag{48b}$$

The exact solution $\boldsymbol{x}_k^*$ of system (1a)–(1d) can be found using the following theorem.

Theorem 2: If Assumption 2 holds, then the kth component $\boldsymbol{x}_k^* = (\underline{x}_k^*, \overline{x}_k^*)$ of the exact solution $\boldsymbol{x}^*$ is determined as follows.

1) $\underline{x}_k^*$ is equal to the kth component of the solution of

$$A\left(\underline{p}^{(k)}\right) x = b\left(\underline{p}^{(k)}\right). \tag{49a}$$

2) $\overline{x}_k^*$ is equal to the kth component of the solution of

$$A\left(\overline{p}^{(k)}\right) x = b\left(\overline{p}^{(k)}\right) \tag{49b}$$

where the vector $\underline{p}^{(k)}$ and $\overline{p}^{(k)}$ are determined according to (48a)–(48b).

The theorem follows directly from the above considerations about the monotonicity of $dx/dp^{(l)}$.

On the basis of the aforegoing we have the following procedure for determining one component x_k^* of the exact solution x^*. Before initiating the procedure, however, we solve system (28) to get an outer solution x' of (1a)–(1d).

Procedure 3: For a given k, solve system (46) using method M1 to find the interval vector B_k. By (44) compute D_{kl}, $l = 1, \ldots, m$. Check conditions (47). If all of them are satisfied, determine the two real vectors $\underline{y}^{(k)}$ and $\overline{y}^{(k)}$ using (48a)–(48b). Finally, solve systems (49a)–(49b) to get the lower end-point $\underline{x}_k^*$ and the upper end-point $\overline{x}_k^*$ of the kth component x_k^* of the exact solution to system (1a)–(1d).

In some cases, it is possible to determine x_k^* even if not all conditions (47a)–(47b) are satisfied, that is if Procedure 3 is not applicable. Indeed, let (for a fixed k) I_1 denote the set of those indices l for which either (47a) or (47b) holds while I_2 denotes the set of indices l for which (47a)–(47b) is violated. Using (48a)–(48b) in which now $l \in I_1$, we can determine those components of the vectors $\underline{p}$ and $\overline{p}$ which are guaranteed to take on end-point values. Thus, each vector can be partitioned into two parts as follows:

$$p = \left(\underline{p}^{(1)}, p^{(2)}\right) \tag{50a}$$

or

$$p = \left(\overline{p}^{(1)}, p^{(2)}\right) \tag{50b}$$

where only the components $p_l^{(2)}$, $l \in I_2$ of $p_l^{(2)}$ are allowed to take on their values within the corresponding intervals $\boldsymbol{p}_l$ forming the interval vector $\boldsymbol{p}^{(2)}$. Let the index set I_1 have m_1 members and I_2 have m_2 members. Since according to (50a)–(50b) the first m_1 components of p are fixed to end-point values it is seen that the original interval vector $\boldsymbol{p}$ has been reduced to a new m_2-dimensional interval vector $\boldsymbol{p}^{(2)}$ whose components are $\boldsymbol{p}_l$ with $l \in I_2$.

We first consider partition (50a). We shall present a procedure for determining the lower end-point $\underline{x}_k^*$ of x_k^*.

Procedure 4: For a fixed k, find the outer solution $\boldsymbol{x}$ of the following modified system of type (1a)–(1d)

$$A\left(\underline{p}^{(1)}, p^{(2)}\right) x = b\left(\underline{p}^{(1)}, p^{(2)}\right), \quad p^{(2)} \in \boldsymbol{p}^{(2)}. \tag{51a}$$

Also, find the outer solution $\boldsymbol{y}$ of the modified system

$$A\left(\underline{p}^{(1)}, \boldsymbol{p}^{(2)}\right) y = e_k \tag{51b}$$

to get the corresponding interval vector B_k. Thus, we can compute by (43) the elements D_{kl}, $l \in I_2$ and check the monotonicity conditions (47a)–(47b). Now we assume that these conditions are fulfilled. Using (48a) we find the reduced-sized vector $\underline{p}^{(2)}$ whose components are $\underline{p}_l^{(2)}$, $l \in I_2$. Finally, $\underline{x}_{(k)}^*$ is computed as the kth component of the solution of

$$A\left(\underline{p}^{(1)}, \underline{p}^{(2)}\right) x = b\left(\underline{p}^{(1)}, \underline{p}^{(2)}\right). \tag{52}$$

We next present a procedure for determining the upper end-point $\overline{x}_k^*$ of x_k^*.

Procedure 5: It has, essentially, the some structure as Procedure 4. Now we solve systems (51a)–(51b) in which $\underline{p}^{(1)}$ is replaced with $\overline{p}^{(1)}$. Thus, we compute the elements D_{kl} corresponding to $(\overline{p}^{(1)}, \boldsymbol{p}^{(2)})$. Once again we assume that the monotonicity conditions (47a)–(47b) are fulfilled. Using (48a)–(48b) we find the real vector $\overline{p}^{(2)}$ whose components are $\overline{p}_l^{(2)}$, $l \in I_2$. Finally, $\overline{x}_k^*$ is computed as the kth component of the solution of

$$A\left(\overline{p}^{(1)}, \overline{p}^{(2)}\right) x = b\left(\overline{p}^{(1)}, \overline{p}^{(2)}\right). \tag{53}$$

The above approach to computing x_k^* will be referred to as method M3.

Method 3 can be used only if both Procedures 4 and 5 are applicable. If, however, the monotonicity conditions (47a)–(47b) are not fulfilled for all components of $\boldsymbol{p}^{(2)}$, a new attempt can be made to determine x_k^*. With this in mind, we treat $\boldsymbol{p}^{(2)}$ as a new reduced-size parameter vector. We then partition the new $\boldsymbol{p}^{(2)}$ into two parts and apply once again method M3. This new computational scheme will also be referred to as method M3.

Method M3 is also applicable in the general case of nonlinear relation (2). Thus, for function (7) we have once again to appeal to formula (43) where, however, all real variables are to be replaced with their interval counterparts. Afterwards, we employ Procedures 4 and 5 that have been modified accordingly.

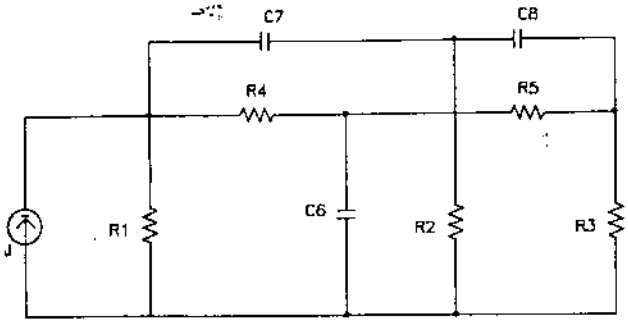

Fig. 1. Notch filter circuit.

VI. Numerical Example

The new methods will be illustrated with the following example. The linear AC circuit considered [11] is shown in Fig. 1. The nominal (center) values of the interval element parameters are

$$R_i^c = 10^4\ \Omega, \quad i = 1, \ldots, 5 \tag{54a}$$

$$C_6^c = 2 \cdot 10^{-4}\ \text{F}, \quad C_7^c = C_7^c = 10^{-4}\ \text{F} \tag{54b}$$

while the fixed quantities are

$$\omega = 10^3\ \text{s}^{-1}, \quad J = 10^{-3}\ \text{A}. \tag{54c}$$

The TA problem to be solved consists in finding the outer, inner, and exact solutions related to the real part V of the output voltage V_3 for several values of the tolerances on R_i and C_i. Thus, the TA problem considered here is of the type (1a)–(1d) and (6).

The numerical experiments are done in a MATLAB environment on a 400-MHz Pentium II PC. The interval arithmetic operations are implemented using the toolbox INTLAB. For space limitation, the numerical results obtained will be reported to only four decimal places.

A. Outer Solutions

Unlike the algorithms using real (complex) calculations, the results obtained by interval algorithms depend on the system describing the problem on hand. Thus, the nominal value of V will be the same notwithstanding whether the system of equations used is set up by nodal analysis, loop analysis or in tableau form. It will be shown that the result for the outer solution, however, is different depending on what type of analysis equations is employed. For the purpose of comparison three types of system of equations will be considered.

System S1: This is a system of equations of the type used in [10] or [6, Ch. 3]. It consists of $2n = 2(n' + m')$ equations where n' and m' is the number of unknown nodal voltages and branch currents, respectively

$$Ax = b \tag{55a}$$

and the interval entries lie only in the first 16 positions along the main diagonal of the coefficient matrix A. The interval coefficients are dependent since

$$a_{ii} = a_{i+m},\ i + m = p_i,\ p_i \in \boldsymbol{p}_i, \quad i = 1, \ldots, m. \tag{55b}$$

For the circuit analyzed $n' = 4$, $m' = 8$ so $n = 24$ and

$$\boldsymbol{p}_i = \boldsymbol{R}_i = R_i^c + t\,[-R_i^c, R_i^c] \tag{56a}$$

where t is the tolerance chosen and $[-R_i^c, R_i^c]$ is a symmetric interval

$$\boldsymbol{p}_i = \boldsymbol{X}_i, \quad i = 6, 7, 8 \tag{56b}$$

with

$$\boldsymbol{X}_i = \frac{1}{\boldsymbol{B}_i}, \quad B_i^c = \omega C_i^c, \quad \boldsymbol{B}_i = B_i^c + t\,[-B_i^c, B_i^c]. \tag{56c}$$

The real part V of the output voltage is represented by the component x_{19} of the real vector x.

System S2: This is the so-called hybrid system of equations [11]–[13]. It has the same structure as system S1, i.e., the interval dependencies are again given by (55b), but now system (55a) has reduced size involving $n = 2m'$ equations. For the circuit studied $n = 16$,

$$\begin{aligned} p_1 &= G_1,\ p_2 = B_6,\ p_3 = G_4,\ p_4 = G_2 \\ p_5 &= R_4,\ p_4 = R_5 \ldots,\ p_7 = X_7,\ p_8 = X_8 \end{aligned} \tag{57}$$

and $G_i = 1/R_i$. Now V is given by x_3.

System S3: In this case, system (55a) is set up using nodal analysis and involves $n = 2n'$ equations. Now

$$a_{ij} = \sum \alpha_{ijk} p_k, \quad i, j = 1, \ldots, n \tag{58a}$$

where α_{ijk} can be 0, +1 or −1. For the circuit investigated $n = 8$ and

$$p_k = G_k,\ k = 1, \ldots, 5, \tag{58b}$$

$$p_k = B_k,\ k = 6, 7, 8. \tag{58c}$$

The output variable V is given by x_3.

We first compare the outer solutions for system S1 obtained by the present method M1 and Hansen's method for two values of the tolerance t. Although a better version [17] of Hansen's method has here been used than that employed in [12] and [13], the comparison of the results obtained by the two methods and listed in Table I shows that the present method outperforms Hansen's method. Indeed, the widths of the intervals $\boldsymbol{V}'$ obtained by M1 are smaller as compared to these corresponding to Hansen's method. These results confirm the theoretical prediction since, unlike Hansen's method, the present method accounts for the dependencies between the elements of the system of equations employed. At the same time, both methods require the same computing time t.

TABLE I
COMPARISON OF THE OUTER SOLUTIONS V' OBTAINED BY THE PRESENT METHOD M1 AND HANSEN'S METHOD

			Present method M1		Hansen's method	
t	System	n	V'(V)	τ (s)	V'(V)	τ (s)
0.05	S1	24	[0.1760, 0.4567]	0.06	[0.1115, 0.5310]	0.06
0.1	S1	24	[-0.0550, 0.6837]	0.06	[-0.2157, 0.8785]	0.06

TABLE II
DATA ON V' OBTAINED BY M1 FOR SYSTEMS S1, S2, S3

t	System	n	V'(V)	τ (s)
0.05	S1	24	[0.1760, 0.4567]	0.06
0.05	S2	16	[0.1787, 0.4477]	0.05
0.05	S3	8	[0.1421, 0.4891]	0.06
0.1	S1	24	[-0.0550, 0.6837]	0.06
0.1	S2	16	[-0.0316, 0.6350]	0.06
0.1	S3	8	[-0.4050, 1.0273]	0.06

In Table II, we present data on the outer solutions $\boldsymbol{V}'$ for two values of t, obtained by the present method M1 using systems S1, S2 and S3. It is seen that for both values of t the outer solution is the narrowest if system S2 is used while the computation time is practically the same.

B. Inner Solutions

The inner solutions $\boldsymbol{V}''$ for V are obtained by Procedures 1 and 2 (method M2). Table III presents data on the width of $\boldsymbol{V}''$, the total number of iterations N needed by both Procedure 1 and Procedure 2, and the execution time ι corresponding to the case where the system of equations used is S1.

The same intervals $\boldsymbol{V}''$ are obtained when systems S2 and S3 are used.

To assess the effectiveness of the present method, we also found the inner solutions for V using the Monte-Carlo method. Table IV lists results corresponding to two values t and two values of the number of trials N_t.

The comparison of the results for $\boldsymbol{V}''$ and ι in Tables III and IV clearly shows that the present method M2 is superior to the Monte-Carlo method. Indeed, method M2 provides wider intervals $\boldsymbol{V}''$ and hence tighter approximations to the exact solutions $\boldsymbol{V}^*$. At the same time, it requires less computation time although M2 was applied to system S1 ($n = 24$) while the Monte-Carlo method was implemented using the smaller system S3 ($n = 8$).

TABLE III
DATA ON THE INNER SOLUTIONS V'' OBTAINED BY THE PRESENT METHOD M2

t	System	n	V'' (V)	N	τ (s)
0.05	S1	24	[0.2122, 0.4340]	5	0.12
0.1	S1	24	[0.1196, 0.5630]	5	0.12

TABLE IV
DATA ON V'' OBTAINED BY THE MOTE-CARLO METHOD

t	System	n	N_t	V'' (V)	τ (s)
0.05	S3	8	1000	[0.2457, 0.3931]	1.43
0.05	S3	8	10000	[0.2354, 0.4002]	14.33
0.1	S3	8	1000	[0.1736, 0.4591]	1.43
0.1	S3	8	10000	[0.1651, 0.4900]	14.39

TABLE V
TWO-SIDED BOUNDS ON THE ENDPOINTS OF THE EXACT SOLUTION V^*

Methods	Bounds on $\underline{V}^*$	Bounds on $\overline{V}^*$	τ(s)
M1 + M2	[0.1787, 0.2122]	[0.4340, 0.4477]	0.17
MC + H	[0.1115, 0.2457]	[0.3931, 0.5310]	1.49

The inner and outer solutions obtained by methods M2 and M1, respectively, provide tight two-sided bounds on each endpoint $\underline{V}^*$ and $\overline{V}^*$ of the exact solution V^*. Table V lists data on these bounds (corresponding to $t = 0.05$, S2 and $N_t = 1000$) obtained by methods M1 and M2, on the one hand, and method Monte-Carlo (MC) and Hansen's method (H), on the other.

C. Exact Solution

Using method M3 (Procedure 4 and 5), the exact solution V^* has been obtained for all systems of equations when $t = 0.05$. Table VI lists results for the endpoints of V^*, the total number of iterations N (corresponding to the determination of both $\underline{V}^*$ and $\overline{V}^*$) and the required computing time τ.

If, however, $t = 0.1$ the exact solution has been reached only for S1 and S2—Table VII.

Method M3 is inapplicable with system S3 since for $t = 0.1$ none of the monotonicity conditions (47a)–(47b) is satisfied. However, if t is reduced to $t = 0.07$, the following result has been obtained (Table VIII).

In this case, method M3 reaches for five iterations only the exact lower endpoint $\underline{V}^*$. It cannot converge to the upper endpoint $\overline{V}^*$ since after 5 iterations conditions (47a)–(47b) remain unsatisfied for $l = 4$ and $l = 5$. However, as is seen in the table, the method provides a good two-sided bound on $\overline{V}^*$.

D. Analysis of the Numerical Results

The comparative analysis of the numerical results obtained can be summarized as follows. The conservatism of the outer solutions depends on the type of system of equations used. According to this criterion, the best results (narrowest solution intervals) are obtained when system S2 is employed. This is seemingly a valid conclusion for low- and medium-size circuits. However, for large-size TA problems, it is hoped that system S3 will be preferable. Further numerical experiments are needed to make a decision on this point. The exact solution is obtained with least computational effort if, again, system S2 is used. It should, however, be verified if this conclusion remains valid for large-scale circuits.

TABLE VI
DATA ON THE EXACT SOLUTION V^* OBTAINED BY THE PRESENT METHOD M3 FOR $t = 0.05$

t	System	n	V^*	N	τ(s)
0.05	S1	24	[0.2122, 0.4340]	4	0.22
	S2	16	[0.2122, 0.4340]	4	0.22
	S3	8	[0.2122, 0.4340]	6	032

TABLE VII
DATA ON V^* OBTAINED BY M3 FOR $t = 0.1$

t	System	n	V^*	N	τ(s)
0.1	S1	24	[0.1196, 0.5630]	7	0.44
	S2	16	[0.1196, 0.5630]	5	0.27

TABLE VIII
DATA ON V^* OBTAINED BY M3 FOR $t = 0.07$

t	System	n	$\underline{V}^*$	Bounds on $\overline{V}^*$	N	τ(s)
0.07	S3	8	0.1736	0.4709 ∓ 0.0241	10	0.55

It should be also stressed that at least for medium-size circuits the numerical efficiency of the present approach seems to be superior to that of the Monte Carlo method. It is expected that this conclusion remains valid also for large-scale circuits.

A final important remark is to be made. It should be noted that in all experiments the inner solution is in fact equal to the exact solution, i.e.,

$$x'' = x \tag{59}$$

(cf. Table III, Tables VI and VII). It is believed that the equality (59) is a generic characteristic for a large class of linear electric circuits. The derivation of a simple criterion establishing the validity of (59) and circumventing the verification of the costly monotonicity conditions (47a)–(47b) would substantially improve the overall efficiency of the new approach to solving the TA problems.

VII. CONCLUSION

A general framework (1a)–(1d) and (2) for treating worst-case TA problems has been considered. The linear algebraic system (1a)–(1d) has elements that are nonlinear functions of a set of parameters belonging to given intervals.

A simple method M1 for determining an outer solution to the TA problem considered has been suggested in Section III. It is

based on Theorem 1 and reduces essentially to setting up and inverting an $n \times n$ real matrix and solving a system of n real linear equations (28), n being the size of the original system (1a)–(1d). The method is self-validating: it is applicable only if the solution to system (28) is positive.

In Section IV, an iterative method M2 (Procedures 1 and 2) is suggested which permits to determine an inner TA solution.

A more involved method M3 (Procedure 3 or Procedures 4 and 5) for computing the exact interval solution of the TA problem considered has been presented in Section V. It is based on the use of Theorem 2 and method M1. The latter is applied to compute tight enclosures D_{kl} of the derivatives dx_k/dp_l according to formula (44). If all derivative monotonicity conditions (47a)–(47b) hold, method M3 is guaranteed to determine the exact solution. In some cases, the method can provide the exact solution even if not all monotonicity conditions (47a)–(47b) are satisfied.

A numerical example has been solved in Section VI. It illustrates the applicability of the above methods to determining an outer, an inner and the exact solution to TA problems. The experimental results seem to show that at least for low- and medium-size circuits the present approach is superior to the Monte Carlo method as regards both accuracy and computing time. Further investigations are needed to decide whether the same conclusion is valid in the case of large-size circuits.

The three methods suggested can be extended to encompass TA problems related to transients in linear circuits. This generalization will be presented in a subsequent publication.

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