INTERNATIONAL JOURNAL OF CIRCUIT THEORY AND APPLICATIONS, VOL. 25, 37–42 (1997)

LETTER TO THE EDITOR

USE OF INTERVAL SLOPES IN IMPLEMENTING AN INTERVAL METHOD FOR GLOBAL NON-LINEAR DC CIRCUIT ANALYSIS

LUBOMIR V. KOLEV AND VALERI M. MLADENOV

Department of Electrical Engineering, Faculty of Automatica, Technical University of Sofia, 1756 Sofia, Bulgaria

INTRODUCTION

Interval methods (see Reference 1, §6.1 and references, cited therein as well as Reference 2) have proved a reliable tool for global analysis of non-linear DC electric circuits (finding all DC operating points of the circuit) as they provide infallible bounds on all the operating points. The earlier methods of this class had, however, high computational complexity, especially for circuits of increased dimensionality.

Various attempts have been undertaken to improve the computational efficiency of the interval methods. The best results—as regards both rate of convergence and memory volume requirements—have been obtained when the circuit description is in the hybrid representation form. For this case an interval method suggested in Reference 2 (a version of method no. 5 in §6.1 of Reference 1) proved to provide the fastest rate of convergence among all known methods, having at the same time comparable memory storage. The basic characteristic of this method is the use of interval derivatives. Indeed, let

$$\boldsymbol{\psi}(\mathbf{x}) = \boldsymbol{\varphi}(\mathbf{x}) - \mathbf{H}\mathbf{x} - \mathbf{s} = \mathbf{0} \tag{1a}$$

where $\boldsymbol{\psi}: \mathbb{R}^n \rightarrow \mathbb{R}^n$ with

$$\varphi_i(x) = \varphi_i(x_i) \tag{1b}$$

be the hybrid description of the circuit investigated, $\varphi_i(x_i)$ being continuously differentiable functions, i.e. $\varphi_i \in C^1$. The method is based on the following iterative procedure:[1,2]

$$Y_i^{(k)} = x_i^{(k)} - D_i^{-1}(X_i^{(k)})\left(\varphi_i(x_i^{(k)}) - h_{ii}x_i^{(k)} - s_i - \sum_{j=1}^{i-1} h_{ij}X_j^{(k+1)} - \sum_{j=i+1}^{n} h_{ij}X_j^{(k)}\right) \tag{2a}$$

$$X_i^{(k+1)} = X_i^{(k)} \cap Y_i^{(k)}, \qquad k \geq 0 \tag{2b}$$

where k is the iteration number, $X_i^{(k)}$ is the ith component of the current box $X_i^{(k)}$, while $D_i(X_i^{(k)})$ is the interval extension of the derivative

$$d_i(x_i^{(k)}) = \frac{\mathrm{d}}{\mathrm{d}x_i}\left[\varphi_i(x_i^{(k)}) - h_{ii}x_i^{(k)}\right] = \varphi_i'(x_i^{(k)}) - h_{ii} \tag{3}$$

It is well known that combined with an appropriate partitioning algorithm of the current box $X^{(k)}$ and the use of extended interval division[1] (see References 1 and 2 for technical details), procedure (2) guarantees global convergence of the method considered even in the case where $D_i(X_i^{(k)})$ contains zero.

In this letter it is suggested to use interval slopes $S_i(X_i^{(k)})$ in (2a) instead of interval derivatives $D_i(X_i^{(k)})$. Theoretical considerations as well as numerical examples show that the new approach leads to a considerable improvement in the numerical efficiency of the resultant modification of the method considered.

CCC 0098–9886/97/010037–06

Received 8 May 1995
Revised 20 December 1995

INTERVAL SLOPES

Interval slopes were introduced in interval computations in 1985 in Reference 3 for the case of rational functions in a single variable. Computation of interval slopes (in fact, of interval extensions of the slopes) is based on recursive formulae (slope arithmetic). The original slope arithmetic[3] was extended to the multivariate case in Reference 4 and subsequently generalized to functions containing irrational components.[4-7] Presently, only the rational part of the function is, however, treated by way of interval slopes, while interval derivatives are used for the irrational components.

Interval slopes were used seemingly for the first time in non-linear DC analysis for the special case of non-linear elements described by rational functions in Reference 8.

In this letter, the traditional approach of using interval (first-order) slopes will be extended to cover the irrational functions also. This will permit the analysis of such a typical class of non-linear DC circuits containing transistors (and diodes) if the Ebers–Moll model of the transistors is used.

In order to generalize the interval slope to the case of irrational functions, we first need an explicit expression for the non-interval (point) slope of a (rational or irrational) function $f\colon D \subseteq \mathbb{R}^1$, $f \in C^1(D)$. The point slope of f is defined at $x, z \in D$ as[3]

$$f[x,z] = \begin{cases} [f(x) - f(z)]/(x,z), & x \neq z \quad (4a) \\ f'(x), & x = z \quad (4b) \end{cases}$$

If f is rational, the traditional approach is to first determine the point slope $f[x,z]$ explicitly as a rational extension in x and z. Then an interval extension $F[X,z]$ of $f[x,z]$ is found with respect to $x \in X$ (where X is an interval) and this is the interval slope of f. If f is irrational, the point slope does not have a finite representation and the above approach is not applicable any more.[3-7]

To circumvent this difficulty, the following approach is suggested here. It is based on direct determination of the range $f[X,z]$ of the point slope, where

$$f[X,z] = \{f[x,z]\colon x \in X\} \tag{5}$$

The new approach will be computationwise effective if the range $f[X,z]$ can be determined with little computational cost. It turns out that this is the case for those functions f (irrational or rational) which are either concave or convex in their domain D. For brevity the sets of these functions will be denoted by W_1 and W_2 respectively. Additionally, let $W = W_1 \cup W_2$. Examples of such functions are $\sqrt{x}$ for $x \geqslant 0$, $\exp(x)$, $\ln(x)$, $\sin(x)$ for $0 \leqslant x \leqslant \pi$ or for $\pi \leqslant x \leqslant 2\pi$, etc. Indeed, if f is concave in $D(f \in W_1)$, it is easily seen that for any $X = [\underline{x}, \bar{x}] \in D$ and any $z \in X$ distinct from $\underline{x}$ or $\bar{x}$ the interval slope S defined as

$$S = f[X,z] \tag{6}$$

is given by the simple formula

$$S = \left[\frac{f(z) - f(\underline{x})}{z - \underline{x}}, \frac{f(\bar{x}) - f(z)}{\bar{x} - z}\right], \quad z \neq \underline{x}, \quad z \neq \bar{x} \tag{7a}$$

If $z = \underline{x}$ or $z = \bar{x}$, then the lower endpoint $\underline{S}$ or the upper endpoint $\bar{S}$ of S is to be calculated as

$$\underline{S} = f'(\underline{x}) \tag{7b}$$

or

$$\bar{S} = f'(\bar{x}) \tag{7c}$$

respectively.

If f is convex in $D(f \in W_2)$, then simmlarly

$$S = \left[\frac{f(\bar{x}) - f(z)}{\bar{x} - z}, \frac{f(z) - f(\underline{x})}{z - \underline{x}}\right], \quad z \neq \underline{x}, \quad z \neq \bar{x} \tag{8a}$$

while

$$\underline{S} = f'(\bar{x}), \qquad z = \bar{x} \tag{8b}$$

or

$$\bar{S} = f'(\underline{x}), \qquad z = \underline{x} \tag{8c}$$

If the component

$$f_i(x_i) = \varphi_i(x_i) - h_{ii}x_i \tag{9}$$

(where the superscript k is dropped for simplicity) in (3) belongs to W, then the corresponding interval slopes S_i over some interval X_i can be most efficiently calculated by (7) or (8).

It should be stressed that

$$S_i(X_i) \subseteq D_i(X_i) \tag{10}$$

where D_i is the extension of the interval derivative of (9) (which is the interval extension of (3)) and usually the inclusion is strict. This in turn makes the interval $Y_i^{(k)}$ in (2a) narrower, which speeds up the convergence rate of the method if $S_i^{-1}(X_i^{(k)})$ is used rather than $D_i^{-1}(X_i^{(k)})$. For this reason the new version of the iterative procedure (2) makes use of the slope $S_i^{-1}(X_i^{(k)})$.

If some component f_i by (9) is not in W within the interval X_i, the simplest approach is to subdivide the interval X_i into subintervals X_{ij}, $j = 1, 2, \ldots, m$, such that $f_i \in W$ for each X_{ij}. Examples show that most often $m = 2$ suffices. For instance, if $f(x) = \sin(x)$ and $X = [-\pi/2, \pi/2]$, then X should be divided into $X_1 = [-\pi/2, 0]$ and $X_2 = [0, \pi/2]$.

ILLUSTRATIVE EXAMPLES

Example 1

We consider the well-known circuit containing four transistors.[9] The hybrid representation (1a) is now chosen of the form

$$\begin{aligned}
\exp(40x_1) - 1 - 510075{\cdot}199x_1 + 957262{\cdot}1492x_2 - 77237{\cdot}6032x_3 + 39152{\cdot}7349x_4 - 712951{\cdot}924 &= 0\\
\exp(40x_2) - 1 + 1087283{\cdot}2583x_1 - 515509{\cdot}9934x_2 + 164640{\cdot}729x_3 - 83458{\cdot}5044x_4 - 2671423{\cdot}626 &= 0\\
\exp(40x_3) - 1 - 77237{\cdot}6032x_1 + 39152{\cdot}7349x_2 - 510075{\cdot}199x_3 + 957262{\cdot}1492x_4 - 909696{\cdot}2857 &= 0\\
\exp(40x_4) - 1 + 164640{\cdot}729x_1 - 83458{\cdot}5044x_2 + 1087283{\cdot}2583x_3 - 515509{\cdot}9934x_4 - 2252040{\cdot}6554 &= 0
\end{aligned} \tag{11}$$

In this example the functions (9) are

$$f_i(x_i) = \exp(40x_i) - 1 - h_{ii}x_i \tag{12}$$

and obviously each $f_i \in W_1$. Thus the interval slopes were computed by (7a).

Usually[4–7] the fixed point z_i is the middle point of X_i. This is, however, not necessarily the best choice. For more flexibility z_i was defined by the formula

$$z_i = \underline{x}_i + \alpha(\bar{x}_i - \underline{x}_i) \tag{13}$$

where α is a suitably chosen constant ranging from zero to unity. Indeed, if $\alpha = 0{\cdot}5$, then z_i is the middle point of X_i; for $\alpha = 0$ and $\alpha = 1$ one gets obviously $z_i = \underline{x}_i$ and $z_i = \bar{x}_i$ respectively.

For the example considered, it turned out that for an accuracy $\varepsilon = 10^{-3}$ (ε is the width of the box enclosing a solution of (11)) the best rate of convergence of the modified method using interval slopes was obtained for $\alpha = 0{\cdot}8$.

To compare the numerical efficiency of the new method with other available interval methods, the number N_i of iterations needed to locate all nine solutions of (11) with precision 10^{-3} are given in Table I. The initial interval box is given by the intervals

$$X_1^{(0)} = [-1{\cdot}1, 0{\cdot}4], \qquad X_2^{(0)} = [-5, 0{\cdot}4], \qquad X_3^{(0)} = [-1{\cdot}6, 0{\cdot}4], \qquad X_4^{(0)} = [-4, 0{\cdot}4] \tag{14}$$

The data for M1 (Krawczyk's method), M2 (the method from Reference 10) and M3 (the method from Reference 2) are known.[2] The datum $N_i = 75$ corresponding to M4 stands for a slightly improved version M3 using interval derivatives and two optimal points (called lower and upper poles). The new method based on interval slopes is designated as M5. It is seen that the use of interval slopes reduces considerably the number N_i, of iterations as compared with the other interval methods based on interval derivatives.

Example 2

In this example a circuit containing 10 tunnel diodes and studied in Reference 11 has a description of the form

$$\begin{aligned}
\varphi_1(x_1) + x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} - 1 &= 0\\
\varphi_2(x_2) + x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} - 2 &= 0\\
\varphi_3(x_3) + x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} - 3 &= 0\\
\varphi_4(x_4) + x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} - 4 &= 0\\
\varphi_5(x_5) + x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} - 5 &= 0\\
\varphi_6(x_6) + x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} - 6 &= 0\\
\varphi_7(x_7) + x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} - 7 &= 0\\
\varphi_8(x_8) + x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} - 8 &= 0\\
\varphi_9(x_9) + x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} - 9 &= 0\\
\varphi_{10}(x_{10}) + x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} - 10 &= 0
\end{aligned} \tag{15}$$

where

$$\varphi_i(x_i) = 2{\cdot}5x_i^3 - 10{\cdot}5x_i^2 + 11{\cdot}8x_i, \qquad i = 1, 2, \ldots, 10 \tag{16}$$

The initial box $X^{(0)}$ is defined by

$$x_i \in [-1, 4], \qquad i = 1, 2, \ldots, 10 \tag{17}$$

The accuracy ε has been chosen to be 0·0001 and the parameter α in (13) determining the points z_i for computing interval slopes is now 0·2. Unlike in Example 1, the functions

$$f_i(x_i) = \varphi_i(x_i) + x_i \tag{18}$$

are not in W for all current intervals X_i arising during the process of dynamically halving the interval box $X^{(0)}$ into subboxes $X^{(k)}$. Three cases are now possible. Let $\tilde{x}_i = 1{\cdot}4$ be the inflection point of $f_i(x_i)$. If $X_i \subseteq [\underline{x}_i, \tilde{x}_i]$, then, as is easily seen, $f_i \in W_2$; if $X_i \in [\tilde{x}_i, \bar{x}_i]$, then $f_i \in W_1$. Finally, if $\tilde{x}_i \in \text{int } X_i$ (int stands for interior), then f_i is generally neither in W nor in W_1 or W_2 within X_i.

Table I. Numerical efficiency of the interval methods considered

Method	M1	M2	M3	M4	M5
N_i	207	143	79	75	46

Table II. Solutions to system (15) for $\varepsilon = 10^{-4}$

Solution	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
1	−0·2794	−0·2224	−0·1603	−0·0917	−0·0144	0·0751	0·1836	0·3279	2·6232	2·7303
2	−0·2703	−0·2125	−0·1494	−0·0795	−0·0005	0·0915	1·7017	0·3581	0·8213	2·7461
3	−0·2798	−0·2229	−0·1608	−0·0922	−0·0150	0·0744	2·2292	0·3266	0·5902	2·7296
4	−0·2824	−0·2257	−0·1638	−0·0956	−0·0189	0·0698	0·1770	2·4725	0·5683	2·7251
5	−0·2912	−0·2352	−0·1742	−0·1071	−0·0320	0·0545	2·0764	0·2919	1·0957	2·7091
6	−0·3355	−0·2829	−0·2261	−0·1644	−0·0962	−0·0195	0·0691	2·2030	2·4711	2·6156
7	−0·3200	−0·2662	−0·2081	−0·1446	−0·0741	0·0057	0·0989	1·6696	2·5204	2·6511
8	−0·3000	−0·2447	−0·1847	−0·1187	−0·0451	0·0392	0·1393	2·4100	1·1660	2·6922
9	−0·2748	−0·2174	−0·1548	−0·0855	−0·0073	0·0835	1·7396	0·3430	0·9220	2·7385

To get the most of the interval slopes as compared with the interval derivatives, the following approach has been adopted for this example. The interval box $X^{(0)}$ is preliminarily partitioned into 1024 subboxes X^ν by dividing each edge $X_i^{(0)}$ into two subintervals using the inflection point $\tilde{x}_i = 1{\cdot}4$ as the upper endpoint of the first subinterval or as the lower endpoint of the second subinterval respectively. Then, as is easily seen, f_i is either in W_1 or in W_2 for any of the subintervals X_i^ν.

For this example two methods have been successively applied to each subbox X_ν, $\nu = 1, 2, \ldots, 1024$. The first method using interval derivatives (based on procedure (2)) and denoted as M6 required a total of $N_i = 524{,}143$ iterations for $\varepsilon = 10^{-4}$ to locate infallibly all nine solutions of (15) available in the initial box (17). The second method used was the new version M5 based on interval slopes. Now all nine solutions were located within the same accuracy $\varepsilon = 10^{-4}$ in a total of $N_i = 116{,}522$ iterations. The solutions obtained are given in Table II. Each component x_i^s, $i = 1, \ldots, 10$, $s = 1, \ldots, 9$ (where s is the number of the solution point in $\mathbb{R}^{10}$ and i is the co-ordinate number of each solution point), is the midpoint of the corresponding solution interval X_i^s whose width is less than or equal to ε.

It is worthwhile mentioning that the method from Reference 11 based on piecewise linear approximation of (16) by subdividing each interval $[-1, 4]$ into 10 equally spaced segments locates only seven solutions of (15) in the same initial box $X^{(0)}$ given by (17).

Finally, the following remark concerning the numerical efficiency of the new approach should be made. It is to be stressed that, owing to (10), the use of interval slopes rather than interval derivatives guarantees a better rate of convergence irrespective of the choice of z_i and hence of $\alpha \in [0, 1]$ in (13). Our experimental evidence has shown the validity of this assertion for various values of α. Initially, α was taken to be 0·5 (z_i being the middle point of the corresponding interval in this case), which led to considerable efficiency improvement as compared with the other interval methods available. Afterwards, starting with $\alpha = 0$ and incrementing it by 0·2 up to $\alpha = 1$, the quasi-optimal values $\alpha = 0{\cdot}8$ for Example 1 and $\alpha = 0{\cdot}2$ for Example 2 were found. It should be borne in mind that the additional amount of computation needed to determine approximately the best choice for α will be compensated advantageously if the method is to be applied repeatedly for the analysis or design of a class of non-linear devices having similar characteristics.

CONCLUSIONS

A new version of an interval method for global analysis of non-linear DC circuits described by its hybrid representation has been suggested. The original method is based on the iterative procedure (2) making use of the interval derivatives $D_i(X_i^{(k)})$. The new version appeals to the use of corresponding interval slopes $S_i(X_i^{(k)})$.

Unlike other known forms of interval slopes, the interval slopes introduced in this letter are applicable to irrational functions also. They provide the narrowest possible width for S_i. Since $S_i \subseteq D_i$ and the inclusion is practically always strict, the use of the interval slopes improves substantially the numerical efficiency of the modified interval method.

Numerical examples involving systems of up to 10 non-linear equations and having up to nine solutions show that the new version reduces the number of iterations needed to locate infallibly, within the accuracy chosen, all the operating points of the circuits investigated.

ACKNOWLEDGEMENTS

The authors are grateful to one of the reviewers for his valuable recommendations. This work was supported by the National Research Fund of Bulgaria under Contract 554/95.

REFERENCES

1. L. V. Kolev, *Interval Methods for Circuit Analysis*, World Scientific, Singapore, 1993.
2. L. V. Kolev and Y. M. Mladenov, 'An interval method for global non-linear DC circuit analysis', *Int. j. cir. theor. appl.*, **22**, 233–241 (1994).
3. R. Krawczyk and A. Neumaier, 'Interval slopes for rational functions and associated centered forms', *SIAM J. Numer. Anal.*, **22**, 604–616 (1985).
4. A. Neumaier, *Interval Methods for Systems of Equations*, Cambridge University Press, London, 1990.
5. S. Zuhe and M. A. Wolfe, 'On interval enclosures using slope arithmetic', *Appl. Math. Comput.*, **39**, 89–105 (1990).
6. E. Hansen, *Global Optimization Using Interval Analysis*, Marcel Dekker, New York, 1983.
7. E. Hansen, 'Computing zeros of functions using generalized interval mathematics', *Interval Comput.*, 3–28 (1994).
8. L. V. Kolev and V. M. Mladenov, 'Calculation of interval extensions of rational functions using interval slopes', *National Youth School of Mathematical Applications in Engineering*, pp 155–157, Varna, 1987 (in Bulgarian).
9. L. O. Chua and A. Ushida, 'A switching-parameter algorithm for finding multiple solutions of nonlinear resistive circuits', *Int. j. cir. theor. appl.*, **4**, 215–239 (1976).
10. L. V. Kolev and V. M. Mladenov, 'An interval method for finding all operating points of non-linear resistive circuits', *Int. j. cir. theor. appl.*, **18**, 257–267 (1990).
11. K. Yamamura, 'Finding all solutions of piecewise-linear resistive circuits using simple sign test', *IEEE Trans. Circuits and Systems*, **CAS-40**, 546–551 (1993).