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A general interval method for tolerance analysis

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***Abstract* - In this paper, an interval method for tolerance analysis of electric circuits is proposed. It is rather general and can be applied to solving tolerance problems both for linear and nonlinear circuits. The problems to be solved can be either of the deterministic, worst-case type (with independent or dependent parameters) or of the probabilistic type (when there are statistical dependencies between the parameters). Under a computationally verifiable condition, the method suggested guarantees to yield an interval solution that encloses the actual set of solutions to the specific tolerance problem considered. Numerical examples related to the worst-case tolerance problem for nonlinear dc circuits seem to indicate that the present method has attractive computational performance.**

***Index Terms* - Tolerance analysis, interval analysis.**

I. Introduction

INTERVAL methods for solving various types of electric circuit tolerance analysis problems have been in existence for over twenty years [1]-[9]. The major part among these treats the worst-case (deterministic) tolerance problem for linear circuits. The linear tolerance problem in probabilistic setting is considered in [5], § 2.5. Papers [7] and [8] address the worst-case tolerance problem for nonlinear electric circuits. The known methods, however, differ considerably from one another depending on the class of circuits analyzed (linear or nonlinear), type of problem to be solved (in deterministic or probabilistic setting) and the type of problem formulation used (as a global constrained optimization problem or in the form of an interval linear or nonlinear system of equations).

In this paper, a general interval method for solving any of the known tolerance analysis problems for both linear and nonlinear circuits is suggested. It is based on a modification and generalization of a method proposed recently in [8]. Unlike [8] where the nonlinear worst-case tolerance problem was analyzed, now the system of equations describing the tolerance problem considered is in rather a general form

$$f(x,p)=0 \qquad (1a)$$

$$p \in \mathbf{p} \qquad (1b)$$

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where f a n'-dimensional vector function, x is a n-dimensional output variable vector, p is a m-dimensional parameter vector and $\mathbf{p}$ is the corresponding interval vector (box). It is assumed that a pair (x^0, p^0) corresponding to the nominal solution is known such that $f(x^0,p^0)=0$ with $p^0 \in \mathbf{p}$; p^0 is usually the center of $\mathbf{p}$.

The solution set $S_f(\mathbf{p})$ of (1) is the set

$$S_f(\mathbf{p}) := \{x : f(x,p)=0, \quad p \in \mathbf{p}\} \qquad (2)$$

The interval hull of $S_f(\mathbf{p})$ will be denoted $\mathbf{x}^*$; any other interval $\mathbf{x}$ such that $\mathbf{x}^* \subset \mathbf{x}$ will be referred to as an interval (outer) bound on $S_f(\mathbf{p})$. In the present paper, the tolerance problem considered is equated to finding a bound $\mathbf{x}$ on S_f. A method for computing $\mathbf{x}$ is suggested which is based on an alternative linear interval enclosure of non-linear functions in a given box [10] - [12].

To simplify presentation, it is henceforth assumed that $n' = n$ (number of equations equals number of output variables).

II. Presentation Of The Method

It is known that a continuous function $g(z_1, \ldots, z_q)$ can be enclosed in a box $\mathbf{z}$ by the following affine linear interval function

$$L_g(z) = \sum_{j=1}^{q} a_j z_j + \mathbf{b} \qquad (3)$$

(a_j are real numbers and $\mathbf{b}$ is an interval) having the property

$$g(z) \in L_g(z), \quad z \in \mathbf{z} \qquad (4)$$

Similar formulae are valid when g is an r-dimensional function. Now

$$L_g(z) = Az + \mathbf{b}, \quad z \in \mathbf{z} \qquad (5)$$

(A is a real matrix and $\mathbf{b}$ is an interval vector) and for the new notation, property (4) is also valid. Constructive procedures for determining A and $\mathbf{b}$ are suggested in [10]-[12]. On account of (5) the linear interval enclosure of (1) in the box $\mathbf{z} = (\mathbf{x}, \mathbf{p})$ will be

$$L_f(x\ ,p) = A^x x + A^p p + \mathbf{b} \quad x \in \mathbf{x}\ ,\ p \in \mathbf{p} \qquad (6)$$

In this section, a method for determining an outer bound x on the solution set $S_f(p)$ of (1) is presented. It consists of two stages: during the first stage, a "good" starting box x^s is determined; the second stage aims at improving x^s by making it narrower.

Stage 1. Let p^0 be the center of p. First, the nominal solution x^0 is found by solving $f(x, p^0) = 0$. Next a narrow box x^0 of width ε_0 centered at x^0 is introduced and, using(6), system (1) is enclosed in z^0 by the linear interval form

$$L_f(x^0, p) = A_0^x x + A_0^p p + b_0 \tag{7a}$$

$$x \in x^0, \quad p \in p \tag{7b}$$

Now, (7a) will be used as a linear approximation of (1) in a larger box $z^1 = (x^1, p)$. Following [10] - [12] the component x^1 of z^1 is determined in the following way

$$x^1 = -(A_0^x)^{-1} b_0' \tag{8a}$$

where

$$b_0' = A_0^p p + b_0 \tag{8b}$$

The first stage can be implemented in two different ways using the following two procedures.

Procedure 1. It is initiated by putting $x^0 = x^1$ and going back to (7).

Procedure 2. It starts as Procedure 1 by computing x^1 using (8). At this point, x^1 is renamed x' and the new x^1 is found by the union

$$x^1 = x' \cup x^0 \tag{8c}$$

Next the iterations continue (as in the previous procedure) from (7) with $x^0 = x^1$.

It is assumed that Procedure 1 (Procedure 2) converges to a stationary interval vector (box) x^s.

In practice, the respective procedure terminates whenever the distance between two successive iterations x^1 and x^0 becomes smaller than an accuracy ε_1. This approximate stationary box denoted as x^a differs, in general, from x^s and may be smaller. Therefore, x^s is constructed in the following way

$$x^s = x_c^a + (1 + \varepsilon_2)[-R, R] \tag{9}$$

where R is the radius of x^a and $\varepsilon_2 \geq 0$.

Stage 2. After the box x^s has been determined by (9) we proceed to the second stage of the present method. Now we try to reduce x^s using the following procedure.

Procedure 3. Let $x^0 = x^s$ and construct the corresponding linear approximation of $f(x, p)$ in (x^0, p) using (7). By (8) find the corresponding box x' and denote it x^1. Next, a new box x^1 is introduced by the intersection

$$x^1 = x' \cap x^0 \tag{10}$$

As before, the iterative process is initiated by putting $x^0 = x^1$ and going back to (7). It is terminated when the distance between two successive boxes becomes smaller than an accuracy ε_3. The corresponding stationary box denoted x is, in fact, the outer solution of the corresponding tolerance problem described by (1).

The distance used in the stopping criterion in Procedures 1 to 3 has been chosen as

$$\mathrm{d} = \max\{|w(x_j^1)| - |w(x_j^0)|\} \tag{11}$$

where w stands for width.

The second stage of the present method permits to computationally test its validity. Indeed, let $x^{(k)}$ be the box obtained at the k th iteration of Procedure 3. If the condition

$$x^{(k)} \subset \mathrm{int}(x^s) \tag{12a}$$

(int denoting interior) is fulfilled for some $k \geq 1$, then

$$S_f \subset x^* \subset x \tag{12b}$$

i.e. the outer solution thus found contains S_f and its interval hull. The proof of (12) (along with other theoretical aspects of this paper's method) will be published elsewhere.

It is seen that the method suggested above can be implemented as:

a) algorithm A1 which is based on Procedures 1 and 3;
b) algorithm A2 which uses Procedures 2 and 3.

Experimental evidence seems to indicate that algorithm A2 requires less iterations than algorithm A1 to solve the tolerance analysis problems considered.

It should be noted that besides being more general, the present method differs from the method of [8] also in the way each iteration of Procedures 1 to 3 is carried out. In [8] this is done by approximately solving a linear interval system where all its elements are interval. The corresponding linear interval system in the new method is much simpler (only the right-hand side is interval) whose exact solution (within round-off errors) is computed by (8). This explains the better computational efficiency of the present method which is confirmed by the examples considered in the next section.

III. Numerical Examples

In this section, two worst-case tolerance examples illustrating the applicability and efficiency of the present method are given. The examples have been solved using algorithms A1 and A2. The algorithms were programmed using the algorithmic language C++. The linear interval enclosures (7) were generated automatically by a procedure that implements the approach suggested in [12].

Example 1. In this example, the system of equations (1) is:

$$10^{-9}(e^{38x_1} - 1) + p_1 x_1 - 1.6722x_2 + 0.6689x_3 - 8.0267 = 0$$

$$1.98 \cdot 10^{-9}(e^{38x_2} - 1) + 0.6622x_1 + p_2 x_2 + 0.6622x_3 + 4.0535 = 0$$

$$10^{-9}(e^{38x_3} - 1) + x_1 - x_2 + p_3 x_3 - 6 = 0 \tag{13a}$$

$$p = (p_1, p_2, p_3) \in ([0.6020, 0.7358], [1.2110, 1.4801], [3.6, 4.4]) \tag{13b}$$

and models a dc electric circuit containing a transistor, a diode and two resistors [9]. We consider the worst-case tolerance problem associated with (13): find an interval (outer) solution x to (13). In this example, the output vector is $x = (x_1, x_2, x_3)$ and the parameter vector is $p = (p_1, p_2, p_3)$. We chose p^0 as the center of p given in (13b). The corresponding nominal solution x^0 was found with accuracy $\varepsilon = 10^{-5}$ using a new nonlinear equations solver (implementing ideas from [11] and [12]):

$$x^0 = (0.5555, \ -3.518, \ 0.4685) \tag{14}$$

(the results in (14) are, however, given only to four decimal places).

Application of algorithms A1 and A2 with $\varepsilon_0 = \varepsilon_1 = \varepsilon_3 = 10^{-4}$ yielded the following results, respectively, for the intervalsolution of the tolerance problem considered:

$$x = ([0.5401 \ \ 0.5682], \ [-3.8926 \ \ -3.1153], \ [0.3483 \ \ 0.5387]) \tag{15}$$

$$x = ([0.5402 \ \ 0.5680], \ [-3.8910 \ \ -3.1194], \ [0.3473 \ \ 0.5331]) \tag{16}$$

For algorithm A1, $\varepsilon_2 = 0.05$ and the fulfillment of (12a) was achieved at $k = 1$ of Procedure 3. For algorithm A2 we chose $\varepsilon_2 = 0$ and nevertheless (12a) was satisfied already at $k = 2$ of Procedure 3. Thus, both bounds (15) and (16) are guaranteed to contain the solution set of (13). where

$$u = v_{12} - v_{13} \tag{18b}$$

and a diode characteristic

$$i_5 = 10^{-9}\left(e^{38u} - 1\right) \tag{19a}$$

with

$$u = v_{13} + e_5 - v_{14} \tag{19b}$$

All the remaining 9 linear resistors have a ±2% tolerance, i.e. their values lie within the interval [98, 102] Ω and define the parameter box p. The components of the output variable vector x are the branch currents i1, i2,..., i11 and the node voltages v12, v13,..., v16. Again, we consider the worst-case tolerance analysis problem associated with this modified circuit.

A numerical difficulty arose in finding the nominal solution x^0 for this example. It is due to the well-known overflow problem caused by the exponential diode nonlinearity. The overflow was overcome by scaling all variables by a factor of 0.01. Afterwards x^0 was found successfully with accuracy $\varepsilon = 10^{-5}$ using the new nonlinear equations solver. The initial box x^0, where x^0 was searched for, was chosen pretty wide: currents were set between (-1) and 1 A and node voltages between 1 and 100 V. A unique nominal solution is:

TABLE I

	Algorithm 1			Algorithm 2		
	Stage 1	Stage 2	Total	Stage 1	Stage 2	Total
N_I	27	8	35	13	7	20

The numbers of iterations corresponding to the two algorithms are listed in Table 1. It is seen that algorithm A2 requires less iterations as compared to algorithm A1.

The same example was solved in [8] by an algorithm similar in structure to algorithm A2 (however, as mentioned in the previous section, each iteration of both the first and second stage of the algorithm is associated with the solution of a corresponding linear interval system and requires more computation than algorithm A2). The following bound was obtained

$$x = ([0.5103, \ 0.5778], \ [-4.352, \ -2.6756], \ [0.3483, \ 0.5898]) \tag{17}$$

It is to be stressed that the bound (17) is more conservative as compared to (15) and (16) and at the same time takes more iterations to be reached: total number of iterations 166 (85 iterations for the first stage and 81 iterations for second stage).

Example 2. This example is a modification of Example 3.2 in [5]. The linear dc circuit of [5] is transformed into a nonlinear circuit by replacing the linear resistors r_3 and r_5 in Fig. 3.2 of [5] with nonlinear elements having respectively: a cubic characteristic

$$i_3 = 10^{-3}\,(2.5\,u^3 - 10.5\,u^2 + 11.8\,u) \tag{18a}$$

$$x^0 = (0.376, \ 0.391 \ \ 0.016, \ 0.272, \ 0.136, \ 0.360, \ -0.070, \ 0.337, \ 0.097, \ 0.167, \ 0.263, \ 62.36, \ 60.88, \ 33.72, \ 16.69, \ 26.35) \tag{20}$$

was located after 124 splits of x^0 and 3949 equation evaluations in the so-called generalized-interval form [12] (we report equation, not system evaluations since not all equations may be required per iteration). Using algorithm A2, the interval (outer) solution around the nominal solution x^0 was computed after 7 unions (stage 1) and 4 intersections (stage 2) for $\varepsilon_0 = \varepsilon_1 = \varepsilon_3 = 10^{-5}$ and $\varepsilon_2 = 0.05$. The left endpoint x^l and right endpoint x^r of the output variable vector are given below:

$$x^l = (0.366, \ 0.380, \ 0.07, \ 0.260, \ 0.123, \ 0.350, \ -0.079, \ 0.327, \ 0.089, \ 0.159, \ 0.254, \ 61.36, \ 59.84, \ 32.93, \ 15.97, \ 25.48) \tag{21}$$

$$x^r = (0.388, \ 0.404, \ 0.027, \ 0.279, \ 0.157, \ 0.369, \ -0.063, \ 0.350, \ 0.104, \ 0.176, \ 0.273, \ 63.28, \ 61.70, \ 34.74, \ 17.48, \ 27.22) \tag{22}$$

where as in (20) only four digits for each component are given.

IV. CONCLUSION

An interval method for tackling various classes of tolerance analysis problems (linear and nonlinear, deterministic with independent or dependent parameters, probabilistic) has been suggested. The method is rather general in its approach and equates the original tolerance problem to that of finding an outer interval solution x to the nonlinear system of equations (1) describing the tolerance problem considered. It is based on a recently suggested linear interval enclosure (5). If the computationally verifiable condition (12a) is satisfied, the method guarantees that the obtained solution x is really an outer solution, i.e. the inclusion (12b) is fulfilled.

A computer program implementing the method has been developed in a C^{++} environment. The numerical results obtained so far (including data not reported here) seem rather encouraging.

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