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A NEW INTERVAL METHOD FOR GLOBAL OPTIMIZATION

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Abstract

Interval methods are iterative methods capable of solving the general nonlinear programming problem globally, providing infallible bounds both on the optimum (optima) and the corresponding solution coordinates. However, their computational complexity grows rapidly with the dimension of the problem and the size of the search domain. In this paper, a new interval method is suggested which seems to have improved numerical efficiency. It is based on the use of a new interval linearization of the nonlinear functions involved. Two algorithms for computing it are presented. The new optimization method incorporates seven procedures that are implemented by way of the new interval linearization. A numerical example illustrating the method suggested is also given

1. Introduction

Applying interval analysis techniques to solving various global optimization problems has been a major deterministic approach over the last decades [1] - [5].
In this paper, a new interval method is suggested for global solution of the following constraint optimization problem.

Minimize

$$\varphi_0(x) \tag{1a}$$

subject to the constraints

$$\varphi_i(x) \le 0, \quad i = 1,2,\dots,r \tag{1b}$$

$$x \in X^{(0)} \subset R^n \tag{1c}$$

where x is a n-dimensional vector and $X^{(0)}$ is a given initial search region (a box). The functions in (1a) and (1b) are assumed to be only continuous in $X^{(0)}$.
It should be stressed that problem (1) presents in an equivalent form the general nonlinear programming problem which also involves equality constraints

$$\varphi_i(x) = 0, \quad i = r+1,\dots,r_1 \tag{1d}$$

Indeed, each equality constraint can be represented equivalently by two inequality constraints.
Let $f(x)$, $x \in X \subset R^p$, $p \ge n$ be a continuously differentiable function. Known interval methods for solving (1) are iterative and are based on the following interval linearization of f in X:

$$L(X) = f(x^c) + \sum_{i=1}^{p} G_i(X)(X_i - x_i^c) \tag{2}$$

where x^c is the centre of X and $G_i(X)$ is either the interval extension of the derivative $g_i(x) = \partial f / \partial x_i$ or the corresponding interval slope [3], [5], [9]. In contrast, the present method appeals to a new interval linearization of f in the form:

$$L(X) = \sum_{i=1}^{p} a_i x_i + B, \quad x_i \in X_i \tag{3}$$

where a_i are real numbers and only B is an interval. The use of (3) in the computational scheme of the new global optimization method leads to improved performance as compared to the other known methods since it permits a better (tighter) enclosure of the original nonlinear functions. Another advantage of the alternative form (3) resides in the fact that it is applicable to nonlinear functions that are only continunuos or even discontinuous.

2. New interval linearization of a nonlinear function

Two algorithms for determining the new interval linearization (3) will be presented in this section

2.1. First algorithm

Let $f(x)$ be a multivariate function $f: D \subset R^p \to R$.
The transformation of a nonlinear function $f(x)$, $x \in X$, to the new linear interval form (3) can be done following the approach suggested in [6]-[8]. If $f(x)$ is in separable form, α_i and B can be determined by a procedure given in [6]. For an arbitrary function $f(x)$ (which is continuous or even discontinuous) (3) can be evaluated by following the approach suggested in [7], [8]. First, f is transformed into a system of equations of the so-called semiseparable form [7] by introducing a certain number m of auxiliary variables. Afterwards, each new semiseparable equation is easily transformed into form (3). Thus, a system of $m+1$ linear interval equations is generated. Finally, the auxiliary variables are eliminated from the latter linear system to yield the linear form (3) corresponding to the original function f.
Thus, the approach outlined above involves the following steps.
Step 1. Transformation to semiseparable form.
A function f is called to be in semiseparable form if [7]

$$f(x) = \sum_{j=1}^{n} f_j(x_j) + \sum_{\substack{k=1 \\ k \neq l}}^{n} \sum_{l=1}^{n} \alpha_{kl} x_k x_l \quad . \tag{4}$$

(some of these terms may be missing). The transformation of an arbitrary function to a set of functions of the semiseparable form (4) can be done by the approach suggested in [7]. This possibility will be illustrated by the following example

E x a m p l e 1. Let

$$f(x) = (x_1^2 + x_2^2 - 1)(x_3^2 - x_2) \tag{5a}$$

with $x \in X$ and

$$X_i = [0,1], \quad i = 1,2,3 \tag{5b}$$

The problem is to convert (5a) into a set of semiseparable functions. With this in mind, we introduce two auxiliary variables x_4 and x_5 to get

$$\begin{aligned} f(x) &= x_4 x_5 \\ x_4 &= (x_1^2 + x_2^2 - 1) \\ x_5 &= (x_3^2 - x_2) \end{aligned} \tag{6}$$

Now all expressions in (6) are in semiseparable form. We need also to evaluate the corresponding ranges

$$X_4 = [-1,1] \tag{7a}$$

$$X_5 = [-1,1] \tag{7b}$$

Step 2. Enclosing the auxiliary variable expressions.

The above example shows that in the general case $f(x)$ will be transformed by the introduction of a given number m of auxiliary variables into a set of $m+1$ functions of semiseparable form

$$f_i(y), \quad i = 0,1,\ldots,m \tag{8}$$

where $f_0 = f$, $f_i = x_{n+i}$, $i = 1,2,\ldots,m$, $y \in R^{n+m}$ is the "augmented" vector of variables. For Example 1, system (8) is given by (6) with

$y = (x_1, x_2, \ldots, x_5)$, $m = 2$, $f_i = x_{n+i}$, $i = 1,2$.

At this point, each function f_i, $i = 0,1,\ldots,m$ is enclosed by a corresponding linear enclosure (3). Thus, for the example considered

$$f_0 = f \in a_4 x_4 + a_5 x_5 + B_0, \quad x_4 \in X_4, x_5 \in X_5 \tag{9}$$

$$f_1 = x_4 \in a_1 x_1 + a_2 x_2 + B_1, \quad x_1 \in X_1, x_2 \in X_2 \tag{10}$$

$$f_2 = x_5 \in -x_2 + a_3 x_3 + B_2, \quad x_3 \in X_3 \tag{11}$$

In the general case

$$f_0 \in \sum_{j=1}^{n+m} a_{0j} x_j + B_0 \tag{12}$$

$$x_{n+i} \in \sum_{j=1}^{n} a_{ij} x_j + B_i, \quad i = 1,2,\ldots,m \tag{13}$$

$$F_i = \sum_{j=1}^{n} a_j X_j + B_i \tag{13'}$$

Step 3. Eliminating the auxiliary variables

In the final step, the auxiliary variables are eliminated using (12) and (13). This possibility will be illustrated by way of Example 1. Substituting (10) and (11) into (9), we get

$$\begin{aligned} f_0 \in\; & a_1 a_4 x_1 + (a_2 a_4 - a_5) x_2 + a_3 a_5 x_3 + \\ & + B_0 + a_4 B_1 + a_5 B_2 \end{aligned} \tag{14}$$

Finally, $f(x)$ given by (5a) has been enclosed in the box X with sides (5b) by the linear interval expression

$$L(x) = a_1' x_1 + a_2' x_2 + a_3' x_3 + B \tag{15a}$$

with

$$a_1' = a_1 a_4, \quad a_2' = a_2 a_4 - a_5, \quad a_3' = a_3 a_5 \tag{15b}$$

$$B = B_0 + a_4 B_1 + a_5 B_2 \tag{15c}$$

2.2. Second algorithm

Now, the only assumption on f is that f is a factorable function [17], i.e. composed of the four arithmetic operations (+, -, *, /) and the unary operations (sin, exp, log, sqrt, abs, etc.). However, to simplify the presentation, the linear interval enclosure (3) will be first determined for the class of polynomial functions. Later on, the approach adopted for polynomial functions will be extended to arbitrary factoriable functions.

A. Polynomial Functions

In this subsection, an algorithm will be suggested for determining the linear interval enclosure (3) for the special case where f is a polynomial function. This algorithm is based on the notion of a generalized interval [3]. We shall introduce a slightly different generalized representation in the following manner.

Let $X = (X_1, \ldots, X_n)$ and

$$X_i = c_i + V_i, \quad i = 1,\ldots n \tag{16a}$$

where c_i is the centre of X_i and V_i is a symmetrical interval

$$V_i = [-R_i, R_i] \tag{16b}$$

R_i being the radius of X_i, i.e.

$$R_i = \overline{x}_i - x_i^c \tag{16c}$$

Definition 1. A generalized interval $\tilde{X}$ is defined as the affine function

$$\tilde{X} = \sum_{i=1}^{n} \alpha_i V_i + c_x + V_x, \quad V_x = [-R_x, R_x] \tag{17}$$

where α_i and c_x are real numbers while V_i and V_x are centred "ordinary" intervals.

Using $\tilde{X}$, any "ordinary" interval can be represented by an appropriate choice of the terms of $\tilde{X}$. Indeed, letting $\alpha_i = 0$ [illegible] and $V_x = 0$ we get $\tilde{X} = V_j +$ [illegible] where X_j is the jth ordinary interval.

Now we shall define the operations of addition and multiplication of generalized intervals (G intervals). Let

$$\tilde{Y} = \sum_{i=1}^{n} \beta_i V_i + c_y + V_y, \quad V_y = [-R_y, R_y] \tag{3.3}$$

be a G interval. Then we have the following rules.

Addition. Let $\tilde{X}$ and $\tilde{Y}$ be two G intervals given by (17) and (18). The sum of $\tilde{X}$ and $\tilde{Y}$ denoted as $\tilde{X} + \tilde{Y}$ is another G interval $\tilde{Z}$:

$$\tilde{Z} = \sum_{i=1}^{n} \gamma_i V_i + c_z + V_z, \quad V_z = [-R_z, R_z] \tag{19a}$$

and

$$\gamma_i = \alpha_i + \beta_i, \quad i = 1,\ldots n \tag{19b}$$

$$c_z = c_x + c_y, \quad R_z = R_x + R_y \tag{19c}$$

Multiplication. The product $\tilde{X} \cdot \tilde{Y}$ of two G intervals $\tilde{X}$ and $\tilde{Y}$ is another G interval $\tilde{Z}$ if:

$$\gamma_i = c_x \beta_i + c_y \alpha_i, \quad i = 1, \dots, n \tag{20a}$$

$$c_z = c_x c_y + \frac{1}{2} \sum_{i=1}^{n} \alpha_i \beta_i R_i^2 \tag{20b}$$

$$R_z = R_x R_y + |c_x| R_y + |c_y| R_x + \sum_{i,j=1}^{n} |\alpha_i \beta_i| R_i R_j + + R_x \sum_{j=1}^{n} |\beta_j| R_j + R_y \sum_{i=1}^{n} |\alpha_i| R_i + \frac{1}{2} \sum_{i=1}^{n} |\alpha_i \beta_i| R_i^2 \tag{20c}$$

(R_i is the radius of X_i).

The proof of (19) and (20) is based on elementary properties of adding, multiplying and centring ordinary intervals and is therefore omitted.

Using the above two operations, any intermediate or final result in evaluating the interval extension of a polynomial function can be represented as a generalized interval. Indeed, multiplying an ordinary or G interval by a constant c is a special case of the multiplication $\tilde{Z} = \tilde{X} \cdot \tilde{Y}$. If $\tilde{X} = c = -1$, we get $\tilde{Z} = -\tilde{Y}$ and the rules for the operation of subtraction follow immediately.

Subtraction. For $\tilde{Z} = \tilde{X} - \tilde{Y}$

$$\gamma_i = \alpha_i - \beta_i, \quad i = 1, \dots, n, \quad c_z = c_x - c_y \tag{21a}$$

$$R_z = R_x + R_y \tag{21b}$$

Example 2. Let $x = (x_1, x_2)$ and

$$f(x) = (x_1 - 2x_2)x_1, \quad x_i \in X_i, \quad i = 1, 2 \tag{22}$$

Find the linear interval enclosure (3) corresponding to (22).

We first introduce the G intervals

$\tilde{X}_1 = \alpha_1 V_1 + 0.V_2 + c_1, \quad \tilde{X}_2 = 0.V_1 + \beta_2 V_2 + c_2$

with $\alpha_1 = 1, \quad \beta_2 = 2$. Then we compute

$$\tilde{Y} = \tilde{X}_1 - \tilde{X}_2 \tag{23}$$

using (21). The final result is obtained as the product

$$F(X) = \tilde{Z} = \tilde{Y}\tilde{X}_1 = \gamma_1 V_1 + \gamma_2 V_2 + c_z - V_z \tag{24}$$

computed by (20).

The linear form (24) represents equivalently (for the example considered) the linear form (3). Indeed, from (16)

$$V_i = X_i - c_i, \quad i = 1, 2 \tag{25}$$

and substituting (25) into (24) we get

$$F(X) = a_1 X_1 + a_2 X_2 + B \tag{26a}$$

with

$$a_1 = \gamma_1, \quad a_2 = \gamma_2, \quad B = c_z - \gamma_1 c_1 - \gamma_2 c_2 + V \tag{26b}$$

B. Factorable functions

The approach suggested in Section 2.2 A will be now extended to arbitrary factorable functions [9]. A function $f: D \subseteq R^n \to R$ is a factorable function (f.f.) if and only if it can be represented by an expression $f(x)$ which is the last element in a finite sequence $(f_i(x))$ of expressions. For the case of $f \in C^1(D)$, the list of admissible expressions is given in [9]. An approach to treating non-differentiable functions is considered in [3], Ch. 14. An alternative idea is suggested in [5], Ch. 6. Here, we shall consider the general situation when $f \in C^1(D)$, $f \in C^0(D)$ or even when some expressions $f_i(x)$ may be discontinuous functions.

Let W denote the set of all building expressions for a given f.f. For our purposes, it is convenient to divide W into two parts W_1 and W_2 such that

$$W = W_1 \cup W_2$$

The set W_1 is made up of expressions which are used to construct a multivariate polynomial function. The set W_2 contains the following three groups of expressions:

(i) the reciprocal value operation

$$f_i(x) = 1/x, \quad x \in X \in R, \quad 0 \notin X \tag{27}$$

(ii) the set Φ containing standard functions Φ_i to be found in high level programming languages, i.e.

$$\Phi = \{\mathrm{sqrt}(\cdot), \exp(.), \ln(\cdot), \sin(\cdot), \mathrm{abs}(\cdot), \dots\} \tag{28}$$

The set Φ can be enlarged as appropriate. For instance, it may include various discontinuous functions. A typical example of such functions is the unit step function $1(x)$ defined as:

$$1(x) = \begin{cases} 1, & \text{for } x > 0 \\ 0, & \text{for } x \le 0 \end{cases} \tag{29}$$

(iii) unary functions $f_i: R \to R$ which may include rational and irrational parts.

The main characteristic of all unary functions $f_i \in W_2$ is the fact that they allow easy computation of a corresponding linear interval enclosure (3).

Now we are in a position to present an algorithm for computing (3) for the case of an arbitrary f.f. We assume that the sequence $f_i(x)$ representing the function at hand f has already been chosen. To simplify the presentation, we assume additionally that the first k expressions $f_i \in W_1$ while the remaining $f_i \in W_2$ (in the general case, the appearance of $f_i \in W_1$ and $f_i \in W_2$ in the sequence $(f_i(x))$ may have a more complex pattern).

On account of the results obtained in subsection 2.2 A it is clear that the linear interval enclosure $F_k(X)$ corresponding to the last expression $f_k(x)$ belonging to W_1 is given by the G interval

$$\tilde{F}_k = \sum_{j=1}^{n} \alpha_{kj} X_j + B_k \tag{30}$$

which has been computed recursively using G intervals $\tilde{F}_j$ corresponding to expressions f_j with $j < k$.

Now consider the first function $f_i \in W_2$, i.e. the function f_{k+1}. According to the construction of the sequence $(f_i(x))$

$$f_{k+1}(x) = f_{k+1}(f_k(x)) \tag{31a}$$

$$f_k(x) \in \tilde{F}_k(X) \tag{31b}$$

Since f_{k+1} is a unary function, it can be enclosed by the interval function

$$F_{k+1} = a_{k+1} \tilde{F}_k + B_{k+1} \tag{32}$$

where a_{k+1} and B_{k+1} are determined in one way or another (depending on whether f_{k+1} is a C^n, C^0 or discontinuous function). Substituting (30) into (32), it is seen that F_{k+1} can be represented as a G interval

$$\tilde{F}_{k+1} = \sum_{j=1}^{n} \alpha_{k+1,j} X_j + B_{k+1} \qquad (33a)$$

with

$$\alpha_{k+1,j} = a_{k+1}\alpha_{kj}, \quad B_{k+1} = a_{k+1}B'_{k+1} + B_k \qquad (33b)$$

Since (31) remains valid if the index $k+1$ is replaced with $i > k+1$, it is clear that the recursive formula (33) also holds for $i > k+1$. Thus, it has been shown that the factoriable function can be enclosed in X by a G interval

$$\tilde{F} = \sum_{j=1}^{n} \alpha_j X_j + B \qquad (34)$$

whose coefficients α_j and additive term B can be determined in a recursive way using only the binary operations (19), (20), (21) (addition, multiplication and subtracting of two G intervals) and (33) (multiplication of a G interval by a scalar).

Example 3 [9]. Find the interval enclosure for

$$f(x) = x_1^2 / \exp(x_2), \quad X_1 = [1, 1.5], \; X_2 = [2, 2.5] \qquad (35)$$

The function $f(x)$ can be defined by the corresponding element $f_3(x)$ from the following sequence

$$f_1(x) = x_1^2 \qquad (36a)$$

$$f_2(x) = 1/\exp(x_2) \qquad (36b)$$

$$f_3(x) = f_1(x) f_2(x_2) \qquad (36c)$$

Applying the above algorithm we first have to compute the enclosure for (36a):

$$F_1(X) = a_1 X_1 + B_1 \qquad (37a)$$

Using Procedure 1 from [7] we have

$$\underline{b}_1 = -a_1^2/4, \quad \overline{b} = -\underline{x}_1 \overline{x}_1 \qquad (37b)$$

In a similar way, we find the enclosure for (36b):

$$F_2(X) = a_2 X_1 + B_2 \qquad (38)$$

where

$\beta = -1/a_2$, $\underline{b}_2 = 1/\beta - a_2 \ln\beta$, $\overline{b}_2 = f_2(\overline{x}_2) - a_2\overline{x}_2$

(In (37) and (38), a_1 and a_2 are the respective slopes.) The expressions (37a) and (38) are then represented in the form of two generalized intervals $\tilde{F}_1$ and $\tilde{F}_2$. Finally, the enclosure $F(X)$ of (35) is given by the product $\tilde{F}_1\tilde{F}_2$, i.e.

$$F(X) = \tilde{F}_1\tilde{F}_2 = \gamma_1 V_1 + \gamma_2 V_2 + B \qquad (39)$$

Using (39), we have obtained

$$F(X) = [0.0367, \; 0.3045] \qquad (40)$$

The same example was solved in [9] using first- and second-order interval derivatives and slopes. The following results have been obtained there:

$$F_{D1} = [-0.013, \; 0.35] \quad F_{S1} = [0.018, \; 0.32] \qquad (41a)$$

$$F_{D2} = [\;0.014, \; 0.32] \quad F_{S2} = [0.021 \; 0.31] \qquad (41b)$$

Comparison of (40) and (41) shows that for the example considered the present algorithm provides a narrower enclosure than the approach used in [9]

3. The new method

It is based on the following set of procedures to be carried out at each iteration. The interval extensions required in every procedure are implemented using the new linear form (3).

Procedure 1. Let $\overline{\varphi}_0$ be a current upper bound of $\varphi_0(x)$, $x \in X$, where $X \subseteq X^{(0)}$. If

$$\Phi_0(X) > \overline{\varphi}_0 \qquad (42)$$

then X is discarded [1], [3].

Procedure 2. Monotonocity test [1], [3]. If X is strictly admissible (s.a.) [3] and for some i

$$G_i(X) > 0 \text{ or } G_i(X) < 0 \qquad (43)$$

where $G_i(X)$ is the interval extension of $\partial\varphi_0/\partial x_i$ then X is discarded.

Procedure 3. Nonconvexity test. If X is s.a. and

$$H_{ii}(X) < 0 \qquad (44)$$

for any $i = 1,\dots,n$, then X is discarded [3]. Here $H_{ii}(X)$ is the interval extension of $\partial^2\varphi_0/\partial x_i^2$.

Procedure 4. Inadmissibility test. If

$$\Phi_i(X) > 0 \qquad (45)$$

for any $i = 1,\dots,r$, X is discarded [3].

Procedure 5. If

$$\Phi_i(X) < 0 \qquad (46)$$

then the corresponding constraint is inactive and can be ignored for the current iteration.

Procedure 6. This procedure is an attempt to reduce the current box $X \in R^n$ using separately each of the inequalities

$$L_0(Y) = \sum_{j=1}^{n} \alpha_j^{(0)} y_j + B_0 \le \overline{\varphi}_0 \le 0 \qquad (47a)$$

$$L_i(Y) = \sum_{j=1} \alpha_j^{(i)} y_j + B_i \le \overline{\varphi}_0 \le 0, \quad i = 1,\dots,n \qquad (47b)$$

where $L_i(Y)$ is the linear interval form (3) corresponding to function $\varphi_i(x)$, $i = 0,\dots,r$.

Procedure 7. This procedure is based on the linearization of the so-called Fritz-John system

$$f_j(x) = u_0 g_0^{(j)}(x) + \sum_{i=1}^{r} u_i g_i^{(j)}(x) = 0, \quad j = 1,\dots,n$$

$$f_j(x) = u_j \varphi_j(x) = 0, \quad j = 1,\dots,r \qquad (48)$$

$$f(m) = \sum_{i=0}^{r} u_i - 1 = 0$$

where e u_i are scalars. This is a nonlinear system of m equations in m unknowns and, in general, $m = n + r + 1$. However, at some iterations m can be reduced by Procedure 2 or/and Procedure 5. Each function $f_j(x)$ in (48) is then linearized in $X \subset R^m$ and the following system is set up

$$Ax = B \qquad (49)$$

where A is constant (real) $m \times n$ matrix while B is an interval vector. System (11) is then solved in an efficient manner by the use of the so-called equationwise constraint propagation [8]

4. Numerical example

We consider the following optimization problem ([3], p. 1[illegible])

Minimize

$$\varphi_0(x) = x_1^6 - 6.3x_1^4 + 12x_1^2 + 6x_1x_2 + 6x_2^2 \qquad (50a)$$

subject to

$$\varphi_1(x) = 1 - 16x_1^2 - 25x_2^2 \le 0$$
$$\varphi_2(x) = 13x_1^3 - 145x_1 + 85x_2 - 400 \le 0 \quad (50b)$$
$$\varphi_3(x) = x_1 x_2 - 4 \le 0$$

There are two global solutions:

$$x_1^* = \pm 0.06604, \quad x_2^* = \pm 0.1929 \quad (51)$$

and the global minimum is

$$\varphi_0^* = 0.1990 \quad (52)$$

The problem has been solved with accuracy $\varepsilon = 10^{-4}$ for various starting boxes $X^{(0)}$. The results are quite encouraging. Thus, for $X_1^{(0)} = X_2^{(0)} = [-1,4]$ the method required N_1=107 iterations to find the global solutions (51), (52) and took t=0.53 sec when run on a 166 MHz Pentium computer. As the volume of the starting box $X^{(0)}$ was enlarged 1000000 times, the run time was increased only 3.562 times.

5. Conclusion

In this paper, the general nonlinear programming problem (or its variants) is addressed. A new interval method for the global solution of the optimization problem considered has been suggested. It is based on an alternative interval linearization of the nonlinear functions involved which is updated at each iteration of the computation process. The interval linearization suggested is more general than other known linearization forms since it is capable of enclosing functions that are only continuous or even discontinuous. The present interval linearization is in the form of an affine interval function where only the additive term is an interval which accounts for its better enclosing properties.

In its present form, the global optimization method suggested is based on the use of seven procedures that are implemented through the new interval linearization. More sophisticated computational schemes, including additional procedures, are however possible. If the optimization problem considered involves a system of nonlinear inequality constraints, one such procedure can solve (in the sense of [3]) this inequality system at each iteration of the iterative process.

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