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AN INTERVAL METHOD FOR GLOBAL INEQUALITY-CONSTRAINT OPTIMIZATION PROBLEMS

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ABSTRACT

An interval method is suggested for globally solving optimization problems of the following type: minimize a given objective function subject to both functional inequality constraints and simple bounds on the variables. The present method appeals to a new interval linearization of each nonlinear function and is based, essentially, on two computation techniques: linear programming and constraint propagation. The use of these techniques in the computational scheme of the present method seems to lead to improved performance as compared to other known interval methods of the same class.

1. INTRODUCTION

Interval methods (methods applying interval analysis techniques) are capable of globally solving various optimization problems with mathematical certainty [1]-[5].

In this paper, a new interval method is suggested for global solution of the following inequality-constraint optimization problem

Minimize

$$\varphi_0(x) \tag{1a}$$

subject to the constraints

$$\varphi_i(x) \le 0, \quad i = 1,2,\ldots,r \tag{1b}$$

$$x \in X^{(0)} \subset R^n \tag{1c}$$

where x is a n-dimensional vector and $X^{(0)}$ is a given initial search region (a box). The function in (1a) is assumed continuously differentiable in $X^{(0)}$ while these in (1b) can be only continuous.

Various methods have been suggested for solving (1) globally [1]-[5]. Let $f(x)$ denote any function in (1). Known interval methods for solving (1) are all based on the linearization of f in X as follows:

$$F(X) = f(x^c) + \sum_{i=1}^{n} G_i(X)(X_i - x_i^c) \tag{2}$$

where x_i^c are the components of the center x^c of the box $X = (X_1, \ldots, X_n)$ with components X_i and $G_i(X)$ is the interval extension of the derivative $g_i(x) = \partial f/\partial x_i$ or, better, the interval extension of the corresponding slope [3] in X (other more sophisticated extensions of the type (2) have also been suggested and used [1]-[5]). An elaborate algorithm for solving (1) by applying (2) is presented in [3]. It is, however, too complicated for practical purposes. Indeed, it involves as an integral part a procedure for solving a nonlinear system derived on account of the so-called Fritz-John conditions [3]. Since (1c) is in fact the short notation of $2n$ inequalities:

$$x_i - \bar{x}_i \le 0, \quad i = 1,\ldots,n \tag{3a}$$

$$\underline{x}_i - x_i \le 0, \quad i = 1,\ldots,n \tag{3b}$$

where $\underline{x}_i$ and $\bar{x}_i$ are the end-points of X_i, the Fritz-John system will be a system of m equations in m unknowns with $m = 3n + r + 1$. To reduce the amount of computation required, an approach called "peeling of the boundary" which skips the inequalities (3) has been proposed in [5]. However, the simpler problem (1a), (1b) is to be solved 3^n times which is still prohibitively inefficient for larger n.

Recently, a new approach to constructing interval methods for global optimization has been proposed in [6]. It is based on a new type of linear interval linearization of $f(x)$ in X in the form:

$$F(X) = \sum_{i=1}^{n} \alpha_i X_i + B, \quad x_i \in X_i \tag{4}$$

where α_i are real numbers and only B is an interval.

In this paper, a method for solving (1) is suggested. It appeals to linearization (4) and uses, essentially, the following two techniques: linear programming and constraint propagation. The combined effect of the use of these techniques in the computational scheme of the new method seems to lead to improved performance as compared to other known interval methods of the same class.

2. LINEAR PROGRAMMING APPROACH

In this section, we formulate the linear programming (LP) problem to be incorporated into the computational scheme of the new combined method.

Let $X \subseteq X^{(0)}$ and

$$g(x) = 0, \quad x \in X \tag{5}$$

where $g(x) = (g_1(x), \ldots, g_n(x))$ is the gradient of $\varphi_0(x)$. The LP problem to be set up and solved at each iteration of the optimization method is related to solving (5) and will be presented in the form of the following subroutine.

Subroutine 1.

It is assumed that using the approach of [7]-[10] the functions $g_i(x)$ of (5) have been linearized by the form (4).

Step 1. Form the (corresponding to X) real matrix $A=\{-a_{ij}\}$ and the interval vector B and set up the system

$$-Ax+B=0\,,\quad x\in X \tag{6}$$

"Correct" B (if possible) by computing

$$B'=AX\,,\quad B=B'\cap B \tag{7}$$

Step 2. (Start of the choice of the objective function for the LP problem) Compute

$$y_c=Ax_c\,,\quad d=y_c-b_c$$

(x_c and b_c are the centers of X and B, respectively) and find

$$d_0=\max\{|d_i|,\quad i=1,...n\}$$

and the corresponding index $l_0=k$. So

$$d_k=d_0 sign(d_k)\,.$$

If $d_k<0$ then go to next step, else go to step 4.

Step 3. (Continuation of the choice of the LP objective function and construction of the LP problem). Form the following LP problem

$$f=b_k^{'}=\sum_{i=1}^{n}c_ix_i=\max \tag{8a}$$

$$c_i=a_{ki}\,,\quad i=1,....,n \tag{8b}$$

$$-\sum_{j=1}^{n}a_{ij}x_j+b_i=0\,,\quad i\neq k\,,\quad i=1,....,n \tag{8c}$$

$$x_i\in X_i\quad,\quad b_i\in B_i \tag{8d}$$

Go to step 5.

Step 4. In this case

$$f=b''=\sum_{i=1}^{n}c_ix_i=\min \tag{9a}$$

and (9b), (9c) and (9d) are as in (8). However, as is well known, problem (9) can be written equivalently as a max problem (8). Thus, the solution of (9) can be carried out in the same way as in the previous step.

Step 5. (Discarding X). Start solving LP problem (8) (or (9)) using the dual LP method. In this method, f for problem (8) is reached from above, i.e.

$$f_l\geq f\,,\quad l=0,1,... \tag{10}$$

where f_l is the value of the objective function at the lth LP iteration. Thus, if at some iteration (for $B_k=[\underline{B}_k,\overline{B}_k]$):

$$f_l<\underline{B}_k \tag{11}$$

then obviously the maximum value

$$f<\underline{B}_k \tag{12}$$

This means that the initial system (6) is incompatible. Also, if at some iteration, the second method shows incompatibility of problem (8), the iterations are stopped and the current X is discarded.

Step 6. (Reducing B_k). If (dealing with (8)) we reach the maximum value f, then obviously

$$f=b_k^{'}<\overline{B}_k \tag{13}$$

and we can update (reduce) B_k by putting

$$B_k=[\underline{B}_k,b_k^{'}] \tag{14a}$$

It is easily seen that in case of problem (9) B_k is updated as follows

$$B_k=[b_k^{''},\overline{B}_k] \tag{14b}$$

Using constraint propagation we can now reduce X as will be shown in the next section.

3. CONSTRAINT PROPAGATION

This technique has already been applied in the context of solving systems of nonlinear equations using linearization (2) [11] or linearization (4) [9].

For the purpose of global optimization we apply the constraint propagation (CP) technique in the following two cases.

A) *In solving system* (5).

In this case, the CP approach is applied when the LP algorithm terminates in Step 6. We start with updating vector B as shown in (14). We then continue with the following subroutine (as in algorithm A6 of [9])

Subroutine 2.

For $i=1$ to n do, for $j=1$ to n do

$$Y_i=\frac{1}{a_{ij}}[B_i-\sum_{k\neq i}^{n}a_{ik}X_k] \tag{15a}$$

$$X_j:=Y_i\cap X_j \tag{15b}$$

As a result the current box is reduced in size.

B) *In handling inequalities* (1b)

Subroutine 3.

Each inequality (1b) is linearized in the form (4) to get

$$L_i(X)=\sum_{j=1}^{n} a_{ij}x_j + B_i \le 0, \quad x_j \in X_j \tag{16}$$

Now for j=1 to n we solve the inequality

$$a_{ij}Y_j + S_j \le 0 \tag{17a}$$

for Y_j as shown in [3] with

$$S_j = \sum_{\substack{k=1 \\ k\ne j}}^{n} a_{ij}X_j + B_i \tag{17b}$$

For each j, we update

$$X_j := Y_j \cap X_j \tag{17c}$$

and use this updated interval in (17b) when $j > 1$.

This process is repeated for all $i = 1,\ldots, r$.

4. ALGORITHM OF THE METHOD

The algorithm of the present iterative method is based on the execution, at each iteration, of several basic procedures.

Let $X \subsetneq X^{(0)}$ be the current box to be analyzed. If X is strictly feasible [3], then the following two procedures are carried out (otherwise the iterative process continues with Procedure 3 to be presented below).

Procedure 1.

(Monotonicity test [1], [3]). If for some i

$$G_i(X) > 0 \text{ or } G_i(X) < 0 \tag{18}$$

where $G_i(X)$ is the interval extension of $\partial\varphi_0/\partial x_i$ then X is discarded. To narrower $G_i(X)$ and, hence, to improve the effectiveness of the monotonicity test, $G_i(X)$ are computed using optimal poles [4].

Procedure 2.

Let $\bar{\varphi}_0$ denote an upper bound on the global minimum φ_0^*, obtained at a previous iteration (at the first iteration $\bar{\varphi}_0$ is set to $+\infty$). Now we compute $\varphi_0(x^c)$ where x^c is the center of X and update $\bar{\varphi}_0$, i.e.

$$\bar{\varphi}_0 = \varphi_0(x^c), \quad \text{if } \varphi_0(x^c) < \bar{\varphi}_0, \tag{19}$$

otherwise $\bar{\varphi}_0$ remains unchanged.

Procedure 3.

(Extended unfeasibility test)

Let

$$\varphi_0(X) := \varphi_0(X) - \bar{\varphi}_0 \tag{20}$$

If for some i from $0,1,\ldots,r$

$$\Phi_i(X) > 0 \tag{21}$$

the current box X is discarded.

If one of the above three tests (18), (19) or (21) is satisfied, a new box X is retrieved from a queue Q of boxes to be processed and the iterative process continues with checking feasibility. Otherwise, we proceed to the following procedure.

Procedure 4.

We call Subroutine 1 to set up and solve LP problem (8) (or (9)).

Procedure 5.

Call Subroutine 2 to apply constraint propagation (CP) to a system of equalities.

Procedure 6.

Call Subroutine 3 to apply CP to a system of inequalities.

Remark. The remaining steps of the algorithm of the present method such as control of the accuracy, splitting the current box whenever necessary etc. are omitted for space reasons.

5. NUMERICAL EXAMPLES

The examples given below have been solved by the following two methods:

- method M1 which is based on constraint propagation using the mean-value form (2) as linearization of each nonlinear equation;

- the present method (denoted as method M2) which appeals to the alternative linearization (4), linear programming and constraint propagation.

The data about the results include: number of splits (denoted as S), number of evaluations of all nonlinear functions (denoted as F), number of all interval gradients (if using M1) or all interval forms (4) (for M2) (denoted by the same symbol G since form (4) requires approximately the same amount of computation as interval gradients) and computer time T in seconds needed to solve the global optimization problem considered within desired accuracy. The examples were run on a relatively slow 166 MHz PC using an interpreter of EXCEL.

Example 1.

The function to be minimized is [3]

$$\varphi_0(x) = x_1^6 - 6.3x_1^4 + 12x_1^2 + 6x_1x_2 + 6x_2^2 \tag{22a}$$

subject to the constraints

$$\begin{aligned} \varphi_1(x) &= 1 - 16x_1^2 - 25x_2^2 \le 0 \\ \varphi_2(x) &= 13x_1^3 - 145x_1 + 85x_2 - 400 \le 0 \\ \varphi_3(x) &= x_1x_2 - 4 \le 0 \end{aligned} \tag{22b}$$

For this example, the absolute accuracy (upper bound minus lower bound on the global minimum) was chosen to be 10^{-5}. We give data about S, F, G and T for two initial boxes whose side are [-2, 4] and [-10^5, 10^5], respectively:

Table 1

	S	F	G	T
M1	1050	8411	8397	7
M2	1050	2115	8437	6
M1	2713	21703	21653	18
M2	2381	4759	19038	14

E x a m p l e 2.

In this example [12]

$$\varphi_0(x) = (x_1 - 10)^2 + 5(x_2 - 12)^2 + x_3^4 + 3(x_4 - 11)^2 +$$
$$10x_5^6 + 7x_6^2 + x_7^4 - 4x_6x_7 - 10x_6 - 8x_7$$
$$\varphi_1(x) = 2x_1^2 + 3x_2^4 + x_3 + 4x_4^2 + 5x_5 - 127 \le 0$$
$$\varphi_2(x) = 7x_1 + 3x_2 + 10x_3^2 + x_4 - x_5 - 282 \le 0$$
$$\varphi_3(x) = 23x_1 + x_2^2 + 6x_6^2 - 8x_7 - 196 \le 0$$
$$\varphi_4(x) = 4x_1^2 + x_2^2 - 3x_1x_2 + 2x_3^2 + 5x_6 - 11x_7 \le 0$$

For this example, the accuracy measure was the relative accuracy (absolute accuracy over absolute value of the mean of upper and lower bounds) and was chosen to be 0.01.

Table 2

	S	F	G	T
IM2	11887	118727	118727	292
IM2	8074	16146	80738	143

The above two examples (as well as others not reported here) confirm the expected effect of enhanced numerical efficiency of the present method as compared with other interval methods of the same class.

6. SUMMARY

A new interval method for globally solving inequality-constrained optimization problems defined as in (1) has been suggested. The method has one major feature which distinguishes it favorably from other known interval methods for global optimization. It consists in the fact that the interval extensions $\Phi_i(X)$ of the functions $\varphi_i(x)$, $i = 0,...,r$, associated with the optimization problem, are determined in a new way, using the recently suggested linear interval form (4). The new form differs from the known form (2) in that the coefficients α_i before the variables x_i are real numbers and only the additive term B is an interval while in (2) all the coefficients are intervals and only the additive term is a real number.

The specific feature of form (4) has permitted the development of two techniques aiming to speed up the computation: linear programming (in solving the associated system of nonlinear equations (5)) and constraint propagation (for handling both (5) and the system of nonlinear inequalities (1b)). Experimental evidence seems to indicate that the new method's performance is better than methods based on the traditional form (2).

An improvement of the numerical efficiency seems possible by generalizing the linear programming approach to the system of nonlinear inequalities associated with the optimization problem considered.

7. REFERENCES

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