Interval analysis, optimisation and robotics

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1 The COPRIN project

- INRIA: National Research Institute in Computer Science and Control Theory
- COPRIN: a team of 6 researchers + 7 PhD/Postdoc/Engineers, created in 2002

Purpose: solving problems with interval analysis, constraints programming, numerical analysis and symbolic computation for application domains such as:

- mechanism theory
- molecular chemistry
- network analysis
- computer vision
- quantum mechanics
- control theory
- ...

using our libraries `Alias`, `IcosAlias`

2 Mathematical viewpoint on robotics problems

Most robotics problems can be basically reduced to:

- **systems solving** problems (involving non-linear equations or inequalities, differential or integral equations)
- **constrained optimization** problems (minimum, maximum, minmax)
- **linear algebra** problems

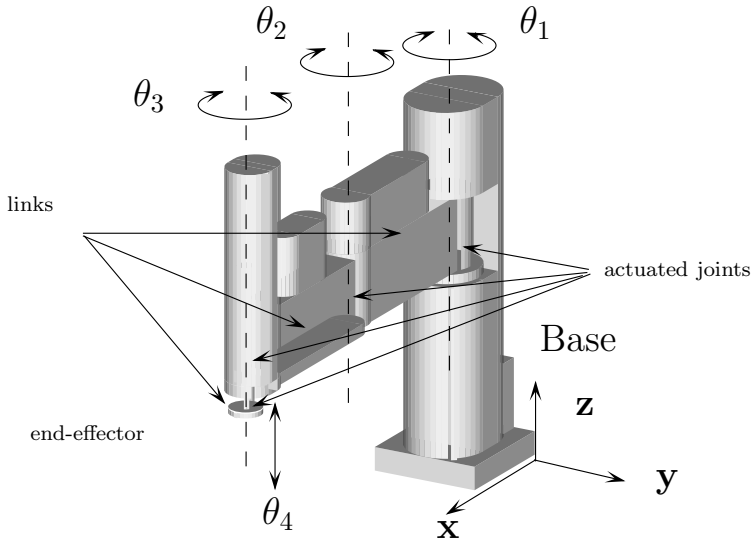
BUT

- most of them have specificities that must be addressed for efficiency and completeness
- functions involved in these problems are complex ⇒ **symbolic computation**
- results must be **certified** (e.g. for safety reasons in medical robotics)
- very large problems ⇒ **distributed implementation**

3 What is a robot manipulator?

Manipulator: a mechanical system whose purpose is to move a particular body (the "hand") to a specific pose by controlling the relative motion of internal bodies through "actuators"

- **X**: parameters describing the pose of the hand **(generalized coordinates)**
- **Θ**: parameters describing the motion of the actuators **(joint variables)**



4 degrees-of-freedom SCARA robot

$$\mathbf{X}=\{x, y, z, \theta_z\}$$
$$\mathbf{\Theta} = \{\theta_1, \theta_2, \theta_3, \theta_4\}$$

4 Kinematics

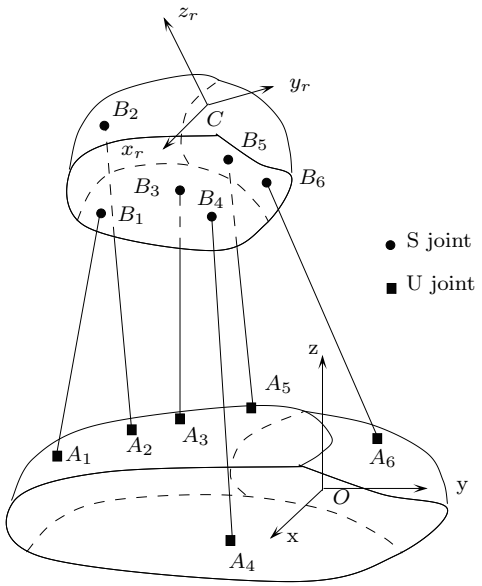
Needed for controlling the robot

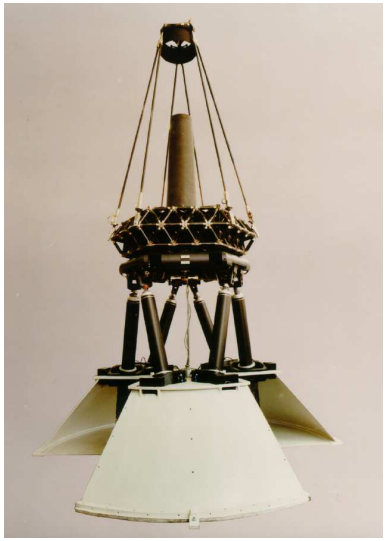
- the robot sensors give **joint variable measurements** $\mathbf{\Theta}_m$
- for control we must have
  - **inverse kinematics**: $\mathbf{X} \rightarrow \mathbf{\Theta} = G(\mathbf{X})$
  - **direct kinematics**: $\mathbf{\Theta}_m \rightarrow \mathbf{X} = F(\mathbf{\Theta}_m)$

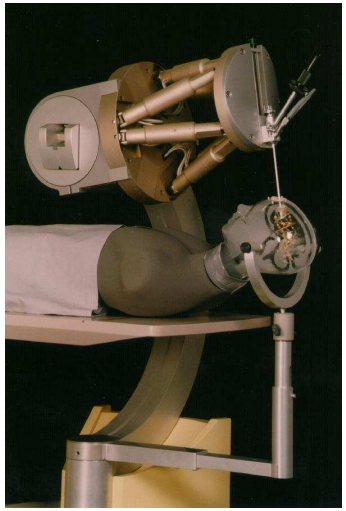
Example: parallel robot

$\mathbf{X}$= **OC**, $\mathbf{R}(\mathbf{x_r}, \mathbf{y_r}, \mathbf{z_r} \rightarrow \mathbf{x}, \mathbf{y}, \mathbf{z})$
$\mathbf{\Theta}$ = leg lengths

We have an explicit inverse kinematics $\mathbf{\Theta} = G(\mathbf{X})$





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Direct kinematics

Problem: being given $\mathbf{\Theta}_m$ find the *actual pose* of the platform

actual pose: pose of the platform when the measurement was performed

Method: solve the inverse kinematic equations $G(\mathbf{X}) = \mathbf{\Theta}_m$

Background:

- this system has multiple real solutions (here up to 40)
- inverse kinematics leads to a system of 6 to 12 equations, according to the choice of the parameters that are used to define the pose of the platform
- unknowns are bounded: $||\mathbf{OC}|| \leq ||\mathbf{OA}|| + \mathbf{\Theta} + ||\mathbf{CB}||$
- additional constraints may have to be considered (limited motion of the passive joints)

There are two different types of direct kinematics problem:

1. off-line: find the current pose without any a-priori information on it
   Computation time is not critical, within limits
2. real-time: determine as quickly as possible the actual pose that should not be "too far" from a previously computed pose.
   - Computation time should be less than the robot controller sampling time (2 to 5 ms)
   - Solutions must be certified and singular or multiple solutions must be detected

4.1 Off-line direct kinematics

- No a-priori information on the actual pose of the platform
- Calculation of **all** solutions

Possible methods

Algebraic formulation

- **elimination** : **numerically unsafe**
- **homotopy** : (**$\geq$ 30 minutes**)
- **Groebner basis** : **certified**,
  $fastest\ method : FgB, between\ 1s\ and\ 30s$

formulation independent

- **interval analysis**

interval analysis second fastest method: based on **distance equations**

$$(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2 = d_{ij}^2 \leftarrow \text{known distance}$$

Algorithm

- **2B and 3B** constraint propagation
- **specific Neumaier and Kantorovitch exclusion test**
  - determine a ball with a unique solution, with a method to compute this solution
  - detect multiple solutions
- **computation time**: 10-60s

but interval analysis is the only method whose computation time may be reduced if

- additional constraints are taken into account
- there are constraints on the search space

But we have still to find an algorithm that will determine the actual pose

4.2 "Real time" direct kinematics

Usual approach: Newton method

BUT Newton is not safe and may leads to the wrong solution with a critical control effect

Safer approach: Newton + interval analysis

$\mathbf{X}_l$: last known position + known maximal velocity of the robot

$$\Downarrow$$

$$\mathbf{X} \in \mathcal{B} = [\mathbf{X}_l - \epsilon, \mathbf{X}_l + \epsilon]$$

1. run **Newton** with the center of the current box as guess
2. if **Newton** does not converge or converge outside $\mathcal{B}$, bissect and reiterate
3. if **Newton** converge toward $\mathbf{X}_n \in \mathcal{B}$ use **Kantorovitch, Neumaier** to
   - **certify** the solution $\mathbf{X}_n$ and find $\mathcal{B}_k$ that includes only $\mathbf{X}_n$
4. if $\mathcal{B} \subset \mathcal{B}_k \rightarrow$ **certified solution** otherwise bissect and reiterate
5. if more than one solution or singular jacobian $\rightarrow$ **STOP!**

$\Rightarrow$ **Certified Newton scheme**

Properties:

- certified Newton can deal with additional constraints (limits on joint motion)
- computation time Newton: from 1.8 to 5 μs **when convergent**
- computation time interval scheme: 300 to 400 μs, well within controller sampling time
- **distributed calculation**

5 Inverse kinematic jacobian matrix

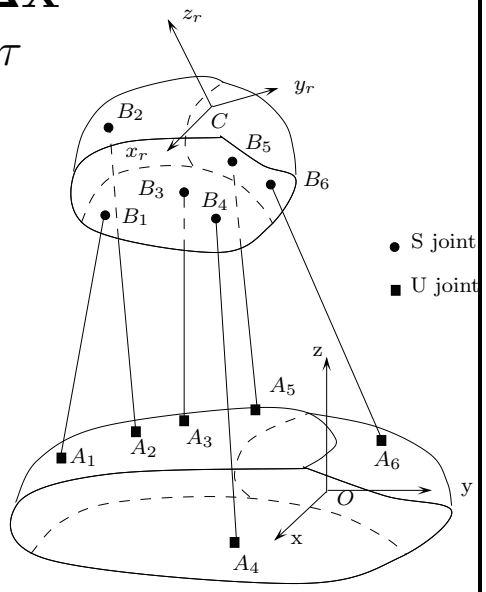
$$\mathbf{\Theta} = G(\mathbf{X})$$
$$\Downarrow$$

- **velocity relation**: $\dot{\mathbf{\Theta}} = J^{-1}(\mathbf{X})\dot{\mathbf{X}}$
- **accuracy relation**: $\mathbf{\Delta\Theta} = J^{-1}(\mathbf{X})\mathbf{\Delta X}$
- **static relation**: $\mathcal{F} = J^{-T}(\mathbf{X})\tau$

J^{-1}: **inverse kinematic jacobian**

$$((\frac{\mathbf{A_iB_i}}{||\mathbf{A_iB_i}||} \qquad \frac{\mathbf{CB_i} \times \mathbf{A_iB_i}}{||\mathbf{A_iB_i}||}))$$

$\rightarrow$ Plücker vector of the lines associated to the legs



Example of problems: accuracy analysis

$$\mathbf{\Delta\Theta_m} = J^{-1}(\mathbf{X})\mathbf{\Delta X}$$

for bounded $\mathbf{\Delta\Theta_m}$ ($|\Delta\Theta_m^i| \leq 1$) shows that $\Delta X_i \leq \epsilon_i$ for any $\mathbf{X}$ in a given workspace

classical problem of computing an outer approximation of the solutions of a **linear interval system** $\mathbf{Ay} = \mathbf{b}$ with $\mathbf{A} = J^{-1}(\mathbf{X})$, $\mathbf{y} = \mathbf{\Delta X}$, $\mathbf{b} = \mathbf{\Delta\Theta}$

BUT

typical of an application problem: $A_{ij} = f_{ij}(\mathbf{X})$ with $\mathbf{X}$ an interval vector

classical interval methods do not take into account the dependency between the f_{ij}

$\Downarrow$

large overestimation of the solution domains

$\Rightarrow$ taking into account the derivatives of the f_{ij} with respect to $\mathbf{X}$ in the Gaussian elimination scheme leads to an algorithm which is 10 to 100 faster than the classical methods

Example of problems: Singularity detection

$$\mathcal{F} = J^{-T}(\mathbf{X})\tau$$
$$\Downarrow$$
$$\tau_i = \frac{|\ldots|}{|J^{-T}|}$$

$\Rightarrow$ if J^{-T} singular, then $\tau_i \rightarrow \infty \Rightarrow$ **robot breakdown**

Problems:

shows that $|J^{-1}(\mathbf{X})| \neq 0$ for any $\mathbf{X}$ in a given workspace (no **singularity**)

$$J^{-1} = ((\frac{\mathbf{A_i B_i}}{||\mathbf{A_i B_i}||} \qquad \frac{\mathbf{CB_i} \times \mathbf{A_i B_i}}{||\mathbf{A_i B_i}||}))$$

$\Rightarrow$ singularity depends also on the location of $\mathbf{A}, \mathbf{B}$ which are uncertain

Algorithm

- find a pose in the workspace and compute numerically the sign of $|J^{-1}|$ (say > 0)
- bisection algorithm to determine if a pose with $|J^{-1}| < 0$ exists in the workspace
  - if yes singularity exists: SINGULARITY, EXIT
  - otherwise no singularity in the workspace

Checking the regularity of an interval matrix

$$\mathbf{A} = \begin{pmatrix} x & x \\ y & 2y \end{pmatrix}$$

with x, y in [1,2]. Clearly $\mathbf{A}$ does not include singular matrix as $|\mathbf{A}| = xy$

we can compute an interval evaluation of the determinant (direct expansion by row or column, Gaussian elimination)

$$\mathbf{A}_I = \begin{pmatrix} [1,2] & [1,2] \\ [1,2] & [2,4] \end{pmatrix} \Rightarrow |\mathbf{A}_I| \in [-2,7] \Rightarrow \textbf{ no direct conclusion}$$

Improvement 1: extremal matrices

Extremal matrices: all scalar matrices with as elements either the lower or upper bound of the corresponding elements in interval matrix

$$\mathbf{A}_I = \begin{pmatrix} [1,2] & [1,2] \\ [1,2] & [2,4] \end{pmatrix} \Rightarrow \mathbf{A}_e^1 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \mathbf{A}_e^2 = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \mathbf{A}_e^3 = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}, \ldots$$

2^{n^2} extremal matrices

Theorem: **if determinant of all extremal matrices have same sign $\Rightarrow$ no singular matrices in $\mathbf{A}_I$**

Improvement by Rex, Rohn (1998): not **all** extremal matrices should be considered, only 2^{2n-1} **but** the set of matrices to test includes

$$\mathbf{A_1} = \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix} \quad \mathbf{A_2} = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix} \quad \mathbf{A_3} = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \ldots$$

$\mathbf{A_1}, \mathbf{A_2}$ have a determinant of opposite sign, $\mathbf{A_3}$ is singular $\Rightarrow$ **no direct conclusion**

Improvement 2: pre-conditioning

Numerical pre-conditioning: $|\mathbf{AK}| = |\mathbf{A}||\mathbf{K}|$

Use $\mathbf{K} = \mathbf{A}^{-1}(Mid(\mathbf{X}))$ as pre-conditioning matrix to get $\mathbf{AK}$ "close" to identity

$$\mathbf{K} = \begin{pmatrix} 4/3 & -2/3 \\ -2/3 & 2/3 \end{pmatrix} \quad \mathbf{AK} = \begin{pmatrix} [0,2] & [-2/3, 2/3] \\ [-4/3, 0] & [2/3, 2] \end{pmatrix}$$

$|\mathbf{K}| = 4/9$

$|\mathbf{AK}| = [-8/9, 44/9] \Rightarrow$ **improvement** on the evaluation width but **no direct conclusion**

Symbolic/numeric pre-conditioning:

off-line

- keep $\mathbf{K}$ symbolic
- compute $\mathbf{AK}$ symbolically
- re-arrange the elements of $\mathbf{AK}$ to decrease the dependency problem

on-line

- plug the numerical values of the element of $\mathbf{K}$ in the symbolic $\mathbf{AK}$

$$(\textbf{off}-\textbf{line})\ \mathbf{AK} = \begin{pmatrix} xk_{11}+xk_{21} & xk_{12}+xk_{22} \\ yk_{11}+2yk_{21} & yk_{12}+2yk_{22} \end{pmatrix} \Rightarrow \mathbf{AK} = \begin{pmatrix} x(k_{11}+k_{21}) & x(k_{12}+k_{22}) \\ y(k_{11}+2k_{21}) & y(k_{12}+2k_{22}) \end{pmatrix}$$

$$(\textbf{on}-\textbf{line})\ \mathbf{K} = \begin{pmatrix} 4/3 & -2/3 \\ -2/3 & 2/3 \end{pmatrix} \Rightarrow \mathbf{AK} = \begin{pmatrix} 2/3[1,2] & 0 \\ 0 & 2/3[1,2] \end{pmatrix}$$

$|\mathbf{AK}| = [4/9, 16/9] \Rightarrow$ **no singular matrix**

Improvement 3: better extremal matrices (Donelan,Merlet,2005)

- assume that some elements of some rows are linearly dependent of some subset of parameters $\{x_i, \ldots, x_j\}$
- construct all matrices H whose rows are independently obtained by setting $x_i, \ldots, x_j$ to their extremal values

Theorem: **if all the determinants of the matrices in H have same sign $\Rightarrow$ no singular matrices in A**

$$\mathbf{A} = \begin{pmatrix} x & x \\ y & 2y \end{pmatrix} \begin{matrix} \leftarrow \text{linear in x} \\ \leftarrow \text{linear in y} \end{matrix} \quad \mathbf{A_1} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \quad \mathbf{A_2} = \begin{pmatrix} 1 & 1 \\ 2 & 4 \end{pmatrix} \quad \mathbf{A_3} = \begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix} \quad \mathbf{A_4} = \begin{pmatrix} 2 & 2 \\ 2 & 4 \end{pmatrix}$$

all determinants have the same sign $\Rightarrow$ no singular matrix

Now we have a large panel of methods to check the regularity of $\mathbf{J}^{-\mathbf{T}}$

- check over a workspace for $\mathbf{X}$
- allow uncertainties on the coordinates of $\mathbf{A}$, $\mathbf{B}$
  - if their range width is small keep them in the matrix element
  - otherwise add these coordinates as new unknowns

Example: robot with base radius 13, platform radius 8, workspace $x, y \in [-5, 5], z \in [45, 50]$

No uncertainty on **A,B**

Orientation ψ, θ, ϕ	`Regular`	`Regular+Rohn`	$\mathbf{J}^{-1}\mathbf{K}$	$\mathbf{J}^{-1}\mathbf{K}$+ `Rohn`	$\mathbf{K}\mathbf{J}^{-1}$
[-1,1]	106.74	0.37	1.09	0.39	0.01
[-2,2]	338.9	0.4	4.3	0.42	0.01
[-5,5]	9076.2	2.6	34.79	2.8	0.01

With uncertainties on **A,B**

- (D): as it in the matrix elements
- (V): added as new variables
- with $\mathbf{KJ}^{-1}$ pre-conditioning

Uncertainty on **A,B**	$x_c, y_c \in [-5, 5]$ $z_c \in [45, 50]$ $\psi, \theta, \phi \in [-5°, 5°]$	$x_c, y_c \in [-5, 5]$ $z_c \in [45, 50]$ $\psi, \theta, \phi \in [-15°, 15°]$	$x_c, y_c \in [-15, 15]$ $z_c \in [45, 50]$ $\psi, \theta, \phi \in [-15°, 15°]$
(D) $\epsilon = \pm 0.05$	0.01	0.23	5.5 (7.32 with `Rohn`)
(V) $\epsilon = \pm 0.05$	0.01	0.63	14.07 (4.54 with `Rohn`)
(D) $\epsilon = \pm 0.1$	0.01	4.47	1540.74 (514.5 with `Rohn`)
(V) $\epsilon = \pm 0.1$	0.02	2.55	2614.55 (402.2 with `Rohn`)

Interval analysis is the only known method allowing a certified singularity verification even with uncertainties on the robot modeling

Examples of problems: Motion verification

Data:

- a trajectory for the robot defined by arbitrary analytical time-function $\mathbf{X}_i = f_i(t)$, $t \in [0,1]$
- possible bounded errors on **A,B**
- possible bounded control errors $\mathbf{X}_i^r = f_i(t) + \epsilon_i$

Objective: check that the real trajectory followed by the robot is **valid**

- the trajectory lie within the workspace of the robot: $\Theta_{min} \leq \Theta(t) \leq \Theta_{max}\ \forall\ t \in [0,1]$
- trajectory is singularity-free
- possibly has to verify other constraints: $\mathbf{G}(\mathbf{X}(t)) \leq 0$

for most usual trajectory the verification is almost real-time

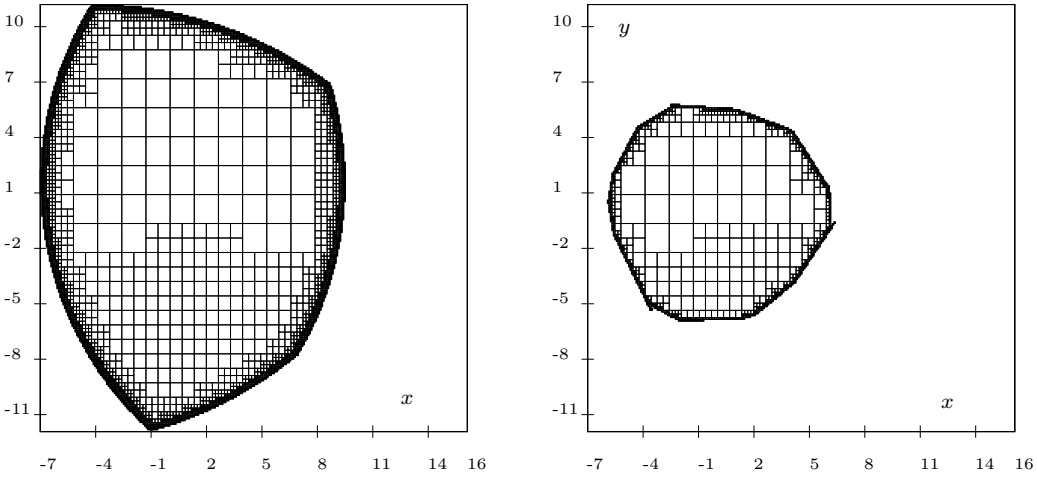
Examples of problems: Workspace analysis

Workspace: set of all poses that can be reached by the hand being given constraints on the mechanism

Constraints examples:

$$\rho_{min} \leq G(\mathbf{X}(t)) \leq \rho_{max} \qquad \frac{\mathbf{AB}.\mathbf{n}}{||\mathbf{AB}||} \geq \cos\alpha$$

We are dealing with a n dimensional variety and interval analysis is able to provide an approximation of the variety



Examples:
cross-section in the $x-y$ plane
of all poses for which
all orientation angles in $[-5°, 5°]$
are allowed

Workspace analysis with additional constraints

Workspace = $\{\mathbf{X}\}$ satisfying kinematic constraints + performance constraints

Examples: if λ_i are the eigenvalues of $J^{-1}J^{-T}$ then

Workspace = $\{\mathbf{X}\}$ satisfying kinematic constraints + $a_i \leq \lambda_i(\mathbf{X}) \leq b_i$

For a box in the workspace:

- $J^{-1}J^{-T}$ is an **interval matrix**
- characteristic polynomial P of $J^{-1}J^{-T}$ has **interval coefficients** $U_i(\mathbf{X})$

Bounds $[\underline{\alpha}, \overline{\alpha}]$ on the eigenvalues of $J^{-1}J^{-T}$:

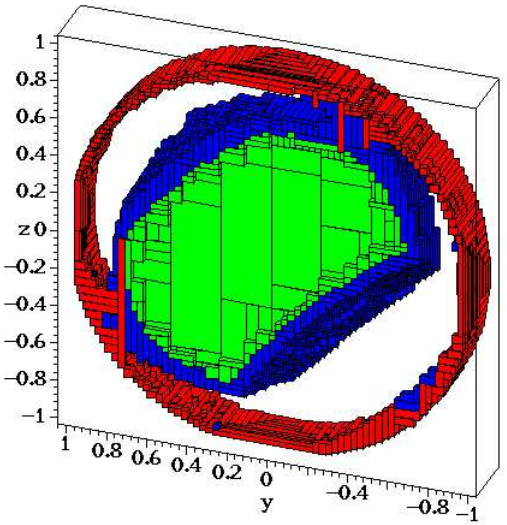
- directly from $J^{-1}J^{-T}$: **Gerschgorin circles, Cassini ovals**
- from the $U_i(\mathbf{X})$: **algebraic geometry theorems**

} adapted for interval coefficients

Note: also useful in control theory

Example: the Orthoglide robot, 3 dof translation robot

Quality index: $1/4 \leq$ eigenvalues of $J^{-1}J^{-T} \leq 4$



Examples of problems: Optimal design

Objectives: find the design parameters of a robot such that it satisfies a list of requirements

Usual approach: minimization of a weighted cost function:

$$\Sigma_i \quad \overbrace{w_i}^{weight} \quad \overbrace{I_i}^{performance} \quad index \in [0,1]$$

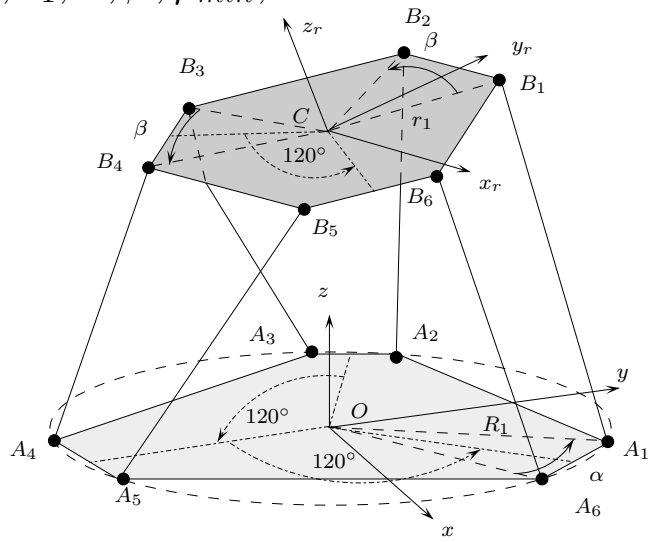
Drawbacks:

- difficulty to figure out the cost function
- result very sensitive to the weights
- provides only one design solution: but the designer is usually not aware of all design constraints
- real robot will differ from the theoretical solution (manufacturing tolerances)

Interval analysis

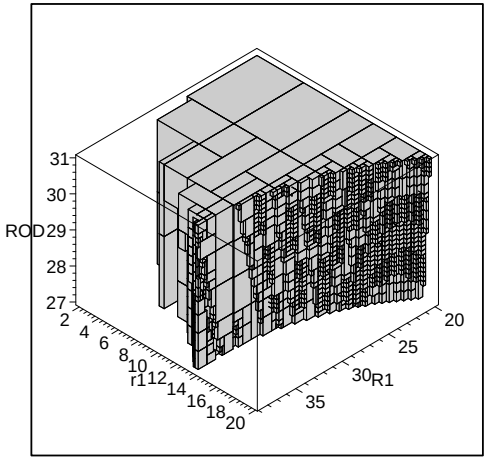
determination of an approximation of **all** parameters values so that the robot satisfies one requirement

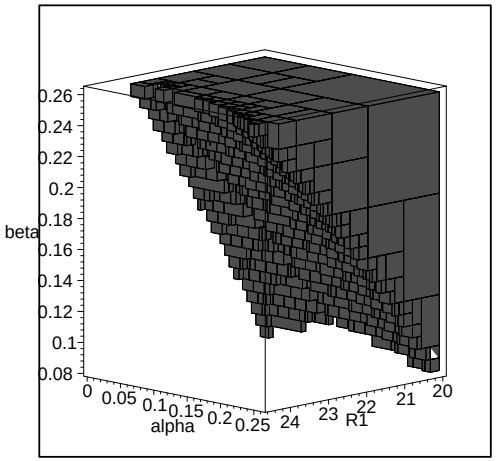
Example: Gough platform with design parameters $\mathcal{P}$: $R_1, r_1, \alpha, \beta, \rho_{min}, S$



Objectives: **find the design parameters such that a set of given poses $\{\mathbf{X}_1, \ldots, \mathbf{X}_n\}$ are included in the workspace of the robot**

$$\rho_{min} \leq G(\mathbf{X}_i) \leq \rho_{min} + S \qquad \frac{\mathbf{AB}.\mathbf{n}}{||\mathbf{AB}||} \geq \cos\alpha$$





- only an approximation is obtained, box with width lower than twice the manufacturing tolerances are neglected ⇒ **robust design solutions**
- computing the intersection of the region obtained for all requirement allows to determine **all design solutions**

deployment movie

wire robot

6 Conclusion: what I learn

- practical applications specificities ⇒ software flexibility
- theorems from other fields must be adapted for intervals: linear algebra, geometry
- practical interval analysis applications are embedded in a complex process

  ⇓

  few users will accept to learn new languages for using interval analysis
  ⇒ better to offer programming interface through general-purpose scientific languages (Matlab, Scilab, Maple, Mathematica)
- symbolic computation **must** be used to improve the efficiency of the solving
- interval analysis should not be presented as a black box because its efficiency is too variable
- tutorial for as many problems as possible should be made available, **interval movie**
- parallel implementation should be generalized

Try our on-line interval solvers: `http://www.inria.fr/coprin`