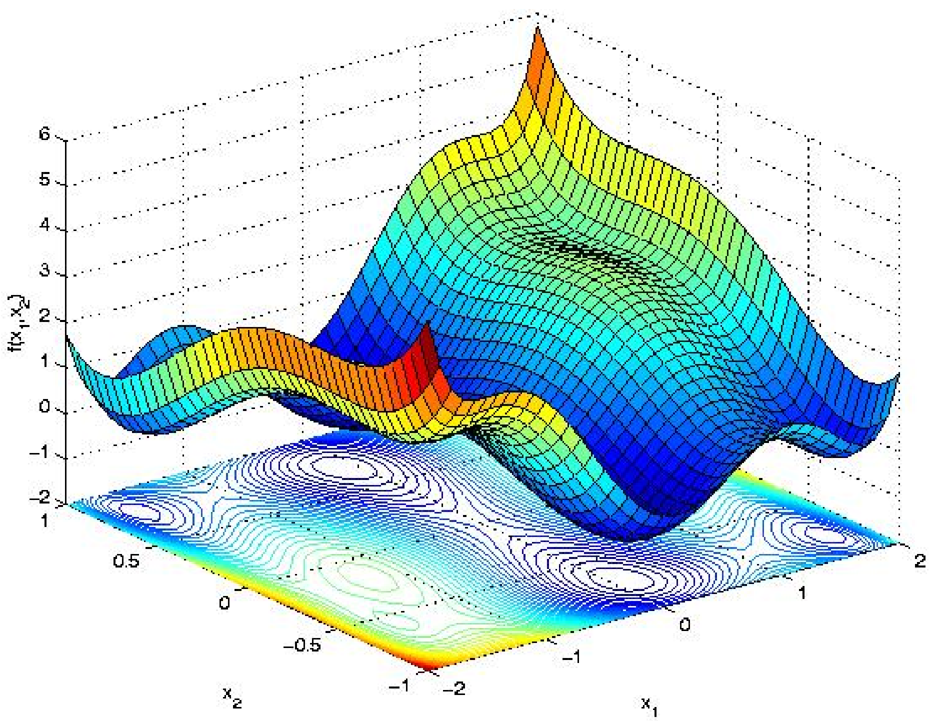
Detecting global optimality and extracting solutions in GloptiPoly

Didier HENRION[1,2]
Jean-Bernard LASSERRE[1]

[1]LAAS-CNRS Toulouse
[2]ÚTIA-AVČR Prague



Part 1
Description of
GloptiPoly

Brief description

GloptiPoly is written as an open-source, general purpose and user-friendly Matlab software

Optionally, problem definition made easier with Matlab Symbolic Math Toolbox, gateway to Maple kernel

Gloptipoly solves small to medium non-convex global optimization problems with multivariable real-valued polynomial objective functions and constraints

Software and documentation available at

`www.laas.fr/~henrion/software/gloptipoly`

Metholodogy

GloptiPoly builds and solves a hierarchy of successive convex linear matrix inequality (LMI) relaxations of increasing size, whose optima are guaranteed to converge asymptotically to the global optimum

Relaxations are built from LMI formulation of sum of squares (SOS) decomposition of positive multivariable polynomials

LMIs solved with Jos Sturm's SeDuMi

In practice convergence is ensured fast, typically at 2nd or 3rd LMI relaxation

LMI relaxation technique

Polynomial optimization problem

$$\begin{array}{ll} \min & g_0(x) \\ \text{s.t.} & g_k(x) \geq 0, \quad k = 1, \ldots, m \end{array}$$

When $p^\star$ is the global optimum, SOS representation of positive polynomial

$$g_0(x) - p^\star = q_0(x) + \sum_{k=1}^{m} g_k(x) q_k(x) \geq 0$$

where unknowns $q_k(x)$ are SOS polynomials similar to Karush/Kuhn/Tucker multipliers

Using LMI representation of SOS polynomials successive LMI relaxations are obtained by increasing degrees of sought polynomials $q_k(x)$

Theoretical proof of asymptotic convergence...
..but no tight degree upper bounds (not yet)

LMI relaxations: illustration

Non-convex quadratic problem

$$\begin{array}{ll} \max & 2x_1^2 + 2x_2^2 - 2x_1x_2 - 2x_1 - 6x_2 + 10 \\ \text{s.t.} & -x_1^2 + 2x_1 \geq 0 \\ & -x_1^2 - x_2^2 + 2x_1x_2 + 1 \geq 0 \\ & -x_2^2 + 6x_2 - 8 \geq 0. \end{array}$$

LMI relaxation built by replacing each monomial $x_1^i x_2^j$ with a new decision variable y_{ij}

For example, quadratic expression

$$-x_1^2 - x_2^2 + 2x_1x_2 + 1 \geq 0$$

replaced with linear expression

$$-y_{20} - y_{02} + 2y_{11} + 1 \geq 0$$

New decision variables y_{ij} satisfy non-convex relations such as $y_{10}y_{01} = y_{11}$ or $y_{20} = y_{10}^2$

LMI relaxations: illustration (2)

Relax these non-convex relations by enforcing LMI constraint

$$M_1^1(y) = \left[\begin{array}{c|cc} 1 & y_{10} & y_{01} \\ \hline y_{10} & y_{20} & y_{11} \\ y_{01} & y_{11} & y_{02} \end{array}\right] \succeq 0$$

Moment or measure matrix of first order relaxing monomials of degree up to 2

We remove the rank constraint on matrix $M_1^1(y)$

First LMI relaxation of original global optimization problem is given by

$$\begin{array}{ll} \max & 2y_{20} + 2y_{02} - 2y_{11} - 2y_{10} - 6y_{01} + 10 \\ \text{s.t.} & -y_{20} + 2y_{10} \geq 0 \\ & -y_{20} - y_{02} + 2y_{11} + 1 \geq 0 \\ & -y_{02} + 6y_{01} - 8 \geq 0 \\ & M_1^1(y) \succeq 0 \end{array}$$

LMI relaxations: illustration (3)

To build second LMI relaxation, we must increase size of moment matrix so that it captures expressions of degrees up to 4

Second order moment matrix reads

$$M_2^2(y) = \left[\begin{array}{c|cc|ccc} 1 & y_{10} & y_{01} & y_{20} & y_{11} & y_{02} \\ \hline y_{10} & y_{20} & y_{11} & y_{30} & y_{21} & y_{12} \\ y_{01} & y_{11} & y_{02} & y_{21} & y_{12} & y_{03} \\ \hline y_{20} & y_{30} & y_{21} & y_{40} & y_{31} & y_{22} \\ y_{11} & y_{21} & y_{12} & y_{31} & y_{22} & y_{13} \\ y_{02} & y_{12} & y_{03} & y_{22} & y_{13} & y_{04} \end{array}\right]$$

Constraints are also relaxed with additional variables

Second LMI features feasible set included in first LMI feasible set, thus providing a tighter relaxation

$$\begin{array}{ll} \max & 2y_{20} + 2y_{02} - 2y_{11} - 2y_{10} - 6y_{01} + 10 \\ \text{s.t.} & \left[\begin{array}{c|cc} -y_{20} + 2y_{10} & \star & \star \\ \hline -y_{30} + 2y_{20} & -y_{40} + 2y_{30} & \star \\ -y_{21} + 2y_{11} & -y_{31} + 2y_{12} & -y_{22} + 2y_{12} \end{array}\right] \succeq 0 \\ & \left[\begin{array}{c|cc} \left(\begin{array}{c} -y_{20} - y_{02} \\ +2y_{11} + 1 \end{array}\right) & \star & \star \\ \hline \left(\begin{array}{c} -y_{30} - y_{12} \\ +2y_{21} + y_{10} \end{array}\right) & \left(\begin{array}{c} -y_{40} - y_{22} \\ +2y_{31} + y_{20} \end{array}\right) & \star \\ \left(\begin{array}{c} -y_{21} - y_{03} \\ +2y_{12} + y_{01} \end{array}\right) & \left(\begin{array}{c} -y_{31} - y_{13} \\ +2y_{22} + y_{11} \end{array}\right) & \left(\begin{array}{c} -y_{22} - y_{04} \\ +2y_{13} + y_{02} \end{array}\right) \end{array}\right] \succeq 0 \\ & \left[\begin{array}{c|cc} -y_{02} + 6y_{01} - 8 & \star & \star \\ \hline \left(\begin{array}{c} -y_{12} + 6y_{11} \\ -8y_{10} \end{array}\right) & \left(\begin{array}{c} -y_{22} + 6y_{21} \\ -8y_{20} \end{array}\right) & \star \\ \left(\begin{array}{c} -y_{03} + 6y_{02} \\ -8y_{01} \end{array}\right) & \left(\begin{array}{c} -y_{13} + 6y_{12} \\ -8y_{11} \end{array}\right) & \left(\begin{array}{c} -y_{04} + 6y_{03} \\ -8y_{02} \end{array}\right) \end{array}\right] \succeq 0 \\ & M_2^2(y) \succeq 0 \end{array}$$

Numerical example (1)

Quadratic problem 3.5 in [Floudas/Pardalos 99]

$$\begin{array}{ll} \text{min} & -2x_1 + x_2 - x_3 \\ \text{s.t.} & x_1(4x_1 - 4x_2 + 4x_3 - 20) + x_2(2x_2 - 2x_3 + 9) \\ & \quad\quad +x_3(2x_3 - 13) + 24 \geq 0 \\ & x_1 + x_2 + x_3 \leq 4, \quad 3x_2 + x_3 \leq 6 \\ & 0 \leq x_1 \leq 2, \quad 0 \leq x_2, \quad 0 \leq x_3 \leq 3. \end{array}$$

To define this problem with GloptiPoly we use the following Matlab/Maple script

```
>> P = defipoly({'min -2*x1+x2-x3',...
['x1*(4*x1-4*x2+4*x3-20)+x2*(2*x2-2*x3+9)' ...
 '+x3*(2*x3-13)+24>=0'],...
'x1+x2+x3<=4', '3*x2+x3<=6',...
'0<=x1', 'x1<=2', '0<=x2', '0<=x3', 'x3<=3'}, ...
'x1,x2,x3');
```

To solve the first LMI relaxation we type

```
>> output = gloptipoly(P)
output =
    status: 0
      crit: -6.0000
       sol: {}
```

Field `status = 0` indicates that it is not possible to detect global optimality with this LMI relaxation, hence `crit = -6.0000` is a lower bound on the global optimum

Numerical example (2)

Next we try to solve the second, third and fourth LMI relaxations

```
>> output = gloptipoly(P,2)
output =
    status: 0
      crit: -5.6923
       sol: {}

>> output = gloptipoly(P,3)
output =
    status: 0
      crit: -4.0685
       sol: {}

>> output = gloptipoly(P,4)
output =
    status: 1
      crit: -4.0000
       sol: {[3x1 double]  [3x1 double]}
>> output.sol{:}
ans =
    2.0000
    0.0000
    0.0000

ans =
    0.5000
    0.0000
    3.0000
```

Both second and third LMI relaxations return tighter lower bounds on the global optimum

Eventually global optimality is reached at fourth LMI relaxation (certified by `status = 1`)

GloptiPoly also returns two globally optimal solutions:

$$x_1 = 2, x_2 = 0, x_3 = 0$$

and

$$x_1 = 0.5, x_2 = 0, x_3 = 3$$

leading to

`crit = -4.0000`

Numerical example (3)

Number of LMI variables (M) and size of relaxed LMI problem (N) increase quickly with relaxation order:

Relaxation	LMI opt	M	N
1	-6.0000	9	24
2	-5.6923	34	228
3	-4.0685	83	1200
4	-4.0000	164	4425
5	-4.0000	285	12936
6	-4.0000	454	32144

..yet fourth LMI relaxation was solved in about 2.5 seconds on a PC Pentium IV 1.6 MHz

Complexity

d: overall polynomial degree ($2\delta = d$ or $d+1$)
m: number of polynomial constraints
n: number of polynomial variables
M: number of LMI decision variables
N: size of LMI

$$
\begin{aligned}
M &= \binom{n+2\delta}{2\delta} - 1 \\
N &= \binom{n+\delta}{\delta} + m \binom{n+\delta-1}{\delta-1}
\end{aligned}
$$

When n is fixed:

- M grows polynomially in $O(\delta^n)$
- N grows polynomially in $O(m\delta^n)$

Features

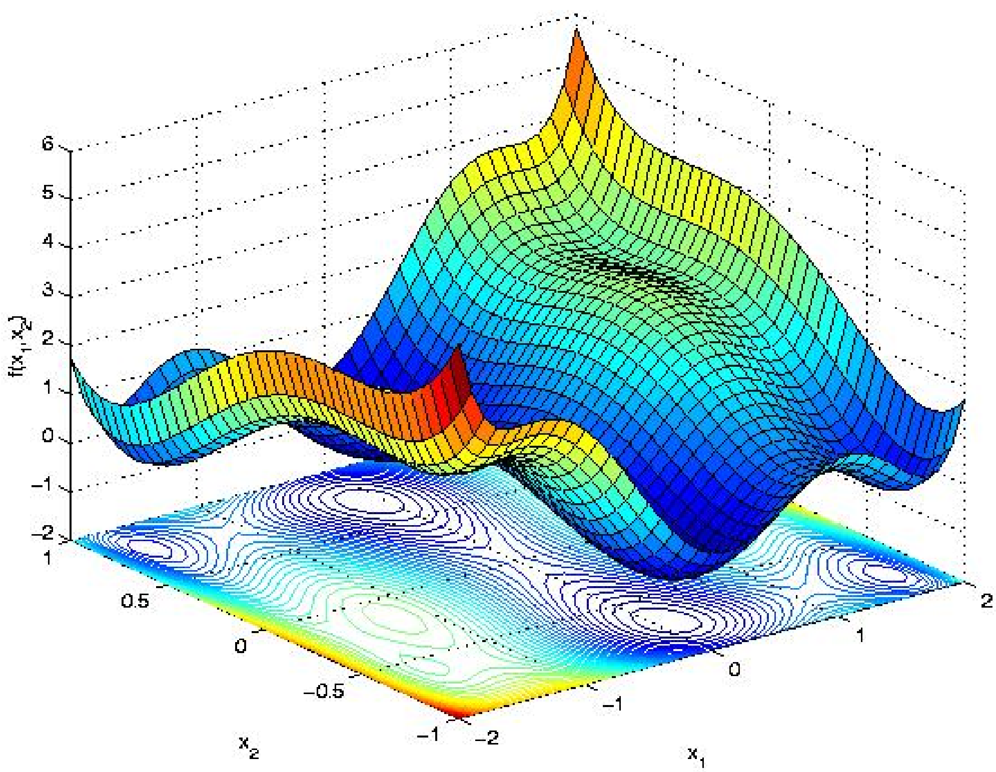
General features of GloptiPoly:

- Certificate of global optimality (rank checks)
- Automatic extraction of globally optimal solutions (multiple eigenvectors)
- 0-1 or ± 1 integer constraints on some of the decision variables (combinatorial optimization problems)
- Generation of input and output data in SeDuMi's format
- Generation of moment matrices associated with LMI relaxations (rank checks)
- User-defined scaling of decision variables (to improve numerical behavior)
- Exploits sparsity of polynomial data

Benchmark examples
Continuous problems

Mostly from Floudas/Pardalos 1999 handbook

About 80 % of pbs solved with LMI relaxation of small order (typically 2 or 3) in less than 3 seconds on a PC Pentium IV at 1.6 MHz with 512 Mb RAM

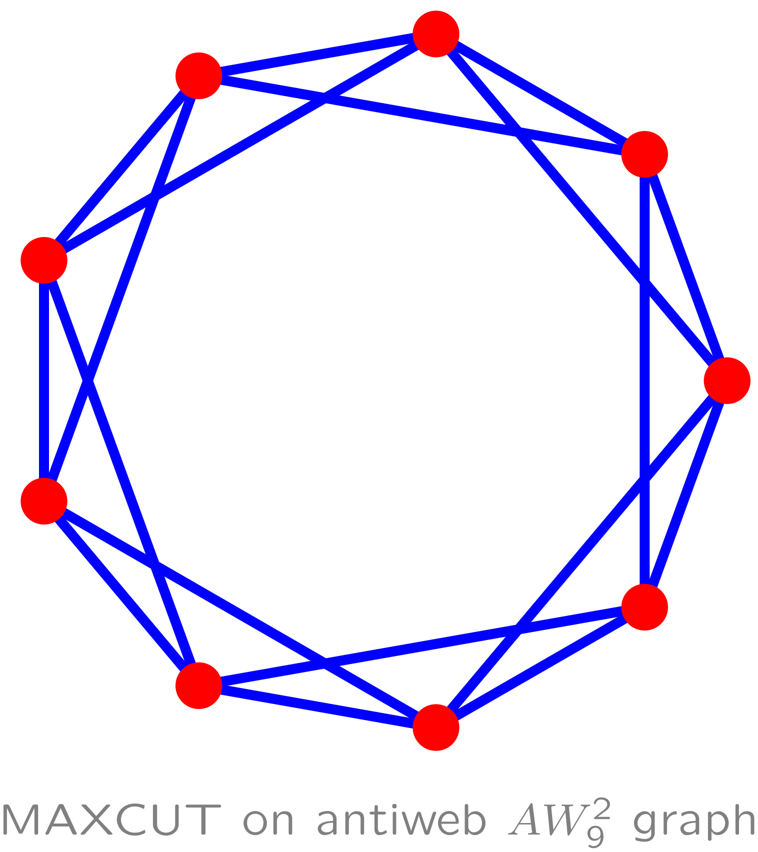


Six-hump camel back function

Benchmark exmaples
Discrete problems

From Floudas/Pardalos handbook and also Anjos' Ph.D (Univ Waterloo)

By perturbing criterion (destroys symmetry) global convergence ensured on 80 % of pbs in less than 4 seconds



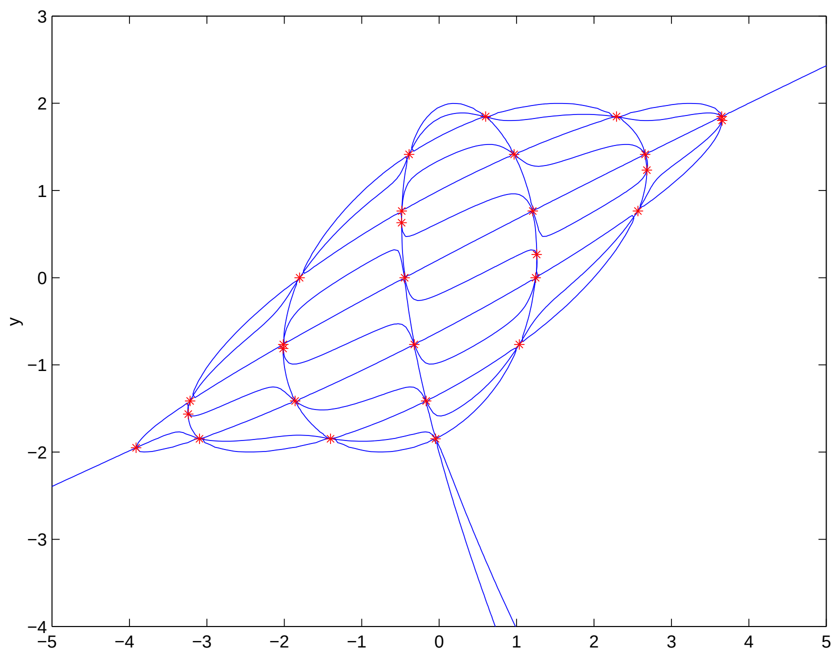
MAXCUT on antiweb AW_9^2 graph

Benchmark examples
Polynomial systems of equations

From Verschelde's database and Frisco INRIA project
Real coefficients & solutions only

Out of 59 systems:
- 61 % solved in t < 10 secs
- 20 % solved in 10 < t < 100 secs
- 10 % solved in t ≥ 100 secs
- 9 % out of memory

No criterion optimized
No enumeration of all solutions



Intersections of seventh and eighth degree polynomial curves

Part 2

Extracting solutions

Detecting global optimality

Global optimization problem

$$\begin{array}{rcll} p^\star & = & \min_x & g_0(x) \\ & & \text{s.t.} & g_i(x) \geq 0,\ i = 1, 2.. \end{array}$$

Let $\deg g_i(x) = 2d_i - 1$ or $2d_i$ and $d = \max_i d_i$

LMI relaxation of order k

$$\begin{array}{rcll} p_k^* & = & \min_y & \sum_\alpha (g_0)_\alpha y_\alpha \\ & & \text{s.t.} & M_k(y) \succeq 0 \\ & & & M_{k-d_i}(g_i y) \succeq 0,\ i = 1, 2.. \end{array}$$

with solution y_k^*

Hierarchy with convergence guarantee

$$p_d^* \leq p_{d+1}^* \leq \cdots \leq p_{k^*}^* = p^*.$$

for (generally) small k^*

Sufficient condition for global optimality
rank of moment matrices

$$\operatorname{rank} M_k(y_k^*) = \operatorname{rank} M_{k-d}(y_k^*)$$

Need for extraction procedures

Because rank condition is only sufficient
any other global optimality certificate
is welcome

Solution extraction procedure suggested
by Arnold Neumaier

Inspired by Chesi, Garulli, Tesi, Vicino
(IEEE CDC 2000)

Based on the work by Corless, Gianni, Trager
(ACM ISSAC 1997)

Key ideas:

- Cholesky decomposition of moment matrix
- column reduced echelon form
- computation of common eigenvalues

Who is Cholesky ?

André Louis Cholesky (1875-1918) was a French military officer (graduated from Ecole Polytechnique) involved in geodesy

He proposed a new procedure for solving least-squares triangulation problems

He fell for his country during World War I

Work posthumously published in
Commandant Benoît. Procédé du Commandant Cholesky. *Bulletin Géodésique*, No. 2, pp. 67-77, Toulouse, Privat, 1924.

Nice biography in
C. Brezinski. André Louis Cholesky. *Bulletin of the Belgian Mathematical Society*, Vol. 3, pp. 45-50, 1996.

I. — NOTICES SCIENTIFIQUES

Commandant BENOIT[1].

NOTE SUR UNE MÉTHODE DE RÉSOLUTION DES ÉQUATIONS NORMALES PROVENANT DE L'APPLICATION DE LA MÉTHODE DES MOINDRES CARRÉS A UN SYSTÈME D'ÉQUATIONS LINÉAIRES EN NOMBRE INFÉRIEUR A CELUI DES INCONNUES. — APPLICATION DE LA MÉTHODE A LA RÉSOLUTION D'UN SYSTÈME DEFINI D'ÉQUATIONS LINÉAIRES.

(Procédé du Commandant Cholesky[2].)

Le Commandant d'Artillerie Cholesky, du Service géographique de l'Armée, tué pendant la grande guerre, a imaginé, au cours de recherches sur la compensation des réseaux géodésiques, un procédé très ingénieux de résolution des équations dites *normales*, obtenues par application de la méthode des moindres carrés à des équations linéaires en nombre inférieur à celui des inconnues. Il en a conclu une méthode générale de résolution des équations linéaires.

Nous suivrons, pour la démonstration de cette méthode, la progression même qui a servi au Commandant Cholesky pour l'imaginer.

1. De l'Artillerie coloniale, ancien officier géodésien au Service géographique de l'Armée et au Service géographique de l'Indo-Chine, Membre du Comité national français de Géodésie et Géophysique.

2. Sur le Commandant Cholesky, tué à l'ennemi le 31 août 1918, voir la notice biographique insérée dans le volume du *Bulletin géodésique* de 1922 intitulé : *Union géodésique et géophysique internationale, Première Assemblée générale, Rome, mai 1922, Section de Géodésie*, Toulouse, Privat, 1922, in-8°, 241 p., pp. 159 à 161.

Cholesky factorization

Extract Cholesky factor V of moment matrix

$$M_k(y_k^\star) = VV'$$

Matrix V has r columns, corresponding to r globally optimal solutions $x_j^\star$, $j = 1, 2, \ldots, r$ (provided global optimum was reached)

Denote

$$v = \left[1 \; x_1 \; x_2 \; \ldots \; x_n \; x_1^2 \; x_1x_2 \; \ldots \; x_1x_n \; x_2^2 \; x_2x_3 \; \ldots \; x_n^2 \; \ldots \; x_n^k\right]'$$

a basis for polynomials of degree at most k

By definition of the moment matrix:

$$M_k(y_k^\star) = V^\star(V^\star)'$$

where

$$V^\star = \left[\begin{array}{cccc} v_1^* & v_2^* & \cdots & v_r^* \end{array}\right]$$

and $v_j^\star$ is polynomial basis v evaluated at solution $x_j^\star$

Extracting solutions amounts to finding linear transformation between Cholesky factors V and $V^\star$

Reduction to column echelon form

Next step is reduction of V into column echelon form

$$U = \begin{bmatrix} 1 & & & & \\ \star & & & & \\ 0 & 1 & & & \\ 0 & 0 & 1 & & \\ \star & \star & \star & & \\ & \vdots & & \ddots & \\ 0 & 0 & 0 & \cdots & 1 \\ \star & \star & \star & \cdots & \star \\ & \vdots & & & \vdots \\ \star & \star & \star & \cdots & \star \end{bmatrix}$$

by Gaussian elimination with column pivoting

Each row in U = monomial x_α in basis v

Pivot entry in U = monomial x_{β_j} in generating basis of the set of solutions

In other words, denoting

$$w = \begin{bmatrix} x_{\beta_1} & x_{\beta_2} & \ldots & x_{\beta_r} \end{bmatrix}'$$

it holds

$$v = Uw$$

for all solutions x_j^*, $j = 1, 2, \ldots, r$

Multiplication matrices

For each first degree monomial x_i extract from U the r-by-r multiplication matrix N_i containing coefficients of product monomials $x_i x_{\beta_j}$ in generating basis w, i.e. such that

$$N_i w = x_i w \quad i = 1, 2, \ldots, n$$

If monomial $x_i x_{\beta_j}$ is not represented in U, then extraction algorithm fails and order k of LMI relaxation must be increased

Given matrices N_i finding scalars x_i is an eigenvalue problem

Extracting solutions amounts to solving an eigenvalue problem

Eigenvectors w are shared by matrices N_i so it is a particular common eigenvalue problem

Common eigenvalue problem

Build combination of multiplication matrices

$$N = \sum_{i=1}^{n} \lambda_i N_i$$

where λ_i are random positive numbers (summing up to one)

Compute ordered Schur decomposition

$$N = QTQ'$$

where

$$Q = \left[\begin{array}{cccc} q_1 & q_2 & \cdots & q_r \end{array} \right]$$

is orthogonal and T upper triangular

Finally, due to orthogonality of vectors q_i, ith entry in solution vector $x_j^\star$ is given by

$$(x_j^*)_i = q_j' N_i q_j, \; i = 1, 2, \ldots, n, \; j = 1, 2, \ldots, r$$

Number of solutions

No easy way to control number of extracted solutions in case of multiple global optima

Number of solutions = rank of moment matrix, but enforcing rank in an LMI is a difficult non-convex problem

By default GloptiPoly minimizes the trace (sum of eigenvalues) of the moment matrix, which may indirectly minimize the rank (number of non-zero eigenvalues)

Practical experiments reveal that low rank moment matrices ensure faster convergence of LMI relaxations to global optimum

First example

Non-convex quadratic optimization

$$\begin{array}{rcll} p^* & = & \max_x & (x_1-1)^2+(x_1-x_2)^2+(x_2-3)^2 \\ & & \text{s.t.} & (x_1-1)^2 \leq 1 \\ & & & (x_1-x_2)^2 \leq 1 \\ & & & (x_2-3)^2 \leq 1 \end{array}$$

First LMI relaxation yields $p_1^\star = -3$ and $\operatorname{rank} M_1(y^*) = 3$, extraction algorithm fails due to incomplete monomial basis

Second LMI relaxation yields $p_2^\star = -2$ and

$$\operatorname{rank} M_1(y^*) = \operatorname{rank} M_2(y^*) = 3$$

so rank condition ensures global optimality

First example: Cholesky factor

Moment matrix of order $k = 2$ reads

$$M_2(y^*) = \begin{bmatrix} 1.0000 & 1.5868 & 2.2477 & 2.7603 & 3.6690 & 5.2387 \\ 1.5868 & 2.7603 & 3.6690 & 5.1073 & 6.5115 & 8.8245 \\ 2.2477 & 3.6690 & 5.2387 & 6.5115 & 8.8245 & 12.7072 \\ 2.7603 & 5.1073 & 6.5115 & 9.8013 & 12.1965 & 15.9960 \\ 3.6690 & 6.5115 & 8.8245 & 12.1965 & 15.9960 & 22.1084 \\ 5.2387 & 8.8245 & 12.7072 & 15.9960 & 22.1084 & 32.1036 \end{bmatrix}$$

Positive semidefinite with rank 3

Cholesky factor

$$V = \begin{bmatrix} -0.9384 & -0.0247 & 0.3447 \\ -1.6188 & 0.3036 & 0.2182 \\ -2.2486 & -0.1822 & 0.3864 \\ -2.9796 & 0.9603 & -0.0348 \\ -3.9813 & 0.3417 & -0.1697 \\ -5.6128 & -0.7627 & -0.1365 \end{bmatrix}$$

satisfies

$$M_k(y_k^\star) = VV'$$

First example: column echelon form

Gaussian elimination on V yields

$$U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 3 & 0 \\ -4 & 2 & 2 \\ -6 & 0 & 5 \end{bmatrix} \begin{matrix} 1 \\ x_1 \\ x_2 \\ x_1^2 \\ x_1 x_2 \\ x_2^2 \end{matrix}$$

$$\begin{matrix} 1 & x_1 & x_2 \end{matrix}$$

which means that solutions to be extracted satisfy the system of polynomial equations

$$\begin{aligned} x_1^2 &= -2 + 3x_1 \\ x_1 x_2 &= -4 + 2x_1 + 2x_2 \\ x_2^2 &= -6 + 5x_2 \end{aligned}$$

in polynomial basis 1, x_1, x_2

First example: extraction

Multiplication matrices of monomials x_1 and x_2 in polynomial basis 1, x_1, x_2 are extracted from U:

$$N_1 = \begin{bmatrix} 0 & 1 & 0 \\ -2 & 3 & 0 \\ -4 & 2 & 2 \end{bmatrix}, \quad N_2 = \begin{bmatrix} 0 & 0 & 1 \\ -4 & 2 & 2 \\ -6 & 0 & 5 \end{bmatrix}$$

Random linear combination

$$N = 0.6909 N_1 + 0.3091 N_2$$

Schur decomposition of $N = QTQ'$ yields

$$Q = \begin{bmatrix} 0.4082 & 0.1826 & -0.8944 \\ 0.4082 & -0.9129 & -0.0000 \\ 0.8165 & 0.3651 & 0.4472 \end{bmatrix}$$

Projections of orthogonal columns of Q onto N yield the 3 expected globally optimal solutions

$$x_1^* = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad x_2^* = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \quad x_3^* = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Second example: minimum trace LMI

Polynomial system of equations

$$\begin{array}{rcl} x_1^2 + x_2^2 & = & 1 \\ x_1^3 + (2 + x_3)x_1x_2 + x_2^3 & = & 1 \\ x_3^2 & = & 2 \end{array}$$

No objective function, so GloptiPoly minimizes the trace of the moment matrix

Extraction on 2nd LMI relaxation fails due to incomplete basis

3rd LMI relaxation yields two globally optimal solutions

$$x_1^* = \begin{bmatrix} 0.5826 \\ -0.8128 \\ -1.4142 \end{bmatrix} \qquad x_2^* = \begin{bmatrix} -0.8128 \\ 0.5826 \\ -1.4142 \end{bmatrix}$$

Second example: zero objective function

With zero objective function GloptiPoly at the 3rd LMI relaxation yields

$\text{rank} M_1(y^*) = 4 \neq \text{rank} M_2(y^*) = \text{rank} M_3(y^*) = 6$

so rank condition cannot ensure optimality

However, extraction algorithm returns 6 globally optimum solutions

$$x_1^* = \begin{bmatrix} -0.8128 \\ 0.5826 \\ -1.4142 \end{bmatrix} \quad x_2^* = \begin{bmatrix} 0.5826 \\ -0.8128 \\ -1.4142 \end{bmatrix}$$

$$x_3^* = \begin{bmatrix} 0.0000 \\ 1.0000 \\ -1.4142 \end{bmatrix} \quad x_4^* = \begin{bmatrix} 1.0000 \\ 0.0000 \\ -1.4142 \end{bmatrix}$$

$$x_5^* = \begin{bmatrix} 0.0000 \\ 1.0000 \\ 1.4142 \end{bmatrix} \quad x_6^* = \begin{bmatrix} 1.0000 \\ 0.0000 \\ 1.4142 \end{bmatrix}$$

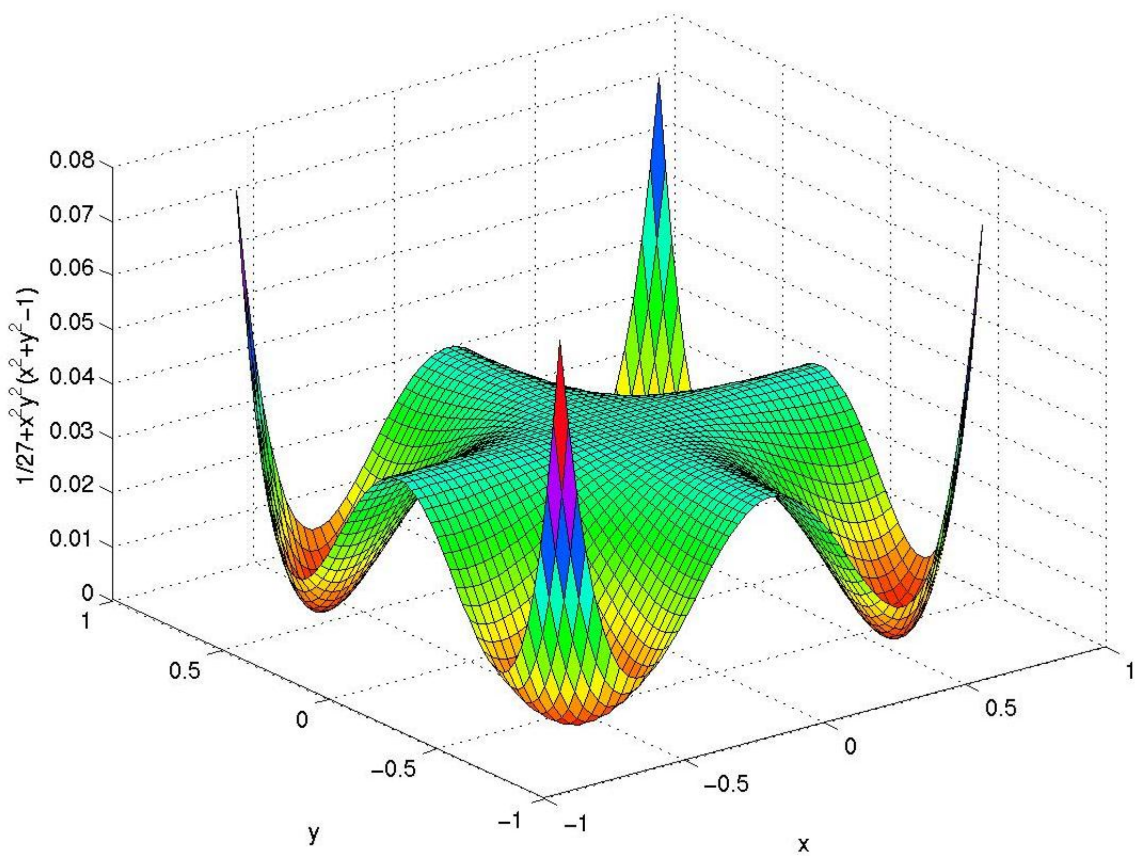
thus proving global optimality of LMI

Motzkin-like polynomial

Polynomial

$$\frac{1}{27} + x^2y^2(x^2 + y^2 - 1)$$

vanishes at $|x| = |y| = \sqrt{3}/3$ and
remains globally non-negative for real x and y
but cannot be written as an SOS



Some kind of magic in GloptiPoly

GloptiPoly finds approximate SOS decomposition of Motzkin polynomial

With 8th LMI relaxation we obtain

$$\frac{1}{27} + x^2y^2(x^2+y^2-1) = \sum_{i=1}^{32} a_i^2 q_i^2(x,y) + \varepsilon r(x,y)$$

where $\|q_i\|_2 = \|r\|_2 = 1$
and $\varepsilon \leq 10^{-8} < a_i^2$, $\deg q_i \leq 8$

Cone of SOS polynomials is dense in set of polynomials nonnegative over box [-1,1]

Numerical inaccuracy helps finding higher degree SOS polynomial close to Motzkin polynomial

Constrained Motzkin polynomial

Additional redundant constraint

$$x^2 + y^2 \leq R^2$$

with $R^2 > 2/3$ (includes the 4 global minima)

For $R = 1$ at the 3rd LMI relaxation we obtain

$$\frac{1}{27} + x^2y^2(x^2 + y^2 - 1) = \sum_{i=1}^{6} a_i^2 q_i^2(x,y) + (R^2 - x^2 - y^2) \sum_{i=1}^{2} b_i^2 r_i^2(x,y)$$

where $\deg q_i \leq 3$, $\deg r_i \leq 2$

R^2	1	2	3	4	$\cdots$	∞
LMI	3	4	5	6	$\cdots$	8

Relevance of feasibility radius
in SDP solver and GloptiPoly

Conclusions

GloptiPoly is a general-purpose software with a user-friendly interface

Pedagogical flavor, black-box approach,
no expert tuning required to cope with very distinct applied maths and engineering pbs

Automatic detection of global optimality and extraction of solutions

Not a competitor to highly specialized codes for solving polynomial systems of equations or large combinatorial optimization pbs

Numerical conditioning (Chebyshev basis) and problem structure (Hankel/Toeplitz matrices) deserve further study

See also Parrilo's SOSTOOLS software

Further news

Major extension of GloptiPoly planned (hopefully) for winter 2003

- Matlab classes for multivariate polynomials and moment matrices
- general moment problems
- performance analysis for stochastic systems in ecology and finance
- robust control problems
- relaxations of robust LMIs

Research efforts

- bilinearity in decision variables
- tailored interior-point algorithms

Regularly updated information at

`www.laas.fr/~henrion`
`www.laas.fr/~lasserre`