Equivalence of Infinite Horizon Optimization Problems and Global Optimization Problems *

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Abstract

We show how to transform an infinite horizon optimization problem into a one-dimensional global optimization problem over a closed and bounded feasible region whose objective function is Hölder continuous with known parameters. The deep connection elicited between the two areas of study introduces several opportunities for cross-fertilization which we exploit within this paper.

Key Words and Phrases global optimization, infinite horizon optimization, Hölder continuous functions

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1 Introduction

Our objective is to show the equivalence of a class of infinite horizon optimization problems and a *solvable* global optimization problem, namely, the problem of finding an optimum of a Hölder function on a subset of real numbers.

Throughout, $\aleph$ denotes the set of natural numbers, and $\Re$ denotes the set of real numbers.

2 Mathematical Model

In this section, we define the class of infinite horizon problems and the global optimization problem, the pair of which we intend to show the equivalency.

Define the space of all strategies, Z, as an infinite product space of {0,1}, i.e.,

$$Z = \prod_{n=1}^{\infty} \{0, 1\}.$$

Therefore, for each strategy $s \in Z$, s is an infinite sequence of 0's and 1's, i.e.,

$$s \equiv (s_n) \qquad \text{where } s_n \in \{0, 1\}, \quad \forall n \in \aleph.$$

For future reference, $n \in \aleph$ is referred to as period n, and s_n is referred to as the decision for period n.

Equip Z with matric d, making a matric space (Z, d), where

$$d(s, t) = \sum_{n=1}^{\infty} \frac{|s_n - t_n|}{2^n}, \quad \forall s, t \in Z.$$

This metric induces the topology of componentwise convergence on Z, and Z is compact under this topology [1].

Let S be a nonempty closed subset of the metric space (Z, d), called the set of feasible strategies. Since Z is compact, S is also compact.

Denote the undiscounted cost function for period n associated with a strategy s by $c(s, n)$, i.e., $c : S \times \aleph \to \Re$. Denote the total discounted cost function associated with a strategy s by $\tilde{c}(s)$. Then $\tilde{c} : S \to \Re$, defined by

$$\tilde{c}(s) = \sum_{n=1}^{\infty} \frac{c(s, n)}{(1 + r)^n}, \tag{1}$$

where $r > 0$. Note that r can be interpreted as a discount rate.

We make two assumptions.

1. For each $s \in S$ and $n \in \aleph$, $c(s, n)$ depends only on s_1 to s_n and does not depend upon s_k where $k > n$. Moreover, for any $s, t \in S$, if $s_k = t_k$, for all $k = 1$ to n, then
$$c(s, n) = c(t, n).$$
2. The undiscounted cost function c is deterministic, and
$$|c(s, n)| \leq B(1 + \gamma)^n, \quad \forall s \in S, \forall n \in \aleph, \tag{2}$$
where $B > 0$, and $0 < \gamma < r$. Note that γ can be interpreted as a growth rate of the cost over time.

The class of infinite horizon optimization problems under consideration is the problem to find an optimal strategy s_* that

Program 1

$$\min \tilde{c}(s)$$
$$s.t. \quad s \in S.$$

Denote the set of all optimal solutions to Program 1 by S_*.

We intend to show that Program 1 is equivalent to a global optimization problem, which is the following Program 2.

Program 2

$$\min f(y)$$
$$s.t. \quad y \in [0, 1] \subseteq \Re.$$

where the objective function f is a Hölder function, i.e.,

$$|f(y) - f(z)| \leq M|y - z|^\alpha, \quad \forall y, z \in [0, 1], \tag{3}$$

for some positive real number M and real number $0 < \alpha \leq 1$. Many solution methods to Program 2 have been proposed in global optimization literature(see e.g. Horst and Tuy [4], Gourdin et. al. [3], and Shubert [7]). Therefore, Program 2 can be considered a solvable global optimization problem.

By equivalence of Program 1 and Program 2, we mean that one can find a transformation that transforms each problem of Program 1 to another problem of Program 2 in such a way that the set of all optimal solutions of the transformed problem can be mapped into the set of all optimal solutions of the original problem. Conversely, one can find another transformation that transforms each problem of Program 2 to that of Program 1 in the same manner. The condition that the set of all optimal solutions of the transformed problem can be mapped to that of the original problem is proposed so that, instead of solving the original problem, one can solve the transformed problem and transform the solution to that of the original problem. In this article, we show the equivalency of those two problems by showing that there exist such transformations.

3 Transformation from Global Optimization Problems to Infinite Horizon Optimization Problems

Starting with a global optimization problem, Program 2, transforming a global optimization problem into an infinite horizon optimization problem is relatively easy. Let the set of feasible strategies S be the entire strategy space Z. Observe that each strategy s in S is a sequence of one and zero, and hence it can be used to represent a real number in the interval [0,1] by the binary expansion. For example, the sequence $(0, 1, 1, 1, \ldots)$ can be used to represent 0.0111_2 which is one half. Formally, we can define a function $\tilde{\sigma} : S \to [0, 1]$ by

$$\tilde{\sigma}(s) = \sum_{n=1}^{\infty} \frac{s_n}{2^n}, \quad \forall s = (s_n) \in S. \tag{4}$$

Note that $\tilde{\sigma}$ is an onto function, but it is not one-to-one, since $(0, 1, 1, 1, \ldots)$ and $(1, 0, 0, 0, \ldots)$ map to the same point.

Define the total discounted cost $\tilde{c}_1 : S \to \Re$, which is the objective function of an infinite horizon optimization, by

$$\tilde{c}_1(s) = f(\tilde{\sigma}(s)), \quad \forall s \in S \tag{5}$$

where f is the objective function of Program 2.

We then define another infinite horizon optimization problem.

Program 3

$$\min \tilde{c}_1(s)$$

$$s.t. \quad s \in S.$$

We see that the set of all optimal solutions of Program 2 is the image of the function $\tilde{\sigma}$ over the set of all optimal solutions of Program 3. We would like to adopt Program 3 as our transformation of Program 2. However, one may concern that Program 3 might not be well defined because it might not be possible to construct a consistent undiscounted cost function, i.e., the undiscounted cost that satisfies all assumptions we made in previous sections and results in the total discounted cost $\tilde{c}_1$ via (1). We show that this apprehension has no grounds.

Define the partial sum $\sigma : S \times \aleph \to [0,1]$ by

$$\sigma(s,n) = \sum_{k=1}^{n} \frac{s_k}{2^k}, \quad \forall s \in S, \forall n \in \aleph.$$

We then define the undiscounted cost function $c_1 : S \times \aleph \to \Re$ by

$$c_1(s,n) = \begin{cases} f(\sigma(s,1))(1+r) & \text{if } n = 1 \\ (f(\sigma(s,n)) - f(\sigma(s,n-1)))(1+r)^n & \text{if } n > 1. \end{cases}$$

Defined in this way, $c_1(s,n)$ depends only on s_1 to s_n. c_1 defines $\tilde{c}_1$ in accordance with (1). Furthermore, for each s, if $n > 1$,

$$\begin{aligned} |c_1(s,n)| &= |f(\sigma(s,n)) - f(\sigma(s,n-1))|(1+r)^n \\ &\leq M \left(\frac{1}{2^n}\right)^{\alpha} (1+r)^n \\ &\quad \text{(by Hölder property of } f\text{)} \\ &= M \left(\frac{1+r}{2^{\alpha}}\right)^{n}. \end{aligned}$$

It implies that $c_1(s,n)$ satisfies (2). Hence, Program 3 does not violate any assumptions that we made, and hence can be adopted as a transformation of the global optimization problem, Program 2.

4 Transformation from Infinite Horizon Optimization Problems to Global Optimization Problems

Starting with Program 1, to transform an infinite horizon optimization problem into a global optimization problem, we first map all strategies into real numbers by the function x. The function x maps Z into the interval $[0,1]$, i.e., $x : Z \to [0,1]$ defined by

$$x(s) = \sum_{n=1}^{\infty} \frac{s_n}{3^n}, \quad \forall s = (s_n) \in Z. \tag{6}$$

Note that x is a one-to-one mapping. Denote the subset of $\Re$ that is the image of x over Z by X. Defined in this way, X is the set of all real numbers in $[0, 1)$ that, when represented in the base-3 expansion, all digits are 0 or 1. (Note that when we refer to the base-3 expansion of a real number, we always refer to the base-3 expansion of that real number that does not end in a string of 2's. For example, 1/3's base-3 expansions are both $0.1000\ldots_3$ and $0.0222\ldots_3$. However, when we refer to the base-3 expansion of 1/3, we are referring to $0.1000\ldots_3$. This convention assures us that x *is one-to-one*. Had we adopted a binary representation, we would lose this one-to-one property.)

Definition 1 *A set C is a Cantor set if C is a subset of $\Re$ that has all the following properties:*

1. *C is nonempty,*
2. *C is closed and bounded,*
3. *All points in C are accumulation points of C,*
4. *C is nowhere dense.*

From mathematical analysis, X is a Cantor set with Lebesgue measure zero. This also implies the compactness of X. Denote the image of x over the set of feasible strategies S by Y, called *the set of feasible solutions* or simply *the feasible region.*

$x : Z \to X$ is a continuous mapping. To see this, observe that, for s and t in Z,

$$|x(s) - x(t)| = \left| \sum_{n=1}^{\infty} \frac{s_n - t_n}{3^n} \right| \leq \sum_{n=1}^{\infty} \frac{|s_n - t_n|}{3^n} \leq \sum_{n=1}^{\infty} \frac{|s_n - t_n|}{2^n} = d(s, t),$$

where the first inequality follows from the triangular inequality. Therefore, if a sequence (s_n) converges to s in Z, then $(x(s_n))$ converges to $x(s)$ in X. This implies the continuity of x. Hence, Y is compact, since S is compact and the image of a compact set under a continuous map is compact. Y also has Lebesgue measure zero, since X has Lebesgue measure zero and $Y \subset X$.

Since the function x defined in (6) is a one-to-one mapping, the inverse function x^{-1} exists. Define an objective function $f : Y \to \Re$ by

$$f(y) = \tilde{c}(x^{-1}(y)), \quad \forall y \in Y. \tag{7}$$

where $\tilde{c}$ is the total discounted cost of Program 1. By so doing, Program 1 is transformed to

Program 4

$$\min f(y)$$

$$s.t. \quad y \in Y \text{ compact } \subseteq \Re.$$

Denote the set of all optimal solutions to Program 4 by Y_* and their corresponding optimal value by f_*. Hence, $S_* = x^{-1}(Y_*)$.

The following Theorem 1 reveals the property of the objective function f over Y. Lemma 1 is needed to prove Theorem 1.

Lemma 1 $\forall y, z \in X, |y - z| \leq 1/3^k$, *where* $k \in \aleph$ *and* $k > 1$, *implies that the decisions from* 1*st period to* $k - 1$*st period of* $x^{-1}(y)$ *and* $x^{-1}(z)$ *are the same.*

Proof. Let $k > 1$ and let $s = x^{-1}(y)$ and $t = x^{-1}(z)$ where $y, z \in X$. If there exist some decisions from 1st period to $k - 1$st period of s and t that are different, there exists $1 \leq h \leq k - 1$ such that $s_h \neq t_h$, and if $h > 1$, $s_n = t_n$ for $n = 1$ to $h - 1$. In other words, h is the first period that the decision starts being different. Then

$$|y - z| = |x(s) - x(t)| = |\sum_{n=1}^{\infty} \frac{s_n - t_n}{3^n}| \geq |\sum_{n=1}^{h} \frac{s_n - t_n}{3^n}| - |\sum_{n=h+1}^{\infty} \frac{s_n - t_n}{3^n}|.$$

But, $s_n = t_n$ for $n = 1$ to $h-1$, and $s_h \neq t_h$. Furthermore, $|\sum_{n=h+1}^{\infty} \frac{s_n - t_n}{3^n}| \leq \frac{1}{(2)(3^h)}$.

$$|y - z| \geq \frac{1}{3^h} - \frac{1}{(2)(3^h)} = \frac{1}{(2)(3^h)} \geq \frac{1}{(2)(3^{k-1})} > \frac{1}{3^k}.$$

This is a contradiction. ■

Theorem 1 *The objective function* f *is a Hölder function on* Y, *i.e.,*

$$|f(y) - f(z)| \leq M|y - z|^{\alpha}, \quad \forall y, z \in Y,$$

where $0 < \alpha \leq 1$, *and* M *is a positive constant. In particular, we may set* $M = 6B/(1 - \beta)$ *and* $\alpha = \min\{\log_3(1/\beta), 1\}$ *where* $\beta = (1 + \gamma)/(1 + r)$.

Proof. Note that we consider only when $\gamma < r$ so that the objective function can be evaluated. Hence, $0 < \beta = (1 + \gamma)/(1 + r) < 1$.

Fix $y, z \in Y$. There exists $k \in \{0, 1, 2, \ldots\}$ such that

$$1/3^{k+1} < |y - z| \leq 1/3^k.$$

Let $s = x^{-1}(y)$ and $t = x^{-1}(z)$. If $k \geq 2$, by Lemma 1, $s_n = t_n$ for $n = 1$ to $k-1$.

$$
\begin{aligned}
|f(y) - f(z)| &= |\tilde{c}(s) - \tilde{c}(t)| \\
&= |\textstyle\sum_{n=k}^{\infty} c(s,n)/(1+r)^n - \sum_{n=k}^{\infty} c(t,n)/(1+r)^n| \\
&\quad \text{(by Assumption 2)} \\
&\leq |\textstyle\sum_{n=k}^{\infty} c(s,n)/(1+r)^n| + |\sum_{n=k}^{\infty} c(t,n)/(1+r)^n| \\
&\leq \textstyle\sum_{n=k}^{\infty} |c(s,n)/(1+r)^n| + \sum_{n=k}^{\infty} |c(t,n)/(1+r)^n| \qquad 7 \\
&\quad \text{(by the triangular inequality)} \\
&\leq 2\textstyle\sum_{n=k}^{\infty} B[(1+\gamma)/(1+r)]^n \\
&\quad \text{(by Inequality 2)} \\
&= 2B\textstyle\sum_{n=k}^{\infty} \beta^n \\
&= 2B\beta^k/(1-\beta).
\end{aligned}
$$

But, $|y-z| > 1/3^{k+1}$. Then $|y-z|^{\alpha} > (1/3^{\alpha})^{k+1}$ for all $\alpha > 0$. Therefore, $(3^{\alpha})^{k+1}|y-z|^{\alpha} > 1$, and thus

$$|f(y) - f(z)| \leq 2B(\frac{\beta^k}{1-\beta})(3^{\alpha})^{k+1}|y-z|^{\alpha} = \frac{(2)(3^{\alpha})B}{1-\beta}(3^{\alpha}\beta)^k|y-z|^{\alpha}.$$

Let α be such that

$$0 < \alpha \leq \min\{\log_3(1/\beta), 1\} \leq 1.$$

Since $\alpha \leq \log_3(1/\beta)$, we have $3^{\alpha}\beta \leq 1$. Since $\alpha \leq 1$, we have $(2)(3^{\alpha}) \leq 6$. Hence,

$$|f(y) - f(z)| \leq \frac{6B}{1-\beta}|y-z|^{\alpha}.$$

On the other hand, if $k = 0$ or 1, s_n and t_n may be different for all $n \in \aleph$. By (2),

$$|f(y) - f(z)| \leq 2B(\frac{\beta}{1-\beta}) \leq 2B(\frac{\beta}{1-\beta})(3^{\alpha})^2|y-z|^{\alpha} \leq \frac{6B}{1-\beta}|y-z|^{\alpha},$$

for $0 < \alpha \leq \min\{\log_3(1/\beta), 1\}$.

Hence, f is a Hölder function with $0 < \alpha \leq \log_3(1/\beta) \leq 1$, i.e.,

$$|f(y) - f(z)| \leq M|y-z|^{\alpha} \quad \text{where } M = \frac{6B}{1-\beta}.$$

■

Corollary 1 Y_* *is not empty.*

Proof. The objective function f is continuous over the compact set Y. The corollary follows readily from Weierstrass Theorem. ■

Up to this point, we have transformed the original infinite horizon optimization problem in an infinite-dimensional space into another global optimization problem of a Hölder objective function in one-dimensional real space. However, our task has not been completed because the feasible region Y is a compact subset of a Cantor set, not the interval [0,1] as we aim at.

To complete the task, we extend the objective function f to the whole interval $[0, 1]$ in such a way that the extended function preserves the same Hölder condition of f, i.e., f_1 satisfies (3) with the same α and M as those of f. In addition, the extended function preserves all optimum of f over the extended set. The interval [0,1] can then be regarded as *the extended feasible region*. One possible extension is the standard McShane's Lipschitz extension [6].

Define the extended objective function $f_1 : [0, 1] \to \Re$ by

$$f_1(y) = \inf_{z \in Y} \{f(z) + M|y - z|^\alpha\}, \quad \forall y \in [0, 1], \tag{8}$$

We show that f_1 is a Hölder extension of f. Basically, a sufficient condition for f_1 to be an extension of f that preserves the same Hölder condition is that $\rho(y, z) = |y - z|^\alpha$, for all y, z in $\Re$, defines a metric on $\Re$, and in fact it is.

Lemma 2 *For all $a, b \in \Re$ and $0 < \alpha \leq 1$*

$$|a + b|^\alpha \leq |a|^\alpha + |b|^\alpha.$$

Proof. This proof can be found in Bruckner, Bruckner and Thomson [2]. If $\alpha = 1$, this is simply the triangular inequality of the absolute value function. Fix $0 < \alpha < 1$. Let $h : [0, \infty) \to \Re$ defined by

$$h(t) = (1 + t)^\alpha - 1 - t^\alpha.$$

Observe that $h(0) = 0$, and $h'(t) < 0$ when $t > 0$ Therefore, for all $t \geq 0$,

$$(1 + t)^\alpha - 1 - t^\alpha \leq 0.$$

Let $a, b \in \Re$ such that $b \neq 0$. Substituting t in the above inequality by $|a|/|b|$, we have

$$\left(1 + \frac{|a|}{|b|}\right)^\alpha - \left(\frac{|b|}{|b|}\right)^\alpha - \left(\frac{|a|}{|b|}\right)^\alpha \leq 0.$$

Hence,

$$|a+b|^\alpha \leq (|a|+|b|)^\alpha \leq |a|^\alpha + |b|^\alpha.$$

If $b = 0$, the lemma trivially follows. ∎

Proposition 1 *f_1 is a Hölder extension of f over $[0,1]$ preserving the same Hölder condition.*

Proof. We follow McShane [6]. By the definition of f_1, $f_1|_Y = f$. $f_1(y)$ is finite for every y in $\Re$ since, for a fixed z in Y,

$$-\infty < \inf_{x \in Y} f_1(x) \leq f(y) \leq f(z) + M|y-z|^\alpha < \infty,$$

and hence well defined. Let $x, y \in \Re$. For $\epsilon > 0$, the definition of f_1 implies that there exists z in Y such that

$$f_1(x) + \epsilon \geq f(z) + M|x-z|^\alpha,$$

and that

$$f_1(y) \leq f(z) + M|y-z|^\alpha.$$

By Lemma 2,

$$f_1(x) + \epsilon \geq f_1(y) - M|y-z|^\alpha + M|x-z|^\alpha \geq f_1(y) - M|y-x|^\alpha.$$

This is true for all $\epsilon > 0$. Therefore,

$$f_1(x) \geq f_1(y) - M|y-x|^\alpha,$$

or

$$f_1(y) - f_1(x) \leq M|y-x|^\alpha.$$

By switching the roles of x and y, we obtain

$$f_1(x) - f_1(y) \leq M|x-y|^\alpha.$$

Therefore,

$$|f_1(x) - f_1(y)| \leq M|x-y|^\alpha,$$

and hence proved. ∎

Define another mathematical program.

Program 5

$$\min f_1(y)$$

$$s.t. \quad y \in [0,1].$$

Program 5 is the transformed problem that we want. One thing left to show is that Program 5 preserves the set of optimal solutions of Program 4.

Denote the set of all optimal solutions of Program 5 by Y_{1*} and their corresponding optimal value by f_{1*}. We claim that $Y_{1*} = Y_*$ and $f_{1*} = f_*$. To see this, let $y \in [0,1] - Y$. Since Y is closed, the distance of y from Y, i.e., $\inf_{z \in Y} |y - z|$ and hence $\inf_{z \in Y} |y - z|^\alpha$ is strictly positive. Therefore,

$$\begin{aligned} f(y) &= \inf_{z \in Y} \{f(z) + M|y - z|^\alpha\} \\ &\geq \inf_{z \in Y} f(z) + M \inf_{z \in Y} |y - z|^\alpha \\ &> \inf_{z \in Y} f(z) \\ &= f_*. \end{aligned}$$

Note that the last equality is a result of compactness of Y. This proves our claim.

We have transformed the infinite horizon optimization Program 1 to another global optimization Program 5 of Hölder function on [0,1], where the set of all optimal solutions of the original problem is the image of the inverse function x^{-1} over the set of all optimal solutions of the transformed problem.

As to the extension scheme, McShane's extension f_1 is not the only possible extension of f. Linear interpolation is another possible extension that preserves the Hölder condition of f. (See Kiatsupaibul [5].) The set of all optimal solutions of the linear interpolation extension of f is not always equal to the set of all optimal solutions of f, but it always contains that of f. Another feature of the linear interpolation extension is that the extension is differentiable almost everywhere since Y has measure zero. Therefore, it seems that the linear interpolation extension might be more regular than McShane's extension with respect to a specific f. However, with any extension, a transformed extended objective function can be as irregular as that in Example 1.

Example 1 *Consider a stationary cost equipment replacement problem with a one-year-old equipment at the beginning of period one, assuming no maximum physical life. Model this problem by the binary sequences of buy-keep decisions. Assume that the optimal solution of Program 5, denoted by y_*, is unique. Assume further that $y_* \in (0, 1/6) = (0.000..._3, 0.011..._3)$. Then, by stationarity of the cost structure, $y_*(n) = \sum_{i=1}^{n} (1/3)^i + y_*/3^n$ is the unique* *optimal solution of f_1 over the open interval*

$$(\sum_{i=1}^{n}(1/3)^i, \sum_{i=1}^{n} 1/3^i + \sum_{i=n+2}^{\infty} 1/3^i)$$

for $n = 1, 2, \ldots$. Therefore, the function f_1 of this problem has at least a countably infinite number of strict local optimal solutions.

5 Discussion

As we have seen, the Hölder property of the transformation from the infinite horizon optimization to the global optimization relies on the mapping from the strategy space into the real number system. With respect to a particular problem, different mapping yields different properties of the transformed problems. Let us entertain ourselves further on this line of base-3 expansion mapping.

Define f on $X \subseteq \Re^m$ as a Hölder function if f satisfying the condition

$$|f(\underline{y}) - f(\underline{z})| \leq M\|\underline{y} - \underline{z}\|^{\alpha}, \quad \forall \underline{y}, \underline{z} \in X \tag{9}$$

where $\|\cdot\|$ denotes the Euclidean norm, $0 < \alpha \leq 1$, and M is a positive constant. The underlined letter denotes an element in $\Re^m$, i.e.,

$$\underline{y} = (y^1, \ldots, y^m) \quad \text{where } y^i \in \Re, \forall i = 1, \ldots, m.$$

A function f is said to be a Lipschitz function if it is a Hölder function with α being equal to 1, i.e., f satisfies the condition

$$|f(\underline{y}) - f(\underline{z})| \leq M\|\underline{y} - \underline{z}\|, \quad \forall \underline{y}, \underline{z} \in X. \tag{10}$$

A Lipschitz function enjoys a lot of nice properties in analysis. In the preceding sections, when the set of all strategies is mapped into $\Re$, the objective function was shown to be a Hölder function in $\Re$. Whether or not the objective function is also a Lipschitz function depends on the discount factor β. In this section, we show that, if we map the set of all strategies into $\Re^m$, for some $m > 1$, the Lipschitz property of the objective function can be guaranteed. The formal treatment of the statement follows.

Let Z be the space of all strategies. Define the mapping $\underline{x} : Z \to \Re^m$ as

$$\underline{x}(s) = (x^1(s), \ldots, x^m(s))$$

where

$$x^i(s) = \sum_{k=0}^{\infty} \frac{s_{mk+i}}{3^{k+1}}, \quad \forall i = 1, \ldots, m, \forall s = (s_n) \in Z. \tag{11}$$

As an example, consider the mapping when $m = 2$. Let $s = (1, 0, 1, 0, ...) = (s_n) \in S$ where $s_n = 1$ if n is odd, and $s_n = 0$ if n is even. Then $\underline{x}$ maps s to $\underline{x}(s) = (0.111\ldots_3, 0.000\ldots_3) = (1/2, 0) \in \Re^2$. Note that the mapping in the preceding section is a special case of the mapping $\underline{x}$ when $m = 1$.

Similar to the mapping x, $\underline{x}$ is also a one-to-one mapping. Let X^m denote the image of $\underline{x}$ over Z. Then X^m is the set of all points in $[0, 1)^m$ such that all digits in each of m coordinates, when represented by the base-3 expansion, are 0 or 1. One can show that $\underline{x}$ is also a continuous mapping. Hence, X^m is compact. Let Y^m denote the image of $\underline{x}$ over the closed subset S of Z. Y^m is also compact. The following lemma is the counterpart of Lemma 1 for the mapping $\underline{x}$.

Lemma 3 *For a fix dimension m, $\forall \underline{y}, \underline{z} \in X^m$, $\|\underline{y} - \underline{z}\| \leq 1/3^l$, where $l \in \aleph$ and $l > 1$, implies that the decisions from 1^{st} period to $m(l-1)^{st}$ period of $\underline{x}^{-1}(\underline{y})$ and $\underline{x}^{-1}(\underline{z})$ are the same.*

Proof. Let $l > 1$ and let $s = \underline{x}^{-1}(\underline{y})$ and $t = \underline{x}^{-1}(\underline{z})$ where $\underline{y}, \underline{z} \in X^m$. If there exist some decisions from 1^{st} period to $m(l-1)^{\text{st}}$ period of s and t that are different, there exist $1 \leq h \leq m(l-1)$ such that $s_h \neq t_h$, and if $h > 1$, $s_n = t_n$ for $n = 1$ to $h - 1$. In other words, h is the first period that the decision starts being different. Since $1 \leq h \leq m(l-1)$, there exists $0 \leq j \leq l-2$ and $1 \leq i \leq m$ such that $h = mj + i$. Then

$$\begin{aligned}
\|\underline{y} - \underline{z}\| &= \|\underline{x}(s) - \underline{x}(t)\| \\
&\geq |x^i(s) - x^i(t)| \\
&= |\textstyle\sum_{k=0}^{\infty} \frac{s_{mk+i} - t_{mk+i}}{3^{k+1}}| \\
&\geq |\textstyle\sum_{k=0}^{j} \frac{s_{mk+i} - t_{mk+i}}{3^{k+1}}| - |\textstyle\sum_{k=j+1}^{\infty} \frac{s_{mk+i} - t_{mk+i}}{3^{k+1}}|.
\end{aligned}$$

But, $s_n = t_n$ for $n = 1$ to $h - 1 = mj + i - 1$, and $s_{mj+i} = s_h \neq t_h = t_{mj+i}$. Furthermore, $|\sum_{k=j+1}^{\infty} \frac{s_{mk+i} - t_{mk+i}}{3^{k+1}}| \leq \frac{1}{(2)(3^{j+1})}$

$$\|\underline{y} - \underline{z}\| \geq \frac{1}{3^{j+1}} - \frac{1}{(2)(3^{j+1})} = \frac{1}{(2)(3^{j+1})} \geq \frac{1}{(2)(3^{l-1})} > \frac{1}{3^l}.$$

This is a contradiction. ■

Define an objective function $f : Y^m \to \mathbf{R}$ by

$$f(\underline{y}) = \tilde{c}(\underline{x}^{-1}(\underline{y})), \quad \forall \underline{y} \in Y^m \tag{12}$$

Theorem 2 *There exists $m \in \aleph$ such that the objective function f is a Lipschitz function on Y^m, i.e.,*

$$|f(\underline{y}) - f(\underline{z})| \leq M\|\underline{y} - \underline{z}\|, \quad \forall \underline{y}, \underline{z} \in Y^m,$$

where M is a positive constant.

Proof. Fix $\underline{y}, \underline{z} \in Y^m$. There exists $l \in \{0, 1, 2, \ldots\}$ such that

$$1/3^{l+1} < \|\underline{y} - \underline{z}\| \leq 1/3^l.$$

Let $s = \underline{x}^{-1}(y)$ and $t = \underline{x}^{-1}(z)$. If $l \geq 2$, by Lemma 3, $s_n = t_n$ for $n = 1$ to $m(l-1)$.

$$\begin{aligned}
|f(\underline{y}) - f(\underline{z})| &= |\tilde{c}(s) - \tilde{c}(t)| \\
&= |\textstyle\sum_{n=m(l-1)+1}^{\infty} c(s,n)/(1+r)^n - \sum_{n=m(l-1)+1}^{\infty} c(t,n)/(1+r)^n| \\
& \text{(by Assumption 2)} \\
&\leq |\textstyle\sum_{n=m(l-1)+1}^{\infty} c(s,n)/(1+r)^n| + |\sum_{n=m(l-1)+1}^{\infty} c(t,n)/(1+r)^n| \\
&\leq \textstyle\sum_{n=m(l-1)+1}^{\infty} |c(s,n)/(1+r)^n| + \sum_{n=m(l-1)+1}^{\infty} |c(t,n)/(1+r)^n| \\
& \text{(by the triangular inequality)} \\
&\leq 2\textstyle\sum_{n=m(l-1)+1}^{\infty} B[(1+\gamma)/(1+r)]^n \\
& \text{(by Inequality 2)} \\
&= 2B\textstyle\sum_{n=m(l-1)+1}^{\infty} \beta^n \\
&= 2B\beta^{m(l-1)+1}/(1-\beta).
\end{aligned}$$

But, $\|\underline{y} - \underline{z}\| > 1/3^{l+1}$. Therefore, $3^{l+1}\|\underline{y} - \underline{z}\| > 1$, and thus

$$|f(\underline{y}) - f(\underline{z})| \leq 2B(\frac{\beta^{m(l-1)+1}}{1-\beta})3^{l+1}\|\underline{y} - \underline{z}\| = \frac{6B\beta^{1-m}}{1-\beta}(3\beta^m)^l\|\underline{y} - \underline{z}\|.$$

Let m be an integer greater than or equal to $\log_\beta(1/3)$. Then $3\beta^m \leq 1$, and, hence,

$$|f(\underline{y}) - f(\underline{z})| \leq \frac{6B\beta^{1-m}}{1-\beta}\|\underline{y} - \underline{z}\|.$$

On the other hand, if $l = 0$ or 1, s_n and t_n may be different for all $n \in \aleph$. By Assumption 2,

$$|f(\underline{y}) - f(\underline{z})| \leq 2B(\frac{\beta}{1-\beta}) \leq 2B(\frac{\beta}{1-\beta})3^2\|\underline{y} - \underline{z}\| \leq \frac{6B\beta^{1-m}}{1-\beta}(3\beta^m)\|\underline{y} - \underline{z}\|.$$

Let m be an integer greater than of equal to $\log_\beta(1/3)$. Then $3\beta^m \leq 1$, and

$$|f(\underline{y}) - f(\underline{z})| \leq \frac{6B\beta^{1-m}}{1-\beta}\|\underline{y} - \underline{z}\|.$$

Hence, if $m \geq \log_\beta(1/3)$, f is a Lipschitz function with Lipschitz constant $M = (6B\beta^{1-m})/(1-\beta)$. ■

With some fixed dimension m, the transformed objective function f can be forced to be a Lipschitz function. The objective function f can also be extended to a Lipschitz function f_1 over the hyperrectangle $[0,1]^m$ by McShane's extension. With the mapping $\underline{x}$ and the extended objective function f_1, the original Program 1 can be transformed to

Program 6

$$\min f_1(\underline{y})$$

$$s.t. \quad \underline{y} \in [0,1]^m \subseteq \Re^m.$$

In this article, we have shown the equivalency of infinite horizon optimization problems and global optimization problems and some of its properties, without mentioning about the solution methods. In fact, numerical procedures based on branch-and-bound methods can be developed to solve Program 4 and Program 5. (See Kiatsupaibul [5].)

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