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A CONVEXIFICATION METHOD FOR A CLASS OF GLOBAL OPTIMIZATION PROBLEMS WITH APPLICATIONS TO RELIABILITY OPTIMIZATION*

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Abstract

A convexification method is proposed for solving a class of global optimization problems with certain monotone properties. It is shown that this class of problems can be transformed into equivalent concave minimization problems using the proposed convexification schemes. An outer approximation method can then be used to find the global solution of the transformed problem. Applications to mixed-integer nonlinear programming problems arising in reliability optimization of complex systems are discussed and satisfactory numerical results are presented.

Key Words: Global optimization, monotone optimization, concave minimization, reliability optimization.

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1 Introduction

We consider global optimization problems of the form:

$$\begin{aligned} &\max \ f(x) \\ &\text{s.t. } g_j(x) \le c_j, \ j = 1,2,\ldots,m, \\ &\quad x \in X, \end{aligned} \tag{1.1}$$

where $f:\ \mathbb{R}^n \to \mathbb{R}$ and $g_j:\ \mathbb{R}^n \to \mathbb{R}$, $j = 1,\ldots,m$, are continuous functions satisfying the following monotone properties: (a) $f(x)$ and $g_j(x),\ j = 1,\ldots,m$, are increasing functions of each x_i, or (b) $f(x)$ and $g_j(x),\ j = 1,\ldots,m$, are decreasing functions of each x_i. We assume that X is a nonempty closed set.

The problem (1.1) can be viewed as the continuous version of the multidimensional nonlinear knapsack problem. Due to the monotonicity of f and the g_j's, the optimal solution of (1.1) always lies on the boundary of the feasible region. The problem (1.1), however, may have multiple local optimal solutions since $f(x)$ is not necessarily concave and g_j are not necessarily convex. Therefore, problem (1.1) is essentially a global optimization problem.

The literature on the solution methods for global optimization can be classified into two categories. The methods in the first category are devised to cope with general global optimization problems with no special structural property assumed. This category includes deterministic methods (see, e.g., [1][4][6][7][8] and the references therein) and stochastic methods (see. e.g., [3][14][15] and the references therein). The second category of methods confines its applicability to certain structured global optimization problems, in particular, concave minimization, D.C. programming and reverse convex programming ([2][5][6][13]). For concave minimization, the global minimum over a compact convex set is always achieved at an extreme point of the convex set. This prominent feature leads to various implementable algorithms that guarantee a convergence to a global optimal solution of the problem.

The main purpose of this paper is to present a novel convexification transformation method to convert problem (1.1) into a concave minimization problem. The monotone property of (1.1) and the resulting convex feasible region allow us to adopt an outer approximation procedure for solving the resulting concave minimization problem. This convexification method is applicable to a large class of real-world optimization problems arising in reliability network systems where monotonicity is an inherent property. Convexification solution schemes have been recently adopted successfully in some other subjects of optimization, such as in convexifying the perturbation function and Lagrangian function in the dual search methods for nonlinear programming ([9][11]) and in convexifying the noninferior frontier in multiobjective optimization ([10]). In section 2 we establish a general theorem on the relationship between monotonicity and convexity of a real function. Two specific forms of convexification transformation are then proposed. In section 3 the convexification transformation is applied to the functions in problem (1.1). The resulting equivalent problem is a concave minimization problem to which outer approximation method can be used to search for a global optimal solution. In section 4 we show how the proposed convexification methods can be adopted to tackle mixed-integer nonlinear programming problems in reliability network system by combining the outer approximation method with a branch-and-bound strategy.

2 Convexification transformations of monotone functions

A function $h: X \to \mathbb{R}$ is called strictly increasing (decreasing) on X if $h(x)$ is a strictly increasing (decreasing) function of each x_i. Consider the following transformation of function $h(x)$:

$$h_p(y) = T\left(ph\left(\frac{1}{p}t(y)\right)\right), \tag{2.1}$$

where $p > 0$ is a parameter, $T : \mathbb{R}^1 \to \mathbb{R}^1$ is a real function and $t : \mathbb{R}^n \to \mathbb{R}^n$ is a separable 1-1 mapping, i.e., $t(y) = (t_1(y_1), \ldots, t_n(y_n))$ for $y = (y_1, \ldots, y_n)$. The domain of $h_p(y)$ is

$$Y_p = \{y \mid y = t^{-1}(px),\ x \in X\}. \tag{2.2}$$

Let Ω be an open set satisfying $Y_p \subseteq \Omega$ for all $p > 0$.

THEOREM 2.1 *Assume that*

(i) $h \in \mathbb{C}^2(X)$, $t_i \in \mathbb{C}^2(\Omega)$, $i = 1, \ldots, n$, *and* $T \in \mathbb{C}^2(\mathbb{R})$.

(ii) *h is a strictly increasing function on X satisfying*

$$\inf\{\frac{\partial h}{\partial x_i} \mid x \in X,\ i = 1, \ldots, n\} \geq \varepsilon > 0, \tag{2.3}$$

$$\sigma = \inf\{d^T \nabla^2 h(x) d \mid x \in X,\ \|d\|_2 = 1\} > -\infty. \tag{2.4}$$

(iii) *T is a strictly increasing and convex function.*

(iv) *t_i, $i = 1, \ldots, n$, are strictly monotone functions satisfying*

$$\frac{t_i''(y_i)}{t_i'(y_i)^2} \geq \tau > 0, \quad y \in \Omega,\ i = 1, \ldots, n. \tag{2.5}$$

Then there exists a finite $p_0 > 0$ such that $h_p(y)$ is a convex function on any convex subset of Y_p when $p \geq p_0$.

PROOF. It suffices to prove that the Hessian of $h_p(y)$ is a positive semi-definite matrix for any $y \in Y_p$. Let $x = \frac{1}{p}t(y)$ and $w = ph(x)$ for $y \in Y_p$. Then $x \in X$ by the definition of Y_p. From (2.1), we have

$$\frac{\partial h_p}{\partial y_i} = T'(w)\frac{\partial h}{\partial x_i}t_i'(y_i), \quad i = 1, \ldots, n.$$

Thus

$$\frac{\partial^2 h_p}{\partial y_i^2} = T''(w)\left(\frac{\partial h}{\partial x_i}\right)^2 t_i'(y_i)^2 + \frac{1}{p}T'(w)\frac{\partial^2 h}{\partial x_i^2}t_i'(y_i)^2 + T'(w)\frac{\partial h}{\partial x_i}t_i''(y_i), \tag{2.6}$$

$$\frac{\partial^2 h_p}{\partial y_i \partial y_j} = T''(w)\left(\frac{\partial h}{\partial x_i}\right)\left(\frac{\partial h}{\partial x_j}\right)t_i'(y_i)t_j'(y_j) + \frac{1}{p}T'(w)\frac{\partial^2 h}{\partial x_i \partial x_j}t_i'(y_i)t_j'(y_j), \ i \neq j. \tag{2.7}$$

Let

$$A(y) = diag\left(t_1'(y_1), \cdots, t_n'(y_n)\right),$$
$$B(y) = diag\left(\frac{\partial h}{\partial x_1}\frac{t_1''(y_1)}{t'(y_1)^2}, \cdots, \frac{\partial h}{\partial x_n}\frac{t_n''(y_n)}{t_n'(y_n)^2}\right).$$

Combining (2.6) with (2.7) gives

$$\begin{aligned}\nabla^2 h_p(y) &= A(y)(T''(w)\nabla h(x)\nabla h(x)^T + \frac{1}{p}T'(w)\nabla^2 h(x) + T'(w)B(y))A(y) \\ &= A(y)C(x,y,w)A(y),\end{aligned} \tag{2.8}$$

where

$$C(x,y,w) = T''(w)\nabla h(x)\nabla h(x)^T + \frac{1}{p}T'(w)\nabla^2 h(x) + T'(w)B(y).$$

It follows from (2.8) that, if $C(x,y,w)$ is a positive semi-definite matrix, then $\nabla^2 h_p(y)$ is a positive semi-definite matrix.

For any $d \in \mathbb{R}^n$ with $\|d\|_2 = 1$, from (2.3), (2.4) and (2.5), we have

$$\begin{aligned} d^T C(x,y,w)d &= T''(w)(d^T\nabla h(x))^2 + \frac{1}{p}T'(w)d^T\nabla^2 h(x)d + T'(w)d^T B(y)d \\ &\geq \frac{1}{p}T'(w)d^T\nabla^2 h(x)d + T'(w)d^T B(y)d \\ &= T'(w)\left(\frac{1}{p}d^T\nabla^2 h(x)d + \sum_{i=1}^n d_i^2\frac{\partial h}{\partial x_i}\frac{t_i''(y_i)}{t_i'(y_i)^2}\right) \\ &\geq T'(w)\left(\frac{1}{p}\sigma + \varepsilon\tau\right), \end{aligned} \tag{2.9}$$

where the first inequality follows from the fact that T is a convex function and so $T''(w) \geq 0$. Let $p \geq p_0$ with

$$p_0 = \begin{cases} -\frac{\sigma}{\varepsilon\tau}, & \sigma < 0 \\ \text{any positive number}, & \sigma \geq 0 \end{cases} \tag{2.10}$$

Notice that T is strictly increasing and so $T'(w) > 0$. Then, from (2.9), we deduce that $d^T C(x,y,w) d \geq 0$ for any $d \in \mathbb{R}^n$. Therefore, $\nabla^2 h_p(y)$ is a positive semi-definite matrix for all $y \in Y_p$ when $p \geq p_0$. **Q.E.D.**

REMARK 2.1 A sufficient condition for (2.3) and (2.4) is that X is a compact set.

REMARK 2.2 Suppose that $h(x)$ is a strictly decreasing function on X. Then $h(-x)$ is a strictly increasing function on $-X$. Note that function $h_p(y)$ in (2.1) can be rewritten as

$$h_p(y) = T\left(ph\left(-\frac{1}{p}(-t)(y)\right)\right).$$

So, Theorem 2.1 is applicable to the strictly decreasing function $h(x)$ via replacing $h(x)$ in (2.3) and t in (2.5) with $h(-x)$ and $-t$, respectively. This can be generalized further to functions which are strictly increasing with respect to some components and strictly decreasing with respect to others by doing the above sign transformation only to the latter.

In the following, we give two examples of functional forms for t_i which satisfy condition (iv) in Theorem 2.1. Assume α is a parameter, $\alpha \neq 0$. Define

$$q_1(s) = \ln\left(1 + \frac{\alpha}{s}\right), \quad \alpha s > 0, \tag{2.11}$$

$$q_2(s) = -\ln(1 + \alpha s), \quad \alpha s > 0. \tag{2.12}$$

We have

$$q_1'(s) = -\frac{\alpha}{s(s+\alpha)},$$

$$q_1''(s) = \frac{\alpha(2s+\alpha)}{s^2(s+\alpha)^2} \geq \frac{\alpha^2}{s^2(s+\alpha)^2} = q_1'(s)^2, \tag{2.13}$$

$$q_2'(s) = -\frac{\alpha}{1+\alpha s},$$

$$q_2''(s) = \frac{\alpha^2}{(1+\alpha s)^2} = q_2'(s)^2. \tag{2.14}$$

COROLLARY 2.1 *Let h be a twice differentiable and strictly increasing function on*

$$X = \{x \mid l_i \le x_i \le u_i,\ i = 1, \dots, n\} \tag{2.15}$$

with $0 < l_i < u_i < \infty$, $i = 1, \dots, n$. *Assume that T is a twice differentiable, strictly increasing and convex function. Take* $t_i = q_1$ *in (2.11) or* $t_i = q_2$ *in (2.12),* $i = 1, \dots, n$. *Then there exists* $p_0 > 0$ *such that* $h_p(y)$ *is a convex function on* Y_p *when* $p \ge p_0$.

PROOF. Since X is a compact set, there exist $\varepsilon > 0$ and finite σ such that (2.3) and (2.4) hold. We only need to verify that condition (iv) in Theorem 2.1 is satisfied. If $t_i = q_1$ or $t_i = q_2$ in (2.1), $i = 1, \dots, n$, then the domain of $h_p(y)$ is

$$Y_p = \{y \mid t_i^{-1}(pl_i) \le y_i \le t_i^{-1}(pu_i),\ i = 1, \dots, n\}, \quad \alpha < 0,$$

or

$$Y_p = \{y \mid t_i^{-1}(pu_i) \le y_i \le t_i^{-1}(pl_i),\ i = 1, \dots, n\}, \quad \alpha > 0.$$

Obviously, t_i is a strictly monotone function on Y_p. Notice that $y < 0$ for any $y \in Y_p$ when $\alpha < 0$ and $y > 0$ for any $y \in Y_p$ when $\alpha > 0$. Equation (2.13) or (2.14) implies that (2.5) holds with $\tau = 1$ and $\Omega = \{y \mid y_i < 0,\ i = 1, \dots, n\}$ when $\alpha < 0$ or $\Omega = \{y \mid y_i > 0,\ i = 1, \dots, n\}$ when $\alpha > 0$. **Q.E.D.**

For illustration, let us consider a one-dimensional function:

$$h(x) = (x-2)^3 + 2x, \quad x \in X = [1, 3].$$

The plot of $h(x)$ is shown in Figure 1, where $a_j = 1 + 0.2j$, $j = 0, 1, \dots, 10$. Note that $h'(x) = 3(x-2)^2 + 2 \ge 2$ and $h''(x) = 6(x-2) \ge -6$ for $x \in [1, 3]$. Take $t(y) = \ln(1 - \frac{1}{y})$ and $T(w) = \exp(w)$ in (2.1). By Corollary 2.1 and (2.10), $p_0 = -(-6)/2 = 3$. So, any $p \ge 3$ will guarantee the convexity of $h_p(y)$ on $Y_p = [t^{-1}(p), t^{-1}(3p)]$. In practice, p can be chosen to be much smaller than the bound defined in (2.10). Figure 2 shows the convexified plot of $h_p(y)$ with $p = 0.3$, where $b_i = t^{-1}(a_j)$, $i = 0, \cdots, 10$.

Theorem 2.1 and the above discussions reveal that the convexity of a monotone function can be achieved via transformation (2.1) under certain conditions on T and t. The proposed convexification methods involve both the function transformation and the variable transformation. An important feature of the convexification transformation (2.1) is that the variable transform $y \leftrightarrow \frac{1}{p}t(y)$ is a 1-1 monotone mapping between Y_p and X which is crucial for the equivalence between problem (1.1) and the transformed one. These equivalences will be established in Section 3.

3 Concave minimization

In this section, we establish the equivalence between problem (1.1) and its transformed concave minimization problems. Consider the following optimization problem which is a transformation of (1.1):

$$\begin{aligned} &\max\ \Theta_0(f(\theta(y))) \\ &\text{s.t. } \Theta_j(g_j(\theta(y))) \le \Theta_j(c_j),\ j = 1, \ldots, m, \\ &\quad y \in Y, \end{aligned} \tag{3.1}$$

where $\Theta_j :\ \mathbb{R} \to \mathbb{R},\ j = 0, 1, \ldots, m$, and $\theta :\ Y \to X$. Let S and $\tilde{S}$ denote the feasible regions of (1.1) and (3.1), respectively.

THEOREM 3.1 *Assume that $\Theta_j,\ j = 0, 1, \ldots, m$, are strictly increasing functions and θ is an onto mapping with $X = \theta(Y)$. Then*

(i) *$y^* \in Y$ is a global optimal solution to (3.1) if and only if $x^* = \theta(y^*)$ is a global optimal solution to (1.1).*

(ii) *If θ^{-1} exists and both θ and θ^{-1} are continuous mappings, then $y^* \in Y$ is a local optimal solution to (3.1) if and only if $x^* = \theta(y^*)$ is a local optimal solution to (1.1).*

PROOF. (i) Let $x \in X$. Since θ is an onto mapping and $X = \theta(Y)$ there exists $y \in Y$ such that $x = \theta(y)$. Since $\Theta_j,\ j = 1, \ldots, m$, are strictly increasing, we have

$$g_j(x) \le c_j \Leftrightarrow g_j(\theta(y)) \le c_j \Leftrightarrow \Theta_j(g_j(\theta(y))) \le \Theta_j(c_j).$$

So $x \in S \Leftrightarrow y \in \tilde{S}$. Moreover,

$$\Theta_0(f(\theta(y))) \le \Theta_0(f(\theta(y^*))) \Leftrightarrow f(\theta(y)) \le f(\theta(y^*)) \Leftrightarrow f(x) \le f(x^*).$$

Thus y^* is a global optimal solution to (3.1) if and only if x^* is a global optimal solution to (1.1).

(ii) Let y^* be a local optimal solution to (3.1). Then there exists an open neighborhood $B(y^*)$ of y^* such that

$$\Theta_0(f(\theta(y))) \le \Theta_0(f(\theta(y^*))), \quad \forall y \in B(y^*) \cap \tilde{S}. \tag{3.2}$$

Since θ^{-1} is a continuous mapping there must exist an open neighborhood $B(x^*)$ of x^* such that $\theta^{-1}(B(x^*)) \subseteq B(y^*)$. ¿From the proof of (i), $S = \theta(\tilde{S})$. Thus, we have from (3.2) that

$$f(x) \le f(x^*), \quad \forall x \in B(x^*) \cap S,$$

which implies that x^* is a local optimal solution to (1.1). Conversely, we can prove by similar argument that y^* is a local optimal solution to (3.1) if $x^* = \theta(y^*)$ is a local optimal solution to (1.1). **Q.E.D.**

In the reminder of this section, we suppose that X is a box set defined by (2.15), f and g_i are strictly increasing functions on X. Suppose also that T_j, $j = 0, 1, \ldots, m$, and t satisfy the conditions in Theorem 2.1 (with $T = T_j$). Consider the following problem:

$$\begin{aligned} &\max\ \phi(y) = T_0\left(pf\left(\frac{1}{p}t(y)\right)\right) \\ &\text{s.t. } \psi_j(y) = T_j\left(pg_j\left(\frac{1}{p}t(y)\right)\right) \le T_j(pc_j),\ j = 1, \ldots, m, \\ &\quad y \in Y_p, \end{aligned} \tag{3.3}$$

where Y_p is defined by (2.2). It follows from Theorem 3.1 that problem (3.3) is equivalent to problem (1.1). On the other hand, Theorem 2.1 ensures that $\phi(y)$ and $\psi_j(y)$ are convex functions on Y_p when p is larger than certain threshold value. Note that Y_p is also a box set since X is a box set and t is a separable mapping. Therefore, when p is sufficient large, (3.3) is a problem of maximizing a convex function over a general convex set, or equivalently a concave minimization problem ([13]).

The convexification transformation (2.1) can also be used to produce a concave minimization problem from a problem where the objective function and constraint function are strictly decreasing functions (see Remark 2 in Section 2).

We now discuss solution methods for the equivalent concave minimization problem (3.3). For convenience, we describe the method for problem (3.3) where t_i, $i = 1, \ldots, n$, are strictly increasing functions. For example, if we take $t_i = q_1$ defined in (2.11) with $\alpha < 0$, or $t_i = q_2$ defined in (2.12) with $\alpha < 0$, then t_i, $i = 1, \ldots, n$, are strictly increasing functions. It is well-known that a convex function always achieves its maximum value over a polyhedron at one of its vertices. An optimal solution can then be obtained by ranking all the vertices of the polyhedron. For maximizing a convex function over a general convex set, a natural approach is to maximize the convex function over a sequence of polyhedra which enclose the feasible set and approach it in the limit. Hoffman [5] proposed the following outer approximation method which approximates the convex feasible set by successively constructing cutting planes. Let S_p denote the feasible region of problem (3.3). Suppose that we have a fixed interior point $\tilde{y}$ of S_p and an initial enclosing polyhedron with all its vertices known. At the k-th iteration of Hoffman's method, a tighter polyhedron is formed by adding a new cutting plane at the boundary point y_k of S_p which is on the line connecting the interior point $\tilde{y}$ and the current best vertex v_k. The maximum of the function values at vertices of the new polyhedron then provides an improved upper bound on the optimal value. A variation of Hoffman's outer approximation method is proposed in [7]. In this method, the cutting plane is formed at a projection point y_k of v_k to the feasible region with a hope that a deeper cut and thus a tighter enclosing polyhedron is obtained. We note that projecting a point onto a general convex set is equivalent to finding the closest point in l_2 norm on the boundary of the feasible set. This can be done by solving a convex programming with a quadratic objective function.

An illustrative example below shows how a problem in the form of (1.1) with multiple local optimal solutions can be transformed into a concave minimization problem and then solved by an outer approximation.

EXAMPLE 3.1

$$\begin{aligned}
&\max\ f(x) = 4.5(1-0.40^{x_1-1})(1-0.40^{x_2-1}) + 0.2\exp(x_1+x_2-7) \\
&\text{s.t. } g_1(x) = 5x_1x_2 - 4x_1 - 4.5x_2 \le 32, \\
&\quad x \in X = \{x \mid 2 \le x_1 \le 6.2,\ 2 \le x_2 \le 6\}.
\end{aligned}$$

It is clear that f and g_1 are strictly increasing function on X. The problem has three local optimal solutions: $x^1 = (2.26923, 6)$ with $f(x^1) = 3.773461249$, $x^2 = (3.45284, 3.58904)$ with $f(x^2) = 3.857736888$ and $x^3 = (6.2, 2.14339)$ with $f(x^3) = 3.663127142$. As shown in Figure 3, the feasible region S of the problem is not a convex set. Moreover, the global optimal solution x^2 is not even on in the boundary of the convex hull of S. In the transformed problem (3.3), take $T_0(w) = T_1(w) = \exp(w)$ and $t_i(y_i) = \ln(1-\frac{1}{y_i})$, i.e., $t_i = q_1$ $(i = 1, 2)$ with $\alpha = -1$ in (2.11). Then it can be verified that functions $\phi(y)$ and $\psi_1(y)$ in problem (3.3) are convex on Y_p when $p = 0.5$. The feasible set S_p of problem (3.3) is illustrated in Figure 4 where $y^k = t^{-1}(px^k)$, $k = 1, 2, 3$, are the transformed local optimal solutions. We can see that the transformed global optimal solution y^2 now is a boundary point of the convexified feasible set S_p. A procedure based on the outer approximation method in [7] finds an approximate global optimal solution $y^* = (-0.21642, -0.19934)$ of (3.3) after 17 iterations and generating 36 vertices. The point y^* corresponds to $x^* = (\ln(1-\frac{1}{y_1^*}), \ln(1-\frac{1}{y_2^*}))$= $(3.45290, 3, 58899)$, an approximate optimal solution to Example 3.1 with $f(x^*) = 3.857736887$.

4 Applications to reliability optimization

Optimal reliability and redundancy allocation and cost minimization of general system are two major classes of optimization problems in reliability network design (see, e.g., [16][17]).

Given a network with n subsystem, the optimal reliability and redundancy allocation problem is to determine simultaneously the number of redundant components in each of q parallel subsystems and the reliability levels of $n-q$ general subsystems so as to maximize the overall reliability of the system subject to certain resource constraints. The problem is described by the following mixed-integer program:

$$\begin{aligned} &\max\ R_s(x,R) = f(R_1(x_1),\dots,R_q(x_q),R_{q+1},\cdots,R_n) \\ &\text{s.t. } C_j(x,R) \le c_j,\ j=1,\dots,m, \\ &\quad 1 \le a_i \le x_i \le b_i,\ x_j \text{ integer},\ i=1,\dots,q, \\ &\quad 0 < \alpha_i \le R_{q+i} \le \beta_i < 1,\ i=1,\dots,n-q, \end{aligned} \tag{4.1}$$

where $(x,R) = (x_1,\cdots,x_q,R_{q+1},\cdots,R_n)$, x_i represents the number of redundant components in i-th subsystem, R_i is the reliability of i-th subsystem (in particular, $R_i(x_i) = 1-(1-r_i)^{x_i}$, $i=1,\cdots,q$, is the reliability of i-th parallel subsystem with $0 < r_i < 1$), $R_s(x,R)$ is the overall system reliability, $C_j(x,R)$ is the j-th resource consumed; c_j is the total available j-th resource, a_i and b_i are lower and upper integer bounds of x_i respectively.

Let

$$D = \{(x,R) \in \mathbb{R}^n \mid a_i \le x_i \le b_i,\ i=1,\cdots,q,\ \alpha_i \le R_{q+i} \le \beta_i,\ i=1,\cdots,n-q\}.$$

Although the forms of f and the C_j depend on the network structure, f and the C_j are strictly increasing functions of (x,R) on D, but they are not necessarily convex or concave. Thus (4.1) is in general a nonconvex mixed-integer programming problem. Another optimization problem closely related to (4.1) is the cost minimization problem in reliability systems ([17]) which is to minimize the weighted cost of a reliability system subject to a minimum reliability constraint.

The existing solution methods for (4.1) are mainly heuristic due to the combinatorial and nonconvex nature of the problem ([12][16][17]). A fundamental difficulty when using a traditional branch-and-bound method to solve the above reliability problems is that the continuous relaxations are not convex problems. There is therefore no guarantee that a local optimization method will find the global solution, and if it does not a bound generated may be invalid.

The convexification method developed in the previous sections provides an exact and quite efficient method for solving (4.1) and the related cost minimization problem when combining with a branch-and-bound strategy. Relaxing the integral restriction for $x_1,\cdots,x_q$, problem (4.1) can be transformed into a concave minimization problem in the form of (3.3). At each node of the branch and bound tree, a concave minimization subproblem is solved using the outer approximation method with a guarantee of finding a global optimum point within a given accuracy. The variable range in each subproblem has the form

$$D^k = \{(x,R) \in \mathbb{R}^n \mid a_i^k \le x_i \le b_i^k,\ i=1,\ldots,q,\ \alpha_i \le R_{q+i} \le \beta_i,\ i=1,\cdots,n-q\}.$$

Since $D^k \subset D$ we can take the same parameter p for each continuous subproblems when the convexification transformation (2.1) is applied to the subproblems. The following important properties of the problem greatly accelerate the convergence of the branch-and-bound search for problem (4.1):

(i) The outer approximation polyhedron obtained in the last iteration for solving the parent continuous subproblem can be used to form a tight initial polyhedron of all descendant continuous subproblems;

(ii) If the right upper corner point $(b_1^k,\ldots,b_q^k,\beta_1,\cdots,\beta_{n-q})$ of D^k is feasible, then it is an optimal solution to the corresponding subproblem;

(iii) If the left lower corner point $(a_1^k,\ldots,a_q^k,\alpha_1,\cdots,\alpha_{n-q})$ of D^k is infeasible, then the corresponding subproblem is infeasible;

(iv) Let R_z be the reliability value of the current best feasible solution. Then R_z provides an additional stopping criteria when solving the descendant subproblems. In fact, suppose in k-th iteration, v^k is the vertex with maximum overall reliability. If $R_s(t^{-1}(v^k) \le R_z$ then the outer approximation method can be terminated at k-th iteration since, in this case, it is impossible for the subproblem to generate a feasible solution with reliability value being greater than R_z.

We have coded a branch-and-bound procedure for (4.1) and the related cost minimization problem using Fortran 90. In our implementation, the relaxed subproblems are solved by a subroutine based on the outer approximation method in [7]. The most fractional variable is chosen for branching with depth-first node search and backtracking to the best node each time a feasible solution is found. The test problems are from two typical complex systems: Bridge and ARPA networks ([12][17]).

EXAMPLE 4.1 5-link Bridge network optimal reliability and redundancy allocation.

$$\begin{aligned}
&\max\ R_s(x,R) = R_1R_2 + Q_2R_3R_4 + Q_1R_2R_3R_4 + R_1Q_2Q_3R_4R_5 + Q_1R_2R_3Q_4R_5 \\
&\text{s. t. } C_1(x) = x_1x_2 + 2.2x_2x_3 + 1.5x_2x_4 + 2\exp\left(\frac{0.01}{1-R_5}\right) \le 28, \\
&\quad C_2(x) = x_1 + 0.1x_2 + 2x_3 + x_4 + 5\exp\left(\frac{0.01}{1-R_5}\right) \le 25, \\
&\quad C_3(x) = x_1^2 + (x_2-2)^3 + 1.5x_3 + x_4 + 0.6\exp\left(\frac{0.01}{1-R_5}\right) \le 21,
\end{aligned}$$

$$1 \le x_i \le 6,\ x_i \text{ integer},\ i = 1, \cdots, 4,\ 0.50 \le R_5 \le 0.99,$$

where $R_i := R_i(x_i) = 1 - (1 - r_i)^{x_i}$, $i = 1, \cdots, 4$, $Q_i = 1 - R_i$, $i = 1, \cdots, 5$, $r_1 = 0.70$, $r_2 = 0.85$, $r_3 = 0.75$, $r_4 = 0.8$.

EXAMPLE 4.2 7-link ARPA network optimal reliability and redundancy allocation.

$$\begin{aligned}
\max\ R_s(x, R) = {} & R_6R_7 + R_1R_2R_3(Q_6 + R_6Q_7) + R_1R_4R_7Q_6(Q_2 + R_2Q_3) \\
& + R_3R_5R_6Q_7(Q_1 + R_1Q_2) + R_1R_2R_5R_7Q_3Q_4Q_6 \\
& + R_2R_3R_4R_6Q_1Q_5Q_7 + R_1R_3R_4R_5Q_2Q_6Q_7 \\
\text{s. t. } C_1(x) = {} & x_1x_2 + 0.5x_1 \log(1 + x_3) + x_4 + 2x_5 + 0.3 \exp\left(\frac{0.02}{1 - R_6}\right) \\
& + 0.3 \exp\left(\frac{0.01}{1 - R_7}\right) \le 27, \\
C_2(x) = {} & (x_1 + 2x_2 + 1.2x_3) \log(1 + x_1 + x_2 + 2x_3) + 0.4x_4 + 0.2x_5 \exp\left(\frac{0.02}{1 - R_6}\right) \\
& + 0.5 \exp\left(\frac{0.01}{1 - R_7}\right) \le 29, \\
& 1 \le x_i \le 4,\ x_i \text{ integer},\ i = 1, \cdots, 5,\ 0.5 \le R_i \le 0.99,\ i = 6, 7,
\end{aligned}$$

where $R_i := R_i(x_i) = 1 - (1 - r_i)^{x_i}$, $i = 1, \cdots, 5$, $Q_i = 1 - R_i$, $i = 1, \cdots, 7$, $r_1 = 0.70$, $r_2 = 0.90$, $r_3 = 0.80$, $r_4 = 0.65$, $r_5 = 0.70$.

EXAMPLE 4.3 5-link Bridge network cost minimization.

$$\begin{aligned}
& \min\ C_w(x, R) = 0.3C_1(x) + 0.5C_2(x) + 0.2C_3(x) \\
& \text{s.t. } R_s(x, R) \ge 0.999, \\
& \quad 1 \le x_i \le 6,\ x_i \text{ integer},\ i = 1, \cdots, 4,\ 0.50 \le R_5 \le 0.99,
\end{aligned}$$

where $R_s(x, R)$ and C_i $(i = 1, 2, 3)$ are defined in Example 4.1.

EXAMPLE 4.4 7-link ARPA network cost minimization.

$$\begin{aligned}
& \min\ C_w(x, R) = 0.4C_1(x) + 0.6C_2(x) \\
& \text{s.t. } R_s(x, R) \ge 0.999, \\
& \quad 1 \le x_i \le 4,\ x_i \text{ integer},\ i = 1, \cdots, 5,\ 0.5 \le R_i \le 0.99,\ i = 6, 7,
\end{aligned}$$

where $R_s(x, R)$ and C_i $(i = 1, 2)$ are defined in Example 4.2.

Numerical results for Example 4.1 to Example 4.4 are reported in Table 1, where p is the convexification parameters used in the outer approximation method when solving the subproblems, x^* is the optimal solution obtained, R^* is the reliability value at the optimal solution, NLP is the total number of subproblems solved, NV represents the total number of vertices generated in the outer approximation method and CPU times are measured on a SUN SPARCstation 5.

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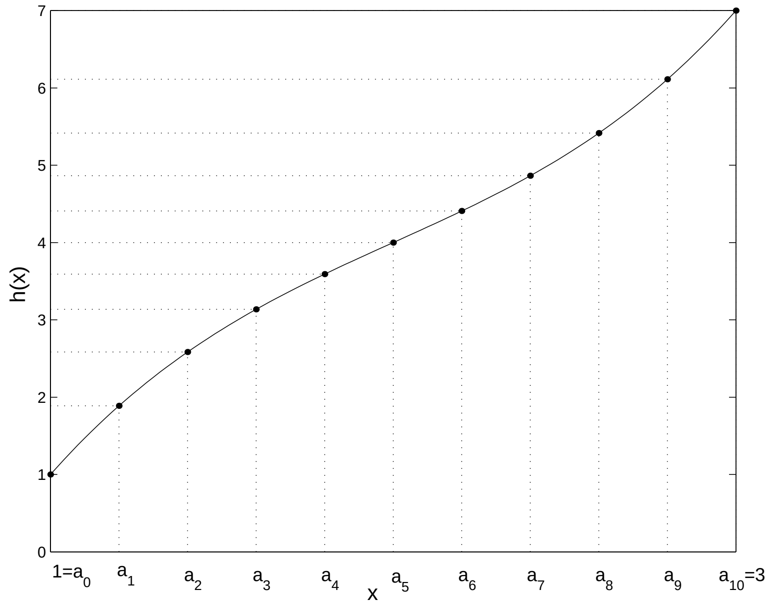

Figure 1: The plot of $h(x)$.

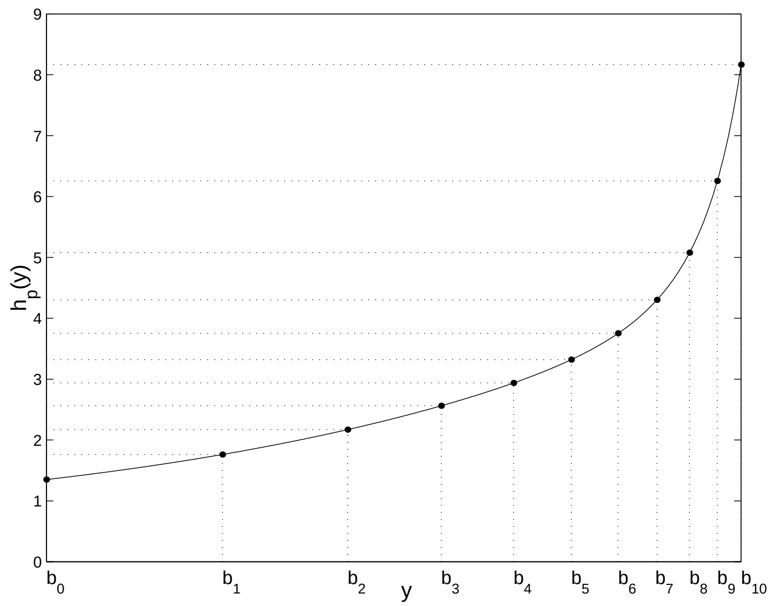

Figure 2: The plot of $h_p(y)$ with $p = 0.3$.

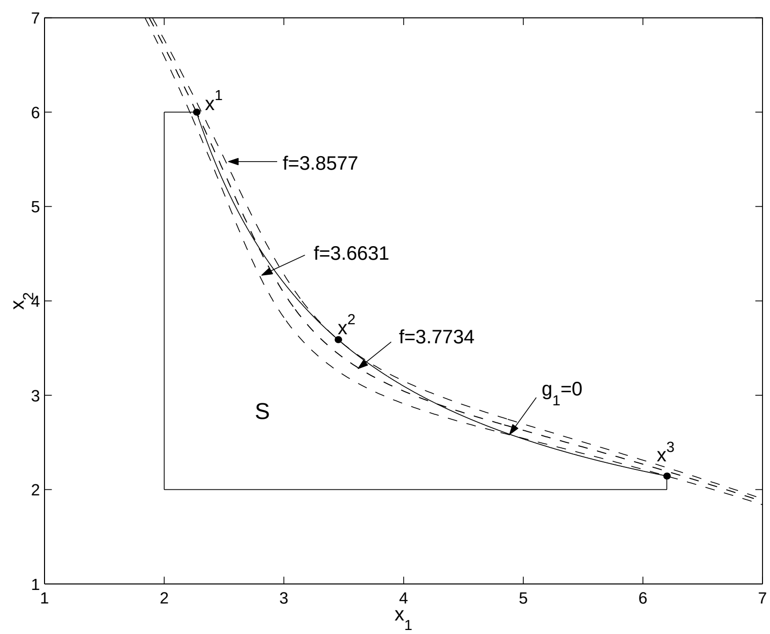

Figure 3: The feasible set S of Example 3.1.

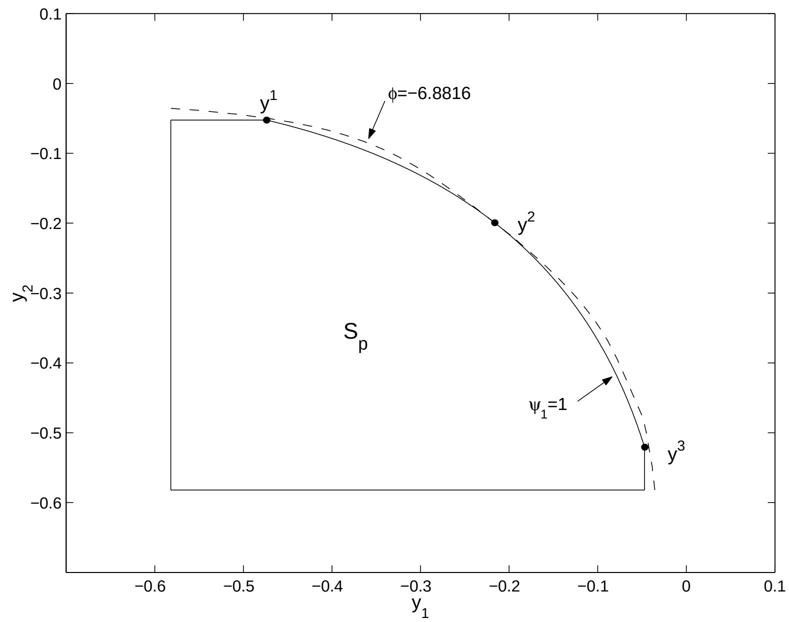

Figure 4: The convexified feasible set S_p of Example 3.1.

Table 1: Numerical results for Example 4.1 to Example 4.4

Problem	p	x^*	R^*	NLP	NV	CPU seconds
Example 4.1	1	(2,1,6,5,0.9396)	0.99992653	10	734	9.84
Example 4.2	1	(4,1,3,4,3,0.9845,0.9899)	0.99974476	13	3494	38.63
Example 4.3	2	(1,1,5,4,0.5)	0.99921211	12	1173	15.97
Example 4.4	2	(3,1,2,2,2,0.9869,0.9900)	0.99900000	17	4150	48.00