t-point counts in distance-regular graphs

Arnold Neumaier

Faculty of Mathematics
University of Vienna, Austria

joint work with

Safet Penjic
University of Primorska, Slovenia

These slides are available at

`http://www.mat.univie.ac.at/~neum/slides/Bled2019.pdf`

We consider distance regular graphs Γ of diameter $d = d_\Gamma$ with v_Γ vertices and intersection array

$$i(\Gamma) = \{b_0, \ldots, b_{d-1}; c_1, \ldots, c_d\}.$$

We write n_i for the number of points at distance i from a point, $n := n_1 = b_0$ for the valency, and

$$a_i := n - b_i - c_i, \quad \lambda := a_1.$$

Many well-known inequalities for the intersection array can be derived in a uniform way using t-point counts $[\Delta]$, normalized counts of the number of ordered subsets isomorphic to a template Δ with certain specified distances between the t vertices of Δ.

In this talk I'll show that by considering t-point counts with $t \leq 6$, the diameter bounds by Ivanov & Ivanov may be derived.

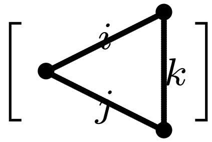
A t-point **type** is an undirected graph Δ with nodes $1, \ldots, t$ whose edges are labelled with integers $\in \{0, 1, \ldots, d_\Gamma\}$; it is called **complete** if any two nodes are joined by a labelled edge. $\Delta_{\mu\nu}$ denotes the label of the edge $\mu\nu$. In drawings, missing labels are taken as having the value 1.

A **configuration** is a finite ordered list $\bar{z} = z_1 \ldots z_t$ of points of Γ. A configuration $\bar{z}$ is of type Δ if

$$d(z_\mu, z_\nu) = \Delta_{\mu\nu} \quad \text{whenever } \mu \sim \nu \text{ in } \Delta.$$

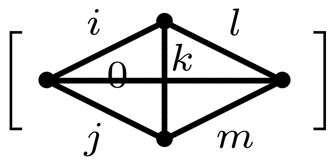
The (rational) number $[\Delta]$ is the number of configurations of type Δ divided by v_Γ. Clearly,

$$[\Delta] \geq 0 \quad \text{for all types } \Delta. \tag{1}$$

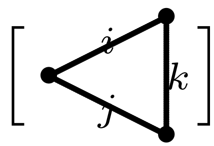


$$= n_k p^k_{ij} = n_i p^i_{jk} = n_j p^j_{ki}. \tag{2}$$

Elimination of distance 0: e.g.,

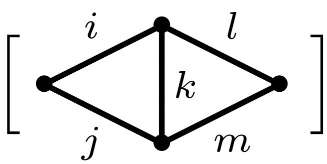


$$= \delta_{il}\delta_{jm}$$

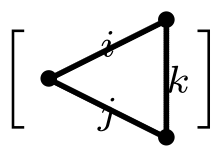


$$= \delta_{il}\delta_{jm} n_k p^k_{ij}. \tag{3}$$

Elimination of nodes of valency 2: e.g.,



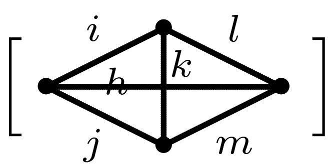
$$= p^k_{lm}$$



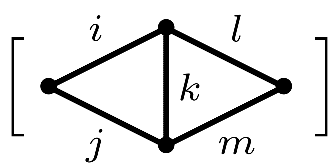
$$= n_k p^k_{ij} p^k_{lm}. \tag{4}$$

Sum over a distance: e.g.,

$$\sum_h$$



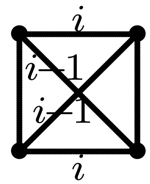
$$=$$



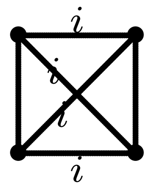
$$. \tag{5}$$

A t-**point count** is a number $[\Delta]$, where Δ is a complete t-point type. Isomorphic types have the same count. As in the special case (5), every $[\Delta]$ can be written as a sum of t-point counts.

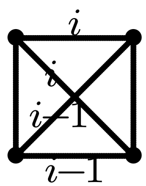
Proposition 1. All 4-point counts containing two disjoint edges can be expressed in terms of the rational numbers

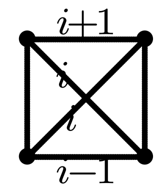
$e_i := \dfrac{1}{n_i c_i}\Big[$ 

$\Big] \geq 0 :$ (6)

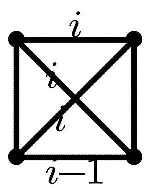
$A_i := \Big[$ 

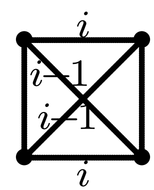
$\Big] = n_i\Big(a_i^2 - c_i(b_{i-1} - e_i) - b_i(c_{i+1} - e_{i+1})\Big),$

$B_i := \Big[$ 

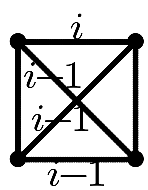
$\Big] = n_i c_i\Big(a_{i-1} - (c_i - c_{i-1} - e_i)\Big),$ $C_i := \Big[$ 

$\Big] = n_i c_i b_i,$

$D_i := \Big[$ 

$\Big] = n_i c_i(b_{i-1} - b_i - e_i),$ $E_i := \Big[$ 

$\Big] = n_i c_i e_i.$

$F_i := \Big[$ 

$\Big] = n_i c_i(c_i - c_{i-1} - e_i),$

Proof. By summing over a distance (see (5) and (4)) we find

$$n_{i+1}c_{i+1}c_i = n_i c_i b_i = C_i,$$

$$n_i c_i a_i = B_i + D_i,$$

$$n_i c_i a_{i-1} = F_i + B_i,$$

$$n_i c_i b_{i-1} = D_i + E_i + C_i,$$

$$n_i c_i^2 = C_{i-1} + E_i + F_i,$$

$$n_i a_i^2 = D_i + A_i + F_{i+1}.$$

Now (6) implies that $E_i = n_i c_i e_i$, and solving the resulting triangular linear system of equations gives the above formulas. □

Corollary 1. Let $x^+ = \max(x, 0)$. Then

$$b_{i-1} \geq b_i, \quad c_i \geq c_{i-1}, \quad (Biggs) \tag{7}$$

$$c_i(a_i - a_{i-1})^+ + b_i(a_i - a_{i+1})^+ \leq a_i^2, \quad (Brouwer\&Lambeck) \tag{8}$$

$$\max\left(0, c_i - c_{i-1} - a_{i-1}, b_{i-1} - b_i - \frac{a_i^2}{c_i}, c_i - c_{i-1} - \frac{a_{i-1}^2}{b_i}\right) \leq e_i, \tag{9}$$

$$e_i \leq \min(b_{i-1} - b_i, c_i - c_{i-1}), \tag{10}$$

Proof. (7), (11), (12), and (8) follow from

$$0 \leq D_i + E_i = n_i c_i (b_{i-1} - b_i),$$

$$0 \leq E_i + F_i = n_i c_i (c_i - c_{i-1}),$$

$$D_i - F_i = n_i c_i (a_i - a_{i-1}),$$

$$(D_i - F_i)^+ + (F_{i+1} - D_{i+1})^+ \leq D_i + F_{i+1} + A_i = n_i a_i^2.$$

(9) follows from $D_i \geq 0$, $F_i \geq 0$, and (10) from $E_i, B_i, A_i \geq 0$. □

Corollary 2.

$$b_{i-1} = b_i \quad \Leftrightarrow \quad D_i = E_i = 0, \tag{11}$$

$$c_{i-1} = c_i \quad \Leftrightarrow \quad E_i = F_i = 0. \tag{12}$$

These relations show that D_i, E_i and F_i are the most restricted and hence the most interesting counts among those of Proposition 1.

When the lower bound for e_i in (9) is positive, we may apply the following results.

Theorem 1. We have

$$e_i > 0,\ c_{s+1} > \min(c_i - c_{i-1}, b_{i-1} - b_i) \quad \Rightarrow \quad c_{i+s} > c_i, \tag{13}$$

$$e_i > 0, \quad i > 1 \quad \Rightarrow \quad c_{2i-1} > c_i. \tag{14}$$

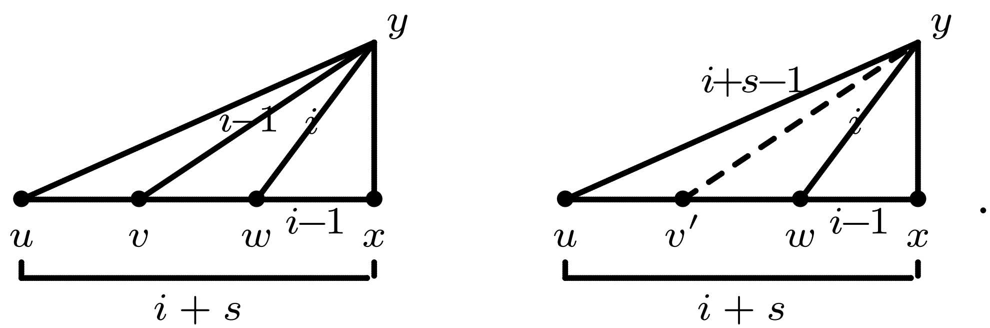
$$e_i > 0 \quad \Rightarrow \quad b_1 \geq b_i + c_{i-1}, \tag{15}$$

$$e_i > 0 \quad \Rightarrow \quad b_{i-1} - b_i + c_i - c_{i-1} \geq \lambda + 2. \tag{16}$$

Proof. If $e_i > 0$ then by (9),

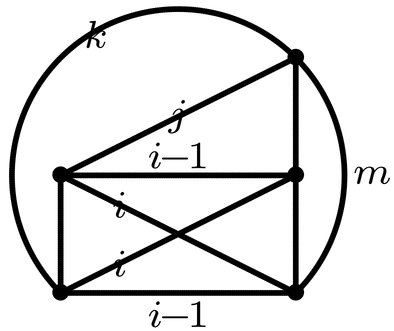
$$e := \min(c_i - c_{i-1}, b_{i-1} - b_i) \geq e_i > 0.$$

Hence we may choose $vwxy$ consistent with



All p^i_{si+1} choices for u yield $d(u, y) = i + s - 1$. Now the number of $v' \in \Gamma(w) \cap \Gamma_s(u)$ with $d(v', y) = i - 1$ is $\leq e$. If also $c_{s+1} > e$, there is some v' with $d(v', y) \geq i$. Now consider $uvxy$ to get $c_{i+s} > c_i$. Thus (13) holds, and (14) follows by taking $s = i - 1 > 0$.

To derive (15) and (16) we use the new counts



$$=: \begin{cases} H_{jk} & \text{if } m = 2, \\ K_{jk} & \text{if } m = 1, \end{cases} \tag{17}$$

which give

$$H_{ii+1} = E_i b_i, \quad H_{i-2i-1} = E_i c_{i-1},$$

$$H_{i-1i-1} + H_{i-1i} + H_{ii-1} + H_{ii} = E_i(b_1 - b_i - c_{i-1}), \tag{18}$$

$$K_{i-1i-1} + H_{i-1i-1} + K_{ii-1} + H_{ii-1} = E_i(c_i - c_{i-1} - 1), \tag{19}$$

$$K_{ii-1} + H_{ii-1} + K_{ii} + H_{ii} = E_i(b_{i-1} - b_i - 1), \tag{20}$$

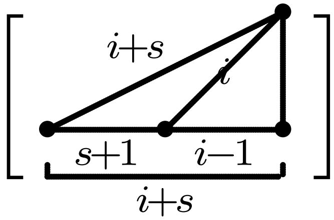
$$K_{i-1i-1} + K_{i-1i} + K_{ii-1} + K_{ii} = E_i\lambda. \tag{21}$$

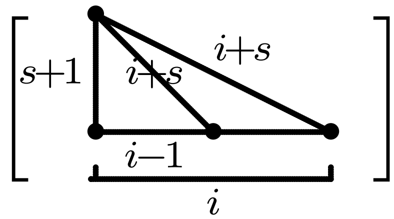
Swapping the two edges gives

$$K_{i-1i} = K_{ii-1}. \tag{22}$$

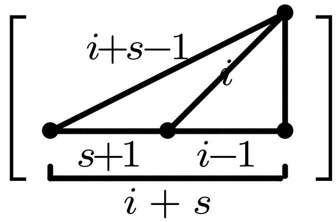
Now (15) follows from the nonnegativity of (18), and (16) follows from (19)+(20)≥(21) which holds in view of (22). □

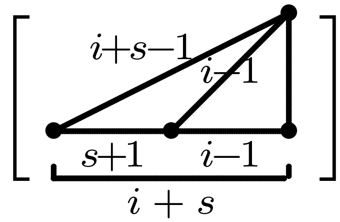
To derive the diameter bound by I & I we first look at certain 4-point counts with a single distance 1 only. We define

$$D_i^s :=$$ 

$$=$$ 

,

$$E_i^s :=$$ 

, $$G_i^s :=$$ 

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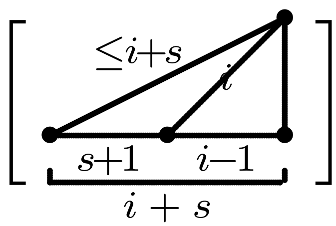
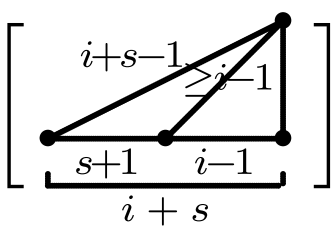
Then

$$D_i^s + E_i^s =$$ 

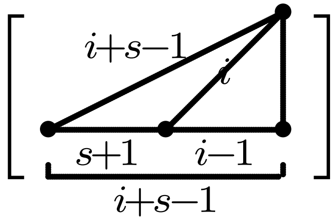
$$= n_{i+s} p^{i+s}_{i-1\, s+1} (b_{i-1} - b_{i+s}),$$

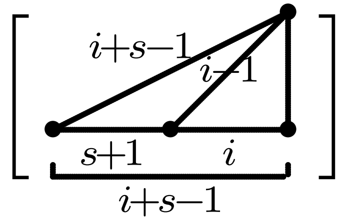
$$E_i^s + G_i^s =$$ 

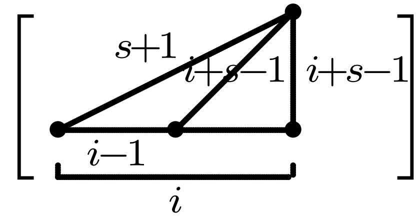
$$= n_{i+s} p^{i+s}_{i-1\, s+1} (c_{i+s} - c_{i-1}).$$

$$D_i^0 = D_i, \quad E_i^0 = E_i,$$

We also define

$F_i^s :=$ 

$=$ 

$=$ 

,

$$F_i^0 = G_i^0 = F_i,$$

Proposition 2. Let $l > s \geq \tau \geq 1$. If

$$b_l = b_{l-s} > b_{s+1} - c_{l-1-s}, \tag{23}$$

$$c_{l+1-t} - c_{l-t} < c_{t+1} \quad \text{for } t = 0, \ldots, \tau, \tag{24}$$

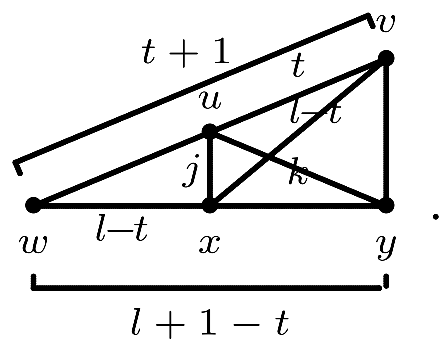
then we have

$$D_{l-t}^t = F_{l+1-t}^t = 0 \quad \text{and } a_l \leq a_{l-1-t} \quad \text{for } t = 0, \ldots, \tau.$$

Proof. (24) for $t = 0$ implies $c_{l+1} = c_l$, hence $F_{l+1}^0 = F_{l+1} = 0$, and (23) implies $b_l = b_{l-1}$, hence $D_l^0 = D_l = 0$. We now prove by induction on $t \leq \tau$ that

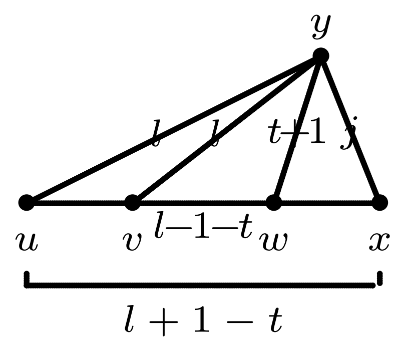
$$D_{l-t}^t = F_{l+1-t}^t = 0. \tag{25}$$

Suppose (25) holds for $t-1$ in place of t. If $F^t_{l+1-t} > 0$ then we can choose $vwxy$ consistent with the diagram



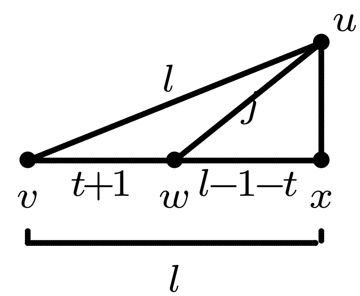
There are c_{t+1} choices for $u \in \Gamma(w) \cap \Gamma_t(v)$. If $k = l - t$ then $j \geq l - t$ (consider uvw); hence (consider $uwxy$) this can happen at most $c_{l+1-t} - c_{l-t}$ times. By (24), this implies that we can choose u such that $k \geq l - t + 1$. Now if $j = l - t$ then $vuxy$ gives $D^{t-1}_{l+1-t} > 0$, if $j = l + 1 - t$ then $uwxy$ gives $0 < E_{l+1-t}$ contradicting $b_{l+1-t} > b_{l-t}$ (note that $t \geq 1$), and if $j = l + 2 - t$ then $uxyv$ gives $F^{t-1}_{l-t} > 0$. In all cases, the induction assumption is violated. Thus we must have $F^t_{l+1-t} = 0$.

If $D^t_{l-t} > 0$ then one can choose $uvwy$ consistent with the diagram



and then choose $x \in \Gamma(w) \cap \Gamma_{l+1-t}(u)$. Among the $b_{l+1-t} = b_e$ choices for x there are at most $b_{s+1} - c_{l-1-s}$ choices with $j = t + 2$. Hence (23) implies at least one choice with $j \leq t + 1$. But then $uwxy$ gives $F^t_{l+1-t} > 0$, contradiction. Thus we must have $D^t_{l-t} = 0$. This shows that (25) holds generally.

Now consider

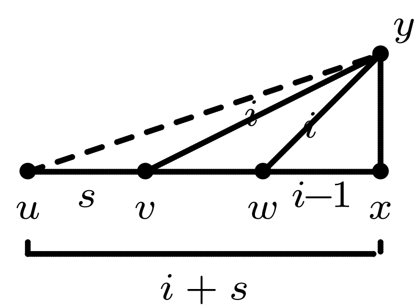


The triangle inequality forbids $j = l - 2 - t$, and $j = l - t$ is forbidden by $D^t_{l-t} = 0$. Hence any of the a_l choices for $u \in \Gamma_l(v) \cap \Gamma(x)$ must be among the a_{l-1-t} choices for $u \in \Gamma_{l-1-t}(w) \cap \Gamma(x)$, giving $a_l \leq a_{l-1-t}$. $\square$

Corollary 3. If $c_{i+s+1} = c_i$, $b_{i+s} = b_i > b_{s+1} - c_{i-1}$ then

$$D_i = D_i^s = F_{i+1}^s = 0 \quad \text{and } a_i \le a_{i-1}.$$

Proof. The assumption implies (23) and (24) for $l = i + s$, $\tau = s$; moreover $a_{i+s} = a_i$. Hence it suffices by Proposition 2 to show that $D_i = 0$. Suppose that $D_i > 0$. Extend a configuration of type D_i by a vertex u as indicated below.



Clearly $d(u, y) \in \{i + s - 1, i + s\}$. But the first choice turns $uwxy$ into a configuration of type D_i^s and the second choice contradicts $c_{i+s} = c_i$. □

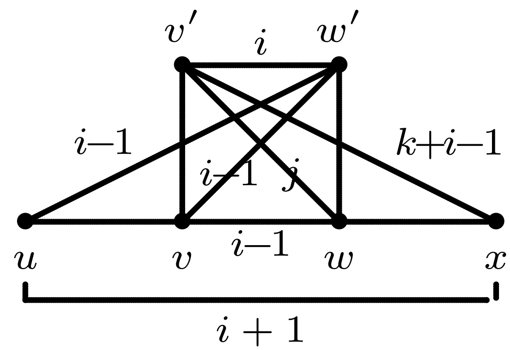
Theorem 2. Suppose that $b_{s+1} < b_i + c_{i-1}$.
(i) If $b_{i-1} > b_i$ and $c_{s+1} > \min(c_i - c_{i-1}, b_{i-1} - b_i)$ then $c_{i+s+1} > c_i$ or $b_{i+s} > b_i$.
(ii) If $a_i \neq a_{i-1}$, $b_{i-1} > c_i - c_{i-1}$ and $c_{s+2} > c_i - c_{i-1} + \min(b_{i-1} - b_i, a_i)$ then $c_{i+s+1} > c_i$ or $b_{i+s+1} < b_i$.

Proof. (i) If $e_i > 0$ then Theorem 1 shows that $c_{i+s} > c_i$. Moreover, $c_{i+s+1} > c_i$ by (7). If $e_i = 0$ the $D_i > 0$ by definition of D_i, contradicting Corollary 3.

(ii) Suppose the conclusion is not valid. Then Corollary 3 implies

$$a_i < a_{i-1}, \quad D_i = D_i^s = F_{i+1}^s = 0. \tag{26}$$

To get a configuration

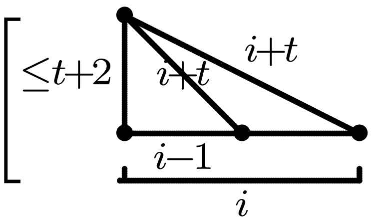


we can choose $uwvxw'v'$ for fixed u in

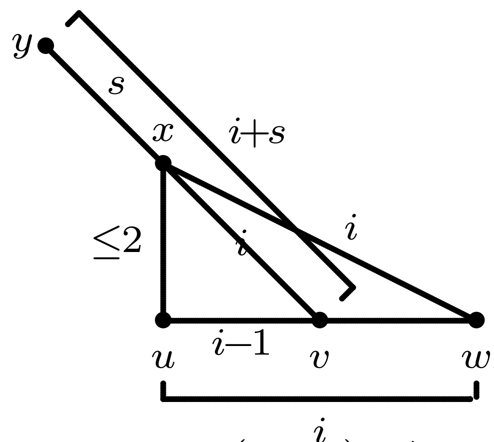
$$n_i c_i b_i (a_{i-1} - a_i)(b_{i-1} - (c_i - c_{i-1})) > 0$$

ways since $w' \in \Gamma(w) \cap (\Gamma_{i-1}(v) \setminus \Gamma_i(u))$ and $v' \in \Gamma(v) \cap \Big(\Gamma_i(w') \setminus (\Gamma_{i-1}(x) \setminus \Gamma_{i-2}(w))\Big)$; hence the configuration exists. The triangle inequality for wvv' and $wv'w'$ gives $j \leq i$ and $j \geq i-1$.

Now we define

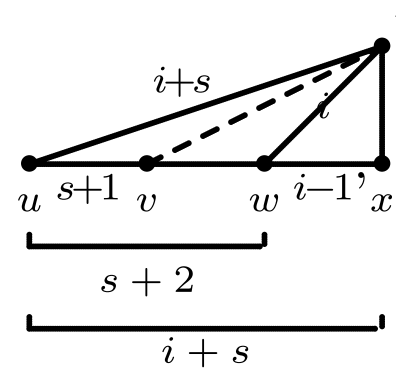
$O_i^t := D_i^t + F_i^{t+1} = \Bigg[$  $\Bigg].$ (27)

If $j = i$ then $v'uw'w$ gives $O_i^0 > 0$. And if $j = i - 1$ then $v'wx$ shows that $k = i$; then $xwv'v'$ gives $O_i^0 > 0$. Thus we always have $O_i^0 > 0$, and since $p^i_{si+s} > 0$, a configuration



exists. Since $c_{i+s} = c_i$ we have $d(y, w) \neq i + s - 1$, hence $d(y, w) = i + s$, and $yuvw$ gives $O_i^s > 0$. Since $D_i^s = 0$ by (26), hence (27) implies $F_i^{s+1} > 0$.

Now we fix $uwxy$ consistent with the configuration



which exists since $F_i^{s+1} > 0$. Among the c_{s+2} choices for $v \in \Gamma(w) \cap \Gamma_{s+1}(u)$ we can have at most $c_i - c_{i-1} + \min(b_{i-1} - b_i, a_i)$ choices with $d(v, y) \leq i$ and $d(v, x) = i$ or $i + 1$. By assumption on c_{s+2}, there is at least one additional choice, which therefore must have $d(v, y) = i$, $d(v, x) = i - 1$ or $d(v, y) = i + 1$, $d(v, x) = i$. Now $uvxy$ yields $D_i^s > 0$ in the first case and $F_{i+1}^s > 0$ in the second case, contradiction. $\square$

Corollary 4. (Ivanov & Ivanov)
If $s > 0$ and $d \geq 2s+2$ then

$$b_{s+1} < b_s \quad \Rightarrow \quad c_{2s+2} > c_{s+1} \quad \text{or} \quad b_{2s+1} < b_{s+1}, \tag{28}$$

$$b_{s+1} = b_s, \quad c_{s+1} > c_s \quad \Rightarrow \quad c_{2s+2} > c_{s+1} \quad \text{or} \quad b_{2s+2} < b_{s+1}. \tag{29}$$

Proof. The Taylor–Levingston inequality $b_s \geq c_{s+1}$ implies the hypothesis of Theorem 2 for $i = s+1$. Now the first implication follows from (i), and the second implication from (ii) of the preceding theorem. □

Theorem 3.(IVANOV & IVANOV).
Suppose that $c_r - b_r > c_1 - b_1$ for some $r > 1$. Then the valency n is bounded by

$$n \geq c_{2^i r} - b_{2^i r} > c_1 - b_1 + i.$$

In particular,

$$d \leq 2^{2n-2-\lambda} r,$$

Proof. By Corollary 4 and the Biggs inequalities (7),

$$t > 1, \quad c_t - b_t > c_{t-1} - b_{t-1} \quad \Rightarrow \quad c_{2t} - b_{2t} > c_t - b_t.$$

Hence the first part follows by induction. If $d \geq 2^{2n-2-\lambda} r - 1$ then we can choose $i = 2n - 2 - \lambda$ and get a contradiction; this yields the second part. □

Thank you for your attention!

These slides are available at

`http://www.mat.univie.ac.at/~neum/slides/Bled2019.pdf`