

# Born's rule and measurement

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**Abstract.** Born's rule in its conventional textbook form applies to the small class of projective measurements only. It is well-known that a generalization of Born's rule to realistic experiments must be phrased in terms of positive operator valued measures (POVMs). This generalization accounts for things like losses, imperfect measurements, limited detection accuracy, dark detector counts, and the simultaneous measurement of position and momentum.

Starting from first principles, this paper gives a self-contained, deductive introduction to quantum measurement and Born's rule, in its generalized form that applies to the results of measurements described by POVMs. It is based on a suggestive definition of what constitutes a detector, assuming an intuitive informal notion of response.

The formal exposition is embedded into the context of a variety of quotes from the literature illuminating historical aspects of the subject. The material presented suggests a new approach to introductory courses on quantum mechanics.

For the discussion of questions related to this paper, please use the discussion forum <https://www.physicsoverflow.org>.

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# 1 The measurement process

*We have developed a pure geometry, which is intended to be descriptive of the relation-structure of the world. The relation-structure presents itself in our experience as a physical world consisting of space, time and things. The transition from the geometrical description to the physical description can only be made by identifying the tensors which measure physical quantities with tensors occurring in the pure geometry; and we must proceed by inquiring first what experimental properties the physical tensor possesses, and then seeking a geometrical tensor which possesses these properties by virtue of mathematical identities.*

*If we can do this completely, we shall have constructed out of the primitive relation-structure a world of entities which behave in the same way and obey the same laws as the quantities recognised in physical experiments. Physical theory can scarcely go further than this.*

Arthur Eddington 1930 [33, p.222]

*The following 'laboratory report' of the historic Stern-Gerlach experiment stands quite in contrast to the usual textbook 'caricatures'. A beam of silver atoms, produced in a furnace, is directed through an inhomogeneous magnetic field, eventually impinging on a glass plate. [...] Only visual measurements through a microscope were made. No statistics on the distributions were made, nor did one obtain 'two spots' as is stated in some texts. The beam was clearly split into distinguishable but not disjoint beams. [...] Strictly speaking, only an unsharp spin observable, hence a POV measure, is obtained.*

Busch, Grabowski and Lahti, 1995 [16, Example 1, p.7]

*If you visit a real laboratory, you will never find there Hermitian operators. All you can see are emitters (lasers, ion guns, synchrotrons and the like) and detectors. The experimenter controls the emission process and observes detection events. [...] Traditional concepts such as "measuring Hermitian operators", that were borrowed or adapted from classical physics, are not appropriate in the quantum world. In the latter, as explained above, we have emitters and detectors, and calculations are performed by means of POVMs.*

Asher Peres, 2003 [75, p.1545f]

All traditional foundations of quantum mechanics depend heavily – far too heavily – on the concept of (hypothetical, idealized) measurements *exactly* satisfying Born's rule, a nontrivial technical rule far from being intuitive. This – almost generally assumed – exact validity without a precise definition of the meaning of the term measurement is probably the main reason why, nearly 100 years after the discovery of the basic formal setting for modern quantum mechanics, these foundations are still unsettled. No other scientific theory has such controversial foundations.

The source of this poor state of affairs is that Born's rule for projective measurements, the starting point of the usual interpretations, constitutes a severe idealization of measurement processes in general. Except in a few very simple cases, it is too far removed from experimental practice to tell much about real measurement, and hence about how quantum physics is used in real applications. But foundations that starts with idealized concepts only do not provide a safe ground for interpreting reality.

More general theoretical descriptions of quantum measurements were introduced in 1970 by DAVIES & LEWIS [27], 45 years after Heisenberg's 1925 paper initiating modern quantum

physics. A very readable account was given 4 years later by ALI & EMCH [4]. Since this paper appeared, another 45 years passed. These general measurement schemes are based on a **discrete POVM** (also called a **discrete resolution of the identity**), a family of finitely many Hermitian positive semidefinite operators  $P_k$  on a Hilbert space summing to 1,

$$\sum_k P_k = 1.$$

This generalization accounts for things like losses, imperfect measurements, limited detection accuracy, dark counts, and the simultaneous measurement of position and momentum. POVMs were soon found useful for concrete applications to the calibration of quantum systems (HELSTROM [41]). BRANDT [12] gives a short, more substantive history. For a fairly concise, POVM-based exposition of the foundations of quantum mechanics see, e.g., ENGLERT [34].

POVMs are indispensable in quantum information theory. Indeed, the well-known textbook by NIELSEN & CHUANG [73] introduces them even before defining the traditional projective measurements. POVMs are also needed to describe quite ordinary experiments without making the traditional textbook idealizations. For example, the original Stern-Gerlach experiment did (in contrast to its textbook caricature) not produce two well-separated spots on the screen but two overlapping lips of silver. This outcome cannot be described in terms of a projective measurement but needs POVMs.

Similarly, joint measurements of position and momentum, which are ubiquitous in engineering practice, cannot be described in terms of a projective measurement. Born's rule in the pre-1970 form does not even have idealized terms for these.

Books featuring POVM measurements include BUSCH et al. [20, 16, 19, 21], DE MUYNCK [28], HOLEVO [43, 44, 45], NIELSEN & CHUANG [73] and PERES [74].

In foundational studies, the projective idealization, taught in nearly <sup>1</sup> every textbook on quantum mechanics, is traditionally taken far too seriously. It counts as the indisputable truth about everything measured on the most fundamental level, to which everyone pays lipservice. But from the practical point of view, this idealization seems to be only a didactical trick to make the formal definitions of quantum mechanics easier to swallow for the newcomer – without needing seemingly abstract notions such as POVMs. On the other hand, Born's rule for projective measurement needs spectral theory up to the spectral theorem, itself not a simple subject. Thus there seems room for improvement even on the didactical level.

The purpose of this paper is to give an intuitive, self-contained approach to quantum measurement in the spirit of Eddington. Born's rule is not postulated as the starting point, but appears as a derived statement valid under the conditions specified in its derivation. This sheds a new light on the understanding of Born's rule and eliminates the most problematic

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<sup>1</sup>see, e.g., NEUMAIER [70] and Footnote <sup>21</sup>

features<sup>2</sup> of its uncritical use.

We do not discuss the question of whether (or which form of) state reduction occurs when subjecting a system to measurement. Neither do we discuss the measurement problem, i.e., how Born's rule can be justified in terms of a unitary dynamics of a larger quantum model containing a measured system and a measuring detector. For such quantum models of POVM measurements see BUSCH & LAHTI [19] and several chapters in BUSCH et al. [16, 21].

In the remainder of this section we give motivations and precise definitions of states (positive linear functionals) and quantum detectors (consisting of a POVM and a scale), and give a precise specification of what is measured by such a detector, covering the most general case. Section 2 gives numerous examples before considering the special case singled out by tradition – that of projective measurements. The section ends with proposing a new design for introductory courses on quantum mechanics, based on the preceding. Section 3 gives a thorough, precise discussion of various aspects of uncertainty in quantum measurements and relates them to the thermal interpretation of quantum physics as defined in my recent book *Coherent quantum physics* (NEUMAIER [68]).

## 1.1 States and their properties

*Quantum mechanics tells us that whatever comes from the emitter is represented by a state  $\rho$  (a positive operator, usually normalized to 1).*  
Asher Peres, 2003 [75, p.1545]

We motivate the formal setting of this paper by considering the polarization of classical light, as in Section 8.6 of NEUMAIER [68], from which a few paragraphs are taken.

A ray (quasimonochromatic beam) of classical polarized light of fixed frequency is characterized by a state, described (MANDEL & WOLF [62, Section 6.2]) by a real **Stokes vector**

$$S = (S_0, S_1, S_2, S_3)^T = \begin{pmatrix} S_0 \\ \mathbf{S} \end{pmatrix}$$

with

$$S_0 \geq |\mathbf{S}| = \sqrt{S_1^2 + S_2^2 + S_3^2},$$

The Stokes vector is a classical observable vector quantity. Equivalently, the state can be described by a **coherence matrix**, a complex positive semidefinite  $2 \times 2$  matrix  $\rho$ . These are related by

$$\rho = \frac{1}{2}(S_0 + \mathbf{S} \cdot \boldsymbol{\sigma}) = \frac{1}{2} \begin{pmatrix} S_0 + S_3 & S_1 - iS_2 \\ S_1 + iS_2 & S_0 - S_3 \end{pmatrix},$$

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<sup>2</sup>All problematic features known to me are collected in my recent book (NEUMAIER [68, Section 14.3]), following the preprint NEUMAIER [64, Section 3.3].

where  $\boldsymbol{\sigma}$  is the vector of Pauli matrices  $\sigma_1, \sigma_2, \sigma_3$ . If we define  $\sigma_0 := 1$  as the identity operator we have

$$S_k = \langle \sigma_k \rangle \quad \text{for } k = 0, \dots, 3,$$

where

$$\langle X \rangle := \text{Tr } \rho X$$

denotes the **q-expectation** of the matrix  $X \in \mathbb{C}^{2 \times 2}$ . In particular,  $\text{Tr } \rho = \langle 1 \rangle = S_0$  is the **intensity** of the beam.

The quotient  $p = |\mathbf{S}|/S_0 \in [0, 1]$  is the **degree of polarization**. Since

$$\det \rho = (S_0^2 - S_3^2) - (S_1^2 + S_2^2) = S_0^2 - \mathbf{S}^2,$$

the fully polarized case  $p = 1$ , i.e.,  $S_0 = |\mathbf{S}|$ , is equivalent to  $\det \rho = 0$ , hence holds iff the rank of  $\rho$  is 0 or 1. In this case we say that the state is **pure**. Thus the pure states correspond precisely to fully polarized beams. In a pure state, the coherence matrix can be written in the form  $\rho = \psi\psi^*$  with a **state vector**  $\psi \in \mathbb{C}^2$  determined up to a phase, and the intensity of the beam is

$$S_0 = \langle 1 \rangle = |\psi|^2 = \psi^*\psi.$$

Thus notions related to a complex Hilbert space of dimension 2 model the simplest quantum phenomenon: A positive definite Hermitian  $\rho$  describes the state of an arbitrary source, the trace of  $\rho$  is the intensity of the source, and certain Hermitian operators represent key quantities. Note the slight difference to density operators, where the trace is required to be one.

Combining two independent sources leads to the addition of the intensities and hence the corresponding densities. Similarly, changing the intensity amounts to a scalar multiplication of the corresponding densities. This provides experimental support for the linearity of typical detector responses, a feature observed in broad generality.

We may therefore generalize from the polarization experiments to experiments involving arbitrary quantum systems. Probably having mastered the above lets every beginning student of quantum physics accept the generalization. It is enough to say that in nearly 100 years of experimental work it was established beyond reasonable doubt that not only photon polarization but an arbitrary quantum system is describable in terms of an arbitrary complex Hilbert space, with a positive definite Hermitian  $\rho$  with finite trace describing the state of an arbitrary source, the trace of  $\rho$  defining the macroscopic intensity of the source, and certain Hermitian operators (with details depending on the quantum system) define key quantities. While this is not a proof, one may refer to authorities when creating foundations for a beginners course, and only make plausibility arguments that are easily grasped.

In the following we develop the time-independent part of the formal core of quantum mechanics – conventional shut-up-and-calculate quantum mechanics in a complex Hilbert space  $\mathbb{H}$ , but without Born's rule. Thus we avoid the traditional a priori link to experimental practice via an ill-defined notion of measurement. Instead, such a link will be established through careful definitions.

We describe a **source** (one of the emitters in the quote by Peres) by a positive semidefinite Hermitian **density operator**  $\rho \in \text{Lin } \mathbb{H}$ ; the source is called **pure** if the density operator has rank 1, and hence is given by  $\rho = \psi\psi^*$  for some **state vector**  $\psi$ . The **state** of the source is the positive linear mapping  $\langle \cdot \rangle$  that assigns to each  $X \in \text{Lin } \mathbb{H}$  its **q-expectation**<sup>3</sup>

$$\bar{X} = \langle X \rangle := \text{Tr } \rho X. \quad (1)$$

More generally, given a fixed state, the **q-expectation**<sup>4</sup> of a vector  $X \in (\text{Lin } \mathbb{H})^m$  with operator components is the vector  $\bar{X} = \langle X \rangle \in \mathbb{C}^m$  with components  $\bar{X}_j = \langle X_j \rangle$ . Its **q-uncertainty** is the nonnegative number

$$\sigma_X := \sqrt{\langle (X - \bar{X})^*(X - \bar{X}) \rangle} = \sqrt{\langle X^*X \rangle - |\bar{X}|^2}. \quad (2)$$

We may also define the **q-covariance matrix**

$$C_X := \langle (X - \bar{X})(X - \bar{X})^* \rangle \in \mathbb{C}^{n \times n},$$

in terms of which

$$\sigma_X = \sqrt{\text{tr } C_X}.$$

The uncertainty relation (due to ROBERTSON [79]), which asserts that for non-commuting Hermitian operators  $A, B \in \text{Lin } \mathbb{H}$ ,

$$\sigma_A \sigma_B \geq \frac{1}{2} |\langle [A, B] \rangle|, \quad (3)$$

follows from the definitions.<sup>5</sup> The q-expectations and q-uncertainties are complex numbers or vectors providing theoretical properties of the state, independent of any notion of measurement. The uncertainty relation also does not involve the notion of measurement. Thus these notions belong to the formal core of quantum mechanics. Thus, a priori, no statistical implication is assumed.

The collection of all q-expectations completely determines the state. Hence any property of the state can be expressed as a function of q-expectations.

<sup>3</sup>Following the convention of ALLAHVERDYAN et al. [3], we add the prefix "q-" to all theoretical quantum notions that suggest by their name a statistical interpretation and hence might confuse the borderline between theory and measurement.

<sup>4</sup>Traditionally, the q-expectation  $\langle X \rangle$  of an operator  $X$  is called the expectation value of  $X$ . But when  $X$  is not normal (and in particular when it is defective, hence has not even a spectral resolution), a statistical interpretation in the traditional sense is impossible. Note that nonnormal q-expectations appear routinely in quantum field theory, e.g., in the definitions of so-called  $N$ -point functions. It is interesting to note that Dirac's 1930 book, which introduced the name "observable" for operators, used this terminology for arbitrary linear operators (DIRAC [32, p.28]). Later editions make the restriction to Hermitian (and implicitly self-adjoint) operators.

<sup>5</sup>Indeed, the relation remains unchanged when subtracting from  $A$  and  $B$  its q-expectation, hence it suffices to prove it for the case where both q-expectations vanish. In this case,  $\langle A^2 \rangle = \sigma_A^2$  and  $\langle B^2 \rangle = \sigma_B^2$ , and the Cauchy-Schwarz inequality gives  $|\langle AB \rangle|^2 \leq \langle A^2 \rangle \langle B^2 \rangle = \sigma_A^2 \sigma_B^2$ , hence  $|\langle AB \rangle| \leq \sigma_A \sigma_B$ . On the other hand, one easily checks that  $i \text{Im} \langle AB \rangle = \frac{1}{2} \langle [A, B] \rangle$ , so that  $\frac{1}{2} |\langle [A, B] \rangle| = |\text{Im} \langle AB \rangle| \leq |\langle AB \rangle|$ . Combining both inequalities gives the assertion.

## 1.2 Detectors, scales, and POVMs

*The theorist's problem is to predict the probability of response of this or that detector, for a given emission procedure. Detectors are represented by positive operators  $E_\mu$ , where  $\mu$  is an arbitrary label whose sole role is to identify the detector. The probability that detector  $\mu$  be excited is  $\text{Tr}(\rho E_\mu)$ . A complete set of  $E_\mu$ , including the possibility of no detection, sums up to the unit matrix and is called a positive operator valued measure (POVM).*

*The various  $E_\mu$  do not in general commute, and therefore a detection event does not correspond to what is commonly called the "measurement of an observable". Still, the activation of a particular detector is a macroscopic, objective phenomenon. There is no uncertainty as to which detector actually clicked. [...]*

Asher Peres, 2003 [75, p.1545]

*The only form of "interpretation" of a physical theory that I find legitimate and useful is to delineate approximately the ensemble of natural phenomena the theory is supposed to describe and to construct something resembling a "structure-preserving map" from a subset of mathematical symbols used in the theory that are supposed to represent physical quantities to concrete physical objects and phenomena (or events) to be described by the theory. Once these items are clarified the theory is supposed to provide its own "interpretation".*

Jürg Fröhlich, 2019 [35, p.3]

To relate q-expectations and q-uncertainties to experimental practice we employ carefully defined rules, thereby providing a clear formal notion of measurement.

To measure aspects of a source we introduce a collection of **detector elements** labelled by labels  $k$  from a finite<sup>6</sup> set  $K$  satisfying the following postulate.

**(DRP): Detector response principle.** A detector element  $k$  responds to a stationary<sup>7</sup> source with density operator  $\rho$  with a rate  $p_k$  depending linearly on  $\rho$ . Each  $p_k$  is positive for at least one density operator  $\rho$ .

**1.1 Theorem.** *If the rates are normalized such that  $\sum p_k = 1$  then there is a unique discrete POVM  $P_k$  ( $k \in K$ ) such that*

$$p_k = \text{Tr} \rho P_k = \langle P_k \rangle \quad \text{for } k \in K. \quad (4)$$

*Proof.* For simplicity, we first assume a Hilbert space with finite dimension  $d$ ; the finiteness restriction is lifted later. By linearity, the rates satisfy

$$p_k = \sum_{i,j} P_{kji} \rho_{ij} \quad (5)$$

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<sup>6</sup>Experimentally realizable detectors always produce only a finite number of possible results; see the examples in Section 2. Idealizations violating this conditions are not discussed in this paper.

<sup>7</sup>A source is stationary if its properties of interest are time-independent. In order that a measured rate has a sensible operational meaning, the source must be reasonably stationary at least during the time measurements are taken. Assuming this allows us to ignore all dynamical issues, including the dynamical differences between isolated systems and open systems. (In particular, we proceed independently of quantum mechanical models for the measurement process itself, which involve microscopical dynamics.) For nonstationary sources, one still gets time-dependent empirical rates of limited accuracy.



for suitable complex numbers<sup>8</sup>  $P_{kji}$ . If we introduce the matrices  $P_k$  with  $(j, i)$  entries  $P_{kji}$ , (5) can be written in the concise form (4). To find the properties of the matrices  $P_k$  we first note that the  $p_k$  are rates of a stationary process. Hence they are nonnegative and sum to a constant. Since  $p_k > 0$  for at least one density operator  $\rho$ , (4) implies that all  $P_k$  are nonzero. Since  $p_k$  is real for all density operators  $\rho$ , we have

$$\mathrm{Tr} \rho P_k^* = \mathrm{Tr} (P_k \rho)^* = \overline{\mathrm{Tr} P_k \rho} = \bar{p}_k = p_k = \mathrm{Tr} \rho P_k.$$

This holds for all density operators  $\rho$ , hence  $P_k^* = P_k$ . Thus the  $P_k$  are Hermitian. Picking arbitrary pure states with  $\rho = \psi\psi^*$  shows that  $P_k$  is positive semidefinite. Summing the rates shows that the sum of the  $P_k$  is a multiple of the identity. Requiring this multiple to be 1 is conventional and amounts to a choice of units for the rates  $p_k$  in such a way that they can be interpreted as detection probabilities. Equivalently, the density operator of the source is normalized to have trace 1.

Thus, in the case where the Hilbert space of the system measured is finite-dimensional, the  $P_k$  form a discrete POVM. It can be shown (DE MUYNCK [28, p.41]) that the same holds in the infinite-dimensional case, but the argument is considerably more abstract. Instead of the matrix argument one uses the fact that each bounded linear functional on the Hilbert space of Hilbert–Schmidt operators can be represented as an inner product, resulting in  $P_k$ s with  $p_k = \mathrm{Tr} \rho P_k$ . By the above arguments they are then found to be positive semidefinite bounded Hermitian operators.  $\square$

Formula (4), derived here from very simple first principles,<sup>9</sup> is a well-known extension of von Neumann’s formulation of Born’s probability formula. It gives the theoretical q-expectations  $\langle P_k \rangle$  a statistical interpretation in terms of response probabilities of a quantum detector.

Note that there is a dual result by BUSH [15] that assumes properties of states in terms of POVMs to prove the existence of a corresponding density operator  $\rho$  satisfying (4).

### 1.3 What is measured?

*One would naturally like to know what is being measured in a measurement.*

Jos Uffink, 1994 [85, p.205]

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<sup>8</sup>These numbers can be found operationally by approximately measuring the rates for at least  $d^2$  density operators  $\rho$  spanning the space of  $d \times d$  matrices and solving the resulting semidefinite linear least squares problem for the coefficients. This process is essential for calibrating detectors and is called **quantum detection tomography**; see, e.g., D’ARIANO et al. [26] or LUNDEEN et al. [60]. Of course, to do this one needs sources with known density operator. In optical applications, textbook quantum optics is used for these.

<sup>9</sup>The usual practice is to assume Born’s rule for projective measurements as a basic premise. Then the POVM setting is postulated and justified in terms of Born’s rule in an artificial extended Hilbert space defined using an appropriate ancilla. This justification is based on Naimark’s theorem (NAIMARK [63]) – also called Neumark’s theorem, using a different transliteration of the Russian originator.

*In a statistical description of nature only expectation values or correlations are observable.*

Christof Wetterich, 1997 [87]

The detection events are usually encoded numerically. A **scale** is an assignment of distinct complex numbers or vectors  $a_k$  to the possible detection elements  $k$ . In concrete settings, the value assigned by the scale to the  $k$ th detection event is whatever has been written on the scale the pointer registering an event points to, or whatever has been programmed to be written by an automatic digital recording device. A **quantum detector**<sup>10</sup> (in the following simply called a **detector**) is defined as a finite collection  $K$  of detection elements  $k \in K$  of which at most one responds at any given time, defining a stochastic process of events, together with a scale  $a_k$  ( $k \in K$ ). The POVM part of a detector description makes no claim about which values are measured.<sup>11</sup> It just says that one of the detector elements making up the detection device responds with a probability given by the trace formula. The quantum effects are in the response of the detector elements, not in the scale used to interpret the responses numerically. The value assigned to the  $k$ th detection event is a purely classical convention, and can be any number  $a_k$  – whatever has been written on the scale the pointer points to, or whatever has been programmed to be written by an automatic digital recording device.

The results of a detector in a sequence of repeated events define a random variable or random vector  $a_k$  (with a dummy index  $k$ ) that allows us to define the **statistical expectation**

$$\mathbf{E}(f(a_k)) := \sum_{k \in K} p_k f(a_k) \quad (6)$$

of any function  $f(a_k)$ . This statistical expectation is operationally approximated by finite sample means of  $f(a)$ , where  $a$  ranges over a sequence of actually measured values. However, the statistical expectation is the usually employed abstraction of this that works with a probabilistic limit of arbitrarily many measured values, so that the replacement of relative sample frequencies by probabilities is justified. Clearly,  $\mathbf{E}$  is linear in its argument. If we

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<sup>10</sup>This may be considered as a technically precise version of the informal notion of an **observer** that figures prominently in the foundations of quantum mechanics. It removes from the latter term all anthropomorphic connotations.

<sup>11</sup> Here we essentially follow the view of SCHROECK [81, 82]. It is slightly different from the more traditional point of view as presented, e.g., in BUSCH et al. [16, 21] or HOLEVO [43, 44, 45]. There the POVM is an appropriate family of positive operators  $\Pi(\Delta)$ , where  $\Delta$  ranges over the subsets of  $\mathbb{C}^m$  (or even only of  $\mathbb{R}$ ). In the notation of the current setting, their POVM is given by

$$\Pi(\Delta) := \sum_{k \in K: a_k \in \Delta} P_k.$$

Since this implies  $\Pi(\{\xi\}) = P_k$  if  $\xi = a_k$  and  $\Pi(\{\xi\}) = \emptyset$  otherwise, the traditional POVM encodes both the detector elements and the scale, and hence fully specifies the detector.

Their terminology when restricted to the commutative case (representing classical physics in the Hilbert space formulation by KOOPMAN [52]) amounts to treating all real random variables as observables (Busch terminology) or generalized observables (Holevo terminology). But this is not the view of classical metrology, where (cf. RABINOWITZ [78] and Subsection 3.4 below) classical observables always have a true value determined by the theoretical description, and all randomness in measurements is assumed to be due to noise in the measurement.

introduce for any family  $x_k$  ( $k \in K$ ) the operators

$$P[x_k] := \sum_{k \in K} x_k P_k, \quad (7)$$

so that in particular,

$$P[1] = 1,$$

we may use (4) to write (6) as

$$\mathbf{E}(f(a_k)) = \langle P[f(a_k)] \rangle. \quad (8)$$

We say that a detector defined by the POVM  $P_k$  ( $k \in K$ ) and the scale  $a_k$  ( $k \in K$ ) **measures the quantity**<sup>12</sup>

$$A := P[a_k] = \sum a_k P_k. \quad (9)$$

When the scale consists of real numbers only, the operator corresponding to the measurement is Hermitian. From (4) and (9) we find the formula

$$\mathbf{E}(a_k) = \text{Tr } \rho A = \langle A \rangle \quad (10)$$

for the statistical expectation of the measurement results  $a_k$  obtained from a source with density operator  $\rho$ . Comparing with (1), we see that the statistical expectation of measurement results coincides with the theoretical q-expectation of  $A$  evaluated in the state  $\langle \cdot \rangle$  of the source. This is **Born's rule in expectation form**, in the context of measurements<sup>13</sup> first stated by VON NEUMANN [72, p.255]. Born's rule gives the purely theoretical notion of a q-expectation a statistical interpretation in terms of expectations of measurement results of a quantum detector.

## 1.4 Informationally complete POVMs

*In this new approach we have a nonuniqueness in places which the old theory accepted as physically significant. This means that, for self-adjoint operators with a well-defined physical meaning like spin, energy, position, momentum, etc., we have now many mathematical formulas (many nonorthogonal resolutions of the identity). The natural question arises: what this means, and how to remove such an ambiguity. Awareness of this nonuniqueness existed in the early papers, but the question was not worked out.*

Marian Grabowski, 1989 [38, p.925]

*This would mean that one can measure all observables of a system in a single experiment, merely by relabeling the outcomes. This would, indeed, offer a radical new solution to the joint measurement problem.*

Jos Uffink, 1994 [85, p.207]

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<sup>12</sup>In traditional terminology (e.g., SCHROECK [81], DE MUYNCK [28, p.360]), one would say that the detector measures the observable represented by the scalar or vector operator  $A$ . Since there is a tradition for using the word 'observable' synonymous with the POVM in the form mentioned in Footnote <sup>11</sup>, and since there are lots of observables – such as spectral widths or intensities – that cannot be represented in this way, we use a more neutral terminology.

<sup>13</sup>The first published statement of this kind seems to be in the paper by LANDAU [55, (4a),(5)], but without any reference to measurement. For a detailed history of the various forms of Born's rule, see NEUMAIER [64, Section 3], [68, Chapter 14].

If the Hilbert space  $\mathbb{H}$  has finite dimension  $d$  and there are  $|K| > d^2$  detector elements, there is a nontrivial relation

$$\sum_{k \in K} \alpha_k P_k = 0$$

with real coefficients  $\alpha_k$  that do not all vanish. Then (7) implies  $P[a_k + \alpha_k \xi] = P[a_k] = A$  for all  $\xi \in \mathbb{R}^m$  (or even  $\xi \in \mathbb{C}^m$ ). Therefore the scale is not uniquely determined by the detector elements and the quantity  $A$  measured.

On the other hand, a POVM  $P_k$  ( $k \in K$ ) is called **informationally complete** if the  $P_k$  span the real vector space  $\mathcal{P}$  of Hermitian operators with finite trace. This is possible only when the Hilbert space  $\mathbb{H}$  has finite dimension  $d$  and then requires  $|K| \geq \dim \mathcal{P} = d^2$  detector elements. By choosing the scale appropriately, an informationally complete POVM allows the measurement of arbitrary vector quantities  $A$  since, by definition, equation (9) can be solved componentwise for the components of the  $a_k$ . A **minimal** informationally complete POVM has  $|K| = d^2$ ; then the POVM and the quantity  $A$  measured uniquely determine the scale.

If the Hilbert space has finite dimension  $d$ , the knowledge of the probabilities  $p_k$  of an informationally complete POVM determines the associated state and hence its density operator. The density operator can be found operationally by approximately measuring the rates and then solving an associated semidefinite linear least squares problem for  $\rho$ . This process is essential for calibrating sources and is called **quantum state tomography**; see, e.g., JEŽEK et al. [48].

## 2 Examples

*The very meaning of the concepts involved – the concept of a simultaneous measurement of position and momentum, and the concept of experimental accuracy – continues to be the subject of discussion. [...] Ordinary laboratory practice depends on the assumption that it is possible to make simultaneous, imperfectly accurate determinations of the position and momentum of macroscopic objects.*

Marcus Appleby, 1998 [7]

Part of the challenge of experimental physics is to devise appropriate preparation and measurement protocols in such a way that experiments with desired properties are possible. Often this is the most difficult aspect of an experiment. On the other hand, it is also a difficult task to work out theoretically, i.e., in terms of statistical mechanics rather than quantum tomography, the right POVM for a given experiment described in terms of an – even idealized – microscopic model; see, e.g., BREUER & PETRUCCIONE [13] and ALLAHVERDYAN et al. [2]. However, on an informal level, it is easy to give numerous relevant examples of detectors in the sense defined above.

## 2.1 Polarization state measurements

In continuation of the introductory example we show here how POVMs model simultaneous measurements of the polarization state  $\rho$  of an optical source; cf. BRANDT [12]. The idea is to split the primary beam emanating from the source by means of a number of beam splitters into a finite number of secondary beams labelled by a label set  $K$ , passing the  $k$ th secondary beam through a filter and detecting individual responses at the  $k$ th detector element, defined as the screen at which the  $k$ th secondary beam ends. If all filters are linear and non-mixing (non-polarizing), passing the  $k$ th filter is described by a mapping  $\rho \rightarrow T_k \rho T_k^*$ , where  $T_k$  is a complex  $2 \times 2$  matrix, the **Jones matrix** (JONES [49]) of the  $k$ th filter. Jones matrices for relevant filters include  $T = \gamma I$  with  $0 < \gamma < 1$  (representing simple attenuation) and  $T = \phi \phi^*$  with a normalized vector  $\phi \in \mathbb{C}^2$  (representing complete polarization in a complex direction  $\phi$ ). We also introduce a null detector 0 accounting for the part of the beams absorbed at the filters, assumed to respond whenever we expect an event but none of the screens shows one. If we assume that the beam splitters are lossless and that the filters are not perfect (so that the Jones matrices are nonsingular), it is not difficult to see that this setting defines a detector with POVM given by

$$P_k = c_k T_k^* T_k, \quad P_0 = 1 - \sum_k P_k$$

for certain constants  $c_k > 0$  depending on the splitting arrangement.

For multiphoton states, the state space and hence the collection of possible filters is much bigger, but using linear quantum-optical networks (LEONHARDT & NEUMAIER [58]) in place of simple filters one can in essentially the same way design detectors with given POVMs.

## 2.2 Joint measurements of noncommuting quantities

Joint measurements of position and momentum are often described in terms of a POVM built from coherent states. An idealized joint measurement of position and momentum was described by a coherent state POVM in ARTHURS & KELLY [8], using infinitely many projectors  $|\alpha\rangle\langle\alpha|$  to all possible coherent states  $|\alpha\rangle$ , where  $\alpha$  is a complex phase space variable. By discretizing this using a **partition of unity** (i.e., a collection of finitely many smooth nonnegative functions  $e_k$  on phase space summing to 1), these projectors can be grouped into finitely many positive operators

$$P_k := \pi^{-1} \int d\alpha e_k(\alpha) |\alpha\rangle\langle\alpha|$$

corresponding to finite resolution measurements, making it look more realistic. This would be suitable as a simple analytic example for presentation in a course. But to check how accurate an actual joint measurement of position and momentum fits this construction for some particular partition of unity would be a matter of quantum tomography!

## 2.3 Realistic measurements of position

In textbooks one often finds idealized hypothetical measurements of operators with a continuous spectrum, for example position operators. Realistic position measurements have limited accuracy and range only, and no sharp boundaries between the position ranges where a particular detector element responds. This can be modeled by POVMs based on a partition of unity on configuration space, analogous to the above construction for coherent states. Details are given in ALI & EMCH [4].

## 2.4 Measuring particle tracks

In experimental practice, measurement is often a fairly complex procedure – far more complex than the idealized statement of Born’s rule would suggest. It involves not just reading a pointer but making a model of the situation at hand, and often involves nontrivial calculations from raw observations and the model description of the quantities that count as measurement results.

As a more complicated, concrete example close to experimental practice we consider a reasonably realistic version of a time projection chamber<sup>14</sup> (TPC) for the measurement of properties of particle tracks. Here obtaining the measurement results requires a significant amount of nontrivial computation, not just pointer readings.

In a TPC, what emanates from the source measured passes an arrangement of wires arranged in  $L$  layers of  $w$  wires each and generates electric current signals, ideally exactly one signal per layer. From these signals, time stamps and positions are computed by a least squares process (via the Kalman filter), assuming that the track is a helix. This is the case for a charged particle in a constant magnetic field, experiencing energy loss in the chamber due to the induced ionizations. From the classical tracks reconstructed by least squares, the momentum is computed in a classical way.

The detector can be described by a POVM with an operator for each of the  $w^L$  possible signal patterns. The value assignment is done by a nontrivial computer program for the least squares analysis and initially produces a whole particle track. Part of the information gathered is discarded; one typically records the computed energy, position and velocity (or momentum if the mass is known). Momentum and energy are the quantities of interest for scattering, but for secondary decays one also needs the decay position. Thus one measures 7-dimensional phase space vectors (including 3 position coordinates, 3 momentum coordinates, and the energy) as a very complicated function of the signal patterns. The POVM operators exist by the general analysis above, though it is not easy to describe them explicitly in mathematical terms. But this is not essential for the basic description, which (according to the introductory quote by Peres) should be given in laboratory terms only.

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<sup>14</sup>The description of the STAR time projection chamber in ANDERSON et al. [5, Section 5.2] mentions only 2 layers, so one has to use linear tracks. The LHC uses more layers and a helical track finder, see AGGLETON et al. [1, Section 5].

## 2.5 Projective measurements

*The standard von Neumann description of quantum measurements applies when all the detector outcomes are well-defined and correspond to precise classical measurement values that unambiguously reflect the state of the system. [...] When measuring a quantum light state, this ideal situation corresponds, for example, to the case of a perfect photon-number detector with unit efficiency and no dark counts. Unfortunately, the quantum measurement description of von Neumann is not valid for measurement schemes that employ real detectors, which are normally affected by several imperfections.*

Zavatta and Bellini, 2012 [89]

Complementing the preceding discussion of concrete measurement arrangements, we now discuss the traditional idealized view of quantum measurements that goes back to Born, Dirac, and von Neumann.

Rather than postulating Born's rule for projective measurements, as done in standard textbooks, we derive it here together with all its ramifications and its domain of validity, from simple, easily motivated definitions. In particular, the spectral notions appear not as postulated input as in traditional expositions, but as consequences of the derivation.

We call a discrete POVM **projective** if the  $P_k$  satisfy the **orthogonality relations**

$$P_j P_k = \delta_{jk} P_k \quad \text{for } j, k \in K. \quad (11)$$

We say that a detector measuring  $A$  performs a **projective measurement** of  $A$  if its POVM is projective. Such projective measurements are unstable under imperfection in the detector. Therefore they are realistic only under special circumstances. Examples are two detection elements behind polarization filters perfectly polarizing in two orthogonal directions, or the arrangement in an ideal Stern–Gerlach experiment for spin measurement. Most measurements, and in particular all measurements of quantities with a continuous spectrum, are not projective.

The orthogonality relations imply that  $AP_k = a_k P_k$ . Since the  $P_k$  sum to 1, any  $\psi \in \mathbb{H}$  can be decomposed into a sum  $\psi = \sum P_k \psi = \sum \psi_k$  of vectors  $\psi_k := P_k \psi$  satisfying the equation  $A\psi_k = AP_k \psi = a_k P_k \psi = a_k \psi_k$ . Therefore  $\psi_k$  (if nonzero) is an eigenvector of  $A$  (or of each component of  $A$  in case the  $a_k$  are not just numbers) corresponding to the eigenvalue  $a_k$  of  $A$ . Since  $P_k^2 = P_k = P_k^*$ , the  $P_k$  are orthogonal projectors to the eigenspaces of the  $a_k$ .

When the  $a_k$  are numbers, this implies that  $A$  is an operator with a finite spectrum. Moreover,  $A$  and  $A^*$  commute, i.e.,  $A$  is a normal operator, and in case the  $a_k$  are real numbers, a Hermitian, self-adjoint operator. (This is the setting traditionally assumed from the outset.) When the  $a_k$  are not numbers, our analysis implies that the components of  $A$  are mutually commuting normal operators with a finite joint spectrum, and if all  $a_k$  have real components only, the components of  $A$  are Hermitian, self-adjoint operators. Thus the projective setting is much more limited with respect to the kind of quantities that it can represent.

For projective measurements, (9) implies

$$f(A^*, A) = \sum_k f(\bar{a}_k, a_k) P_k$$

for all functions  $f$  for which the right hand side is defined. Therefore the modified scale  $f(\bar{a}, a)$  measures  $f(A^*, A)$ , as we are accustomed from classical measurements, and defines a projective measurement of it. But when the components of  $A$  are not normal or do not commute, this relation does not hold.

From the above discussion we conclude in particular that the possible values in a projective<sup>15</sup> measurement of  $A$  are precisely the finitely many eigenvalues of  $A$  (or joint eigenvalues of the components), measured with a probability of  $p_k = \text{Tr } \rho P_k$ . This is the textbook form of **Born's rule**, valid for projective measurements of quantities represented by mutually commuting normal operators with a finite joint spectrum.

In the special case where the spectrum of  $A$  is **nondegenerate**, i.e., all eigenspaces have dimension 1, the orthogonal projectors have the special form  $P_k = \phi_k \phi_k^*$ , where  $\phi_k$  are normalized eigenstates corresponding to the eigenvalue  $a_k$ . In this case, the probabilities take the form

$$p_k = \phi_k^* \rho \phi_k.$$

If, in addition, the source is pure, described by  $\rho = \psi \psi^*$  with the normalized state vector  $\psi \in \mathbb{H}$ , this can be written in the more familiar squared amplitude form

$$p_k = |\phi_k^* \psi|^2. \tag{12}$$

In practice, the orthogonality relations (11) can be implemented only approximately (due to problems with efficiency, losses, inaccurate preparation of directions, etc.). Thus the present derivation shows that measurements satisfying Born's rule (i.e., projective measurements) are always idealizations.

Whenever one simultaneously measures quantities corresponding to noncommuting operators, Born's rule in textbook form does not apply and one needs a POVM that is not projective. The operators corresponding to most measurements discussed in Section 2 do not commute; therefore such joint measurements cannot even be formulated in the textbook setting of projective measurements.

In general, the POVM description of a real device cannot simply be postulated to consist of orthogonal projectors. The correct POVM must be found out by quantum tomography, guided by the theoretical model of the measuring equipment but ultimately just using the formula (4) for probabilities. This formula is a proper extension of Born's rule for probabilities of projective measurements. It cannot be reduced to the latter unless one adds to the description nonphysical stuff – namely imagined ancillas without a physical representation, formally constructed on the basis of Naimark's theorem (cf. Footnote <sup>9</sup>).

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<sup>15</sup>In a general measurement as discussed in Subsection 1.3, the measurement results are usually unrelated to the eigenvalues of  $A$ .



## 2.6 A modern introduction to quantum mechanics

The developments so far are suitable for an introductory course on quantum mechanics. To introduce POVMs without using the standard formulation in the usual terms of observables and states is simpler than to introduce the eigenvalue form of Born's rule in full generality. To explain the correct rule in an introduction to quantum mechanics is easier than writing down Born's rule, because one needs no discussion of the spectral theorem and of the subtle problems with self-adjointness and associated proper boundary conditions. Thus in the foundations, there is no longer an incentive for giving a special, highly idealized case in place of the real thing.

After introducing the basic kinematical framework as in Section 1 and illustrating it as in the preceding subsections, the next step is to motivate the dynamics. Again, classical optics provides the lead. A linear, non-mixing (not depolarizing) instrument (for example a polarizer or phase rotator) is characterized by a complex  $2 \times 2$  **Jones matrix**  $T$ . For example, a perfect **polarizer** has the rank 1 form  $T = \phi\phi^*$ , where  $|\phi|^2 = 1$ . The instrument transforms an in-going beam with density operator  $\rho$  into an out-going beam in the state with density operator  $\rho' = T\rho T^*$ . The intensity of a beam after passing the instrument is  $S'_0 = \text{Tr } \rho' = \text{Tr } T\rho T^* = \text{Tr } \rho T^*T$ .

Passage through an inhomogeneous medium can be modeled by means of many slices consisting of very thin instruments with Jones matrices close to the identity, hence of the form

$$T(t) = 1 + \Delta t K(t) + O(\Delta t^2), \quad (13)$$

where  $\Delta t$  is the very short time needed to pass through one slice and  $K(t)$  is an operator specifying how  $T(t)$  deviates from the identity. If  $\rho(t)$  denotes the density operator at time  $t$  then  $\rho(t + \Delta t) = T(t)\rho(t)T(t)^*$ , so that

$$\frac{d}{dt}\rho(t) = \frac{\rho(t + \Delta t) - \rho(t)}{\Delta t} + O(\Delta t) = K(t)\rho(t) + \rho(t)K(t)^* + O(\Delta t).$$

In the continuum limit  $\Delta t \rightarrow 0$  we obtain the **quantum Liouville equation**

$$\frac{d}{dt}\rho(t) = K(t)\rho(t) + \rho(t)K(t)^*. \quad (14)$$

If the instrument is lossless, the intensities of the in-going and the out-going beam are identical. This is the case if and only if the Jones matrix  $T$  is unitary. Inserting (13) into the equation  $TT^* = 1$  and comparing the coefficient of  $\Delta t$  shows that  $K(t) + K(t)^* = 0$ . Therefore the time-dependent **Hamiltonian** defined by

$$H(t) = i\hbar K(t),$$

is in the lossless case Hermitian, and the quantum Liouville equation takes the special commutator form

$$i\hbar \frac{d}{dt}\rho(t) = [H(t), \rho(t)] \quad (15)$$

of the **von Neumann equation**.

More generally, a linear, mixing (depolarizing) instrument transforms  $\rho$  instead into a sum of several terms of the form  $T\rho T^*$ . It is therefore described by a real  $4 \times 4$  **Mueller matrix** (PEREZ & OSSIKOVSKI [76]) acting on the Stokes vector. Equivalently, it is described by a completely positive linear map (KOSSAKOWSKI [53], CHOI [24]) on the space of  $2 \times 2$  matrices, acting on the polarization matrix. Repeating in this more general situation the derivation of the quantum Liouville equation in a more careful manner, taking into account second order terms, leads to the **Lindblad equation** (LINDBLAD [59]) for the general dynamics of a realistic quantum system.

For the idealized case of beams in a pure state  $\psi$ , further development is possible. For example, the perfect polarizer with  $T = \phi\phi^*$  reduces the intensity of a pure state  $\psi$  to

$$S'_0 = \langle T^*T \rangle = |\phi^*\psi|^2.$$

This is **Malus' law** from 1809 (MALUS [61]). Reinterpreted in terms of detection probabilities, this gives Born's squared amplitude formula for quantum probabilities. An instrument with Jones matrix  $T$  transforms a beam in the pure state  $\psi$  into a beam in the pure state  $\psi' = T\psi$ . If  $\psi(t)$  denotes the pure state at time  $t$  then the same slicing scenario as above gives  $\psi(t + \Delta t) = T(t)\psi(t)$ . Therefore

$$\frac{d}{dt}\psi(t) = \frac{\psi(t + \Delta t) - \psi(t)}{\Delta t} + O(\Delta t) = K(t)\psi(t) + O(\Delta t),$$

giving in the continuum limit  $\Delta t \rightarrow 0$  the **time-dependent Schrödinger equation**

$$i\hbar \frac{d}{dt}\psi(t) = H(t)\psi(t).$$

Thus a polarized quasimonochromatic beam of classical light behaves exactly like a modern quantum bit. We might say that classical ray optics in the form known already in 1852 by STOKES [83] is just the quantum physics of a single qubit passing through a medium, complete with all bells and whistles. This is discussed in some more detail in NEUMAIER [69].

It is now easy to generalize all this to the case of general quantum systems. At this point, it makes sense to go through the considerations of Sections 2.5 and 2.6 of my book (NEUMAIER [68]) to understand the role and limitations of pure states and the Schrödinger equation for general quantum systems.

Proceeding as outlined provides a fully intelligible motivation for all basic features of quantum mechanics and quantum information theory. In contrast to the usual treatments, where the basic features are addressed by just postulating the required items, usually even in a highly idealized form (for pure states and projective measurements), this gives actual understanding, not only the appearance of it.

Real systems oscillate and need an infinite-dimensional Hilbert space. Thus after the qubit, one should introduce the anharmonic oscillator, the second simplest system of fundamental importance. This shows that finite-dimensional Hilbert spaces are not enough and infinitely many dimensions (i.e., functional analysis) are needed.

Now a lot of elementary phenomena (related to boundary conditions, bound states and scattering states, tunneling) can be discussed in terms of exactly solvable problems. Here the Schrödinger equation starts to become important, as a computational tool.

Then one may introduce canonical commutation relations and Ehrenfest's theorem for  $q$ -expectations. As in Chapter 2 of NEUMAIER [68] one may derive the classical limit, where operators may be replaced by their  $q$ -expectations without introducing significant errors. A simple consequence is the **Rydberg–Ritz formula**

$$\hbar\omega = E_j - E_k \tag{16}$$

relating a discrete energy spectrum  $E_0 < E_1 < E_2 < \dots$  to observable spectral lines with frequencies  $\omega$ . The observation by DIRAC [31] that the Poisson bracket is the classical limit of the scaled commutator implies that coupled quantum oscillators are described by a tensor product of Hilbert spaces. This is easily generalized to arbitrary composite systems. At this point, the close connection between classical mechanics and quantum mechanics is established.

Now one can introduce annihilation and creation operators for the harmonic oscillator, and then for a system of  $n$  harmonic oscillators. This motivates bosonic Fock spaces over an  $n$ -dimensional Hilbert space. Proceeding to an infinite number of oscillators, one can turn to the interpretation of bosonic Fock space as the Hilbert space of an arbitrary number of indistinguishable particles. One can then play with the construction and look at fermionic Fock spaces. This is the Hilbert space of  $n$  qubits, important for quantum information theory.

Then one may raise curiosity about Fock spaces over infinite-dimensional Hilbert spaces and relate them to quantum fields and systems of arbitrarily many free particles.

The canonical commutation relations and the Poisson bracket provide first examples of the use of Lie algebras in quantum physics. From there it is only a small step to other important Lie algebras. In particular, one can restrict the Poisson bracket to rigid bodies and obtains a Lie algebra  $so(3)$  whose Lie product is the vector product in 3 dimensions, and whose generators are the components of angular momentum. It is also the Lie algebra  $su(2)$  of the Hermitian quantities with zero trace for the qubit, allowing one to present the germs of representation theory.

The next step would be to discuss approximation methods and scattering theory, but this is already beyond the foundation.

### 3 Measurement uncertainty

*Some hypotheses are dangerous, first and foremost those which are tacit and unconscious. And since we make them without knowing them, we cannot get rid of them. Here again, there is a service that mathematical physics may render us. By the precision which is its characteristic, we are compelled to formulate all the hypotheses that we would unhesitatingly make without its aid.*

Henri Poincaré, 1902 [77, p.151]

This section gives a thorough, precise discussion of various aspects of uncertainty in quantum measurements. There are natural links to the thermal interpretation of quantum physics as defined in my book (NEUMAIER [68]). Subsections 3.3–3.6 use material from Sections 10.6-10.7 of this book.

#### 3.1 Statistical uncertainty

*Aus diesen Gründen ist eine gleichzeitige genaue Beobachtung von  $q$  und  $p$  prinzipiell ausgeschlossen. [...] Man kann aber auch beide Größen in einer einzigen Beobachtung messen, also wohl gleichzeitig, aber nur mit beschränkter Genauigkeit. Bei einer solchen Beobachtung fragt man in der klassischen Theorie nach dem 'Fehler' des gemessenen Wertes. [...] Die 'Beobachtungsfehler' erscheinen in der neuen Theorie als mit der statistischen Unbestimmtheit selbst zusammenschmolzen.*

Earle Kennard 1927 [51, p.340f]

*Results of measurements cannot be absolutely accurate. This unavoidable imperfection of measurements is expressed in their inaccuracy.*

Semyon Rabinovich, 2005 ([78, p.2])

We write  $|x| := \sqrt{x^*x}$  for the Euclidean norm of a vector  $x \in \mathbb{C}^m$ , and generalize it to vectors  $A \in (\text{Lin } \mathbb{H})^m$  with operator components by defining<sup>16</sup> the operator

$$|A| := \sqrt{A^*A}.$$

This allows us to formulate and prove the basic inequality

$$\min_{\xi \in \mathbb{C}^m} \mathbf{E}(|a_k - \xi|^2) = \mathbf{E}(|a_k - \bar{A}|^2) \geq \langle |A - \bar{A}|^2 \rangle = \sigma_A^2 \quad (17)$$

bounding the statistical uncertainty of the measurement results  $a_k$  in terms of the theoretical q-uncertainty of the quantity  $A$  measured. This result, for  $m = 1$  due to HOLEVO [43, (9.8), p.88] and later redicovered by DE MUYNCK & KOELMAN [29] and WERNER [86, Proposition 3(2)], follows by observing that

$$\mathbf{E}(|a_k - \xi|^2) - \mathbf{E}(|a_k - \bar{A}|^2) = \mathbf{E}(|a_k - \xi|^2 - |a_k - \bar{A}|^2) = \mathbf{E}(|\bar{A} - \xi|^2) = |\bar{A} - \xi|^2 \quad (18)$$

is minimal for  $\xi = \bar{A}$ . Now the following proposition (cf. HOLEVO [42, Lemma 13.1] for the case  $m = 1$ ) applies.

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<sup>16</sup>Actually we never need  $|A|$  but only the notation  $|A|^2$  for  $A^*A$ , especially when  $A$  is some composite formula.

**3.1 Proposition.** Given a detector measuring  $A = P[a_k] \in \mathbb{H}^m$ . Then:

(i) For every Hermitian positive semidefinite  $G \in \mathbb{C}^{m \times m}$ , the operator  $P[a_k^* G a_k] - A^* G A$  is positive semidefinite.

(ii) For any  $\xi \in \mathbb{C}^m$  and any state  $\langle \cdot \rangle$ ,

$$\mathbf{E}(|a_k - \xi|^2) \geq \langle |A - \xi|^2 \rangle. \quad (19)$$

*Proof.* Using the POVM  $P_k$  ( $k \in K$ ) of the detector, we write

$$\Delta := \sum_{k \in K} (A - a_k)^* G P_k (A - a_k).$$

(i) For any  $\psi \in \mathbb{H}$ , we define the vectors  $\psi_k := (A - a_k)\psi$  and find  $\psi^* N \psi = \sum \psi_k^* G P_k \psi_k \geq 0$ . Hence  $\Delta$  is positive semidefinite. Since

$$\begin{aligned} \Delta &= \sum \left( A^* G P_k A - a_k^* G P_k A - A^* G P_k a_k + a_k^* G P_k a_k \right) \\ &= A^* G P[1] A - A^* G P[a_k] - P[a_k]^* G A + P[a_k^* G a_k] \\ &= A^* G A - A^* G A - A^* G A + P[a_k^* G a_k] = P[a_k^* G a_k] - A^* G A, \end{aligned}$$

the first part follows.

(ii) In the special case where  $G$  is the identity matrix, we find from (8) that

$$\mathbf{E}(|a_k|^2) = \langle P[|a_k|^2] \rangle = \langle P[a_k^* a_k] \rangle = \langle N + A^* A \rangle = \langle \Delta \rangle + \langle A^* A \rangle \geq \langle A^* A \rangle = \langle |A|^2 \rangle.$$

Since  $P[a_k - \xi] = P[a_k] - \xi = A - \xi$  we may apply this with  $a_k - \xi$  in place of  $a_k$  and  $A - \xi$  in place of  $A$  and find (19).  $\square$

In the special case of projective measurements,  $A = P[a_k]$  satisfies

$$P_k(A - a_k) = P_k A - a_k P_k = \sum_j a_j P_k P_j - a_k P_k = 0,$$

so that  $\Delta := 0$  in the proof of Proposition (3.1). Therefore inequality (17) holds in this case with equality for all states. A converse was proved in KRUSZYŃSKI & DE MUYNCK [54, Proposition 2]. Thus the difference in (17) describes the lack of projectivity. We may view it as a measure of quality of a real detector, as opposed to an idealized projective one. BUSCH et al. [17, (16)] view this difference as a measure of intrinsic noise.

## 3.2 Imperfect measurements

*The fact that actual measurements are always imprecise is well-known and led Poincare to distinguish carefully the "mathematical continuum" from the "physical continuum." In the mathematical continuum the notion of identity satisfies the usual transitivity condition [...] By contrast, this property cannot be assumed for the notion of "indistinguishability" in the physical continuum attached to the raw data of experiments. [...] To pass from the physical continuum to the mathematical continuum requires an idealization, namely that infinitely precise measurements are in principle, if not in fact, attainable.*

Ali and Emch, 1974 [4, p.1545]

*The discrete nature of the reading scale entails that a given measuring apparatus allows only a measurement of a discrete version of the observable under consideration. With this we do not, however, deny the operational relevance of continuous observables. On the contrary, their usefulness as idealisations shows itself in the fact that they represent the possibility of indefinitely increasing the accuracy of measurements by choosing increasingly refined reading scales.*

Busch, Lahti and Mittelstaedt, 1996 ([20, p.81])

The spectral norm of a quantity measured by an arbitrary detector is bounded since the sum (9) is finite and  $\|P_k\| \leq 1$  for all  $k$ . Hence the components of  $A$  are bounded linear operators.<sup>17</sup> In particular, (10) is defined for all density operators  $\rho$ . Note that the same bounded linear operator  $A$  can be decomposed in many ways into a linear combination of the form (9); thus there may be many different detectors with different scales measuring quantities corresponding to the same operator  $A$ .

In practice, one is often interested in measuring a given (bounded or unbounded) operator  $X$  of interest. Designing realistic detectors that allow a high quality measurement corresponding to theoretically important operators is the challenge of high precision experimental physics. Due to experimental limitations, this generally involves both statistical and systematic errors.

If we approximate a (possibly vector-valued) quantity  $X$  by a measurable substitute quantity  $A$  – some such approximation is unavoidable in practice –, we make a systematic error that may depend on the state of the system measured. (18) implies the formula

$$\mathbf{E}(|a_k - \bar{X}|^2) = \mathbf{E}(|a_k - \bar{A}|^2) + |\bar{A} - \bar{X}|^2 \quad (20)$$

for the mean squared error of  $a_k$  as an approximation of  $\bar{X}$ . In particular, the term

$$\Delta := |\bar{A} - \bar{X}| = |\langle X \rangle - \langle A \rangle|,$$

the **bias** due to the substitution of  $A$  for  $X$ , is a lower bound for the **root mean squared error (RMSE)**

$$\varepsilon_X := \sqrt{\mathbf{E}(|a_k - \bar{X}|^2)}.$$

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<sup>17</sup>To extend the notion of measurement to unbounded quantities such as position or momentum – which is outside the scope of the present paper –, one would need to proceed in an idealized fashion, using continuous POVMs for idealized measurements with infinite precision. Then q-expectations are defined only for sufficiently regular density operators. For a proper treatment of the unbounded case see, e.g., the books mentioned at the beginning of this paper.

Unlike the RMSE but like q-expectations, the bias is a theoretical property of a state, independent of measurement. We say that  $A$  is an **unbiased** approximation of  $X$  in all states such that the bias vanishes. If the bias vanishes in all states from an open subset of the set of all states, we necessarily have  $X = A$ . In particular, there are no everywhere unbiased approximations of unbounded operators.<sup>18</sup> Note that in practice,  $\bar{A} \neq \bar{X}$  due to imperfections. In particular, unbiased measurements are necessarily idealizations.

The RMSE  $\varepsilon_X$  measures the uncertainty in the value assigned to  $\bar{X}$ . But because of the broken watch effect,<sup>19</sup> it cannot be regarded as the measured uncertainty in the value assigned to  $X$ . Thus we need to add a systematic uncertainty correction that corrects for the possibility that the uncertainty of  $A$  is less than the uncertainty of  $X$ . The latter has nothing to do with measurement and hence must be a theoretical quantity computable from q-uncertainties. Now

$$\langle |A - \bar{X}|^2 \rangle = \langle |A - \bar{A}|^2 \rangle + |\bar{A} - \bar{X}|^2 = \sigma_A^2 + |\bar{A} - \bar{X}|^2 \quad (21)$$

in analogy to (20), proved by expanding all squares. Comparing this with the formula

$$\langle |X - \bar{A}|^2 \rangle = \langle |X - \bar{X}|^2 \rangle + |\bar{X} - \bar{A}|^2 = \sigma_X^2 + |\bar{X} - \bar{A}|^2$$

obtained by interchanging the role of  $A$  and  $X$ , we see that the natural uncertainty correction to the mean squared error is  $(\sigma_X^2 - \sigma_A^2)_+$ , where  $x_+ := \max(x, 0)$  denotes the positive part of a real number  $x$ . We therefore regard

$$\Delta_X[a_k] := \sqrt{\mathbf{E}(|a_k - \bar{X}|^2) + (\sigma_X^2 - \sigma_A^2)_+}, \quad (22)$$

the square root of the corrected mean squared error, as the **measurement uncertainty** when measurements of  $A$  are performed in place of measurements of  $X$ . Such **imperfect measurements** make sense even for unbounded quantities  $X$  in states with finite q-uncertainty  $\sigma_X$ . By (19) and (21),

$$\begin{aligned} \Delta_X[a_k]^2 &= \mathbf{E}(|a_k - \bar{X}|^2) + (\sigma_X^2 - \sigma_A^2)_+ \\ &\geq \langle |A - \bar{X}|^2 \rangle + \sigma_X^2 - \sigma_A^2 \\ &= \sigma_X^2 + |\bar{A} - \bar{X}|^2 \end{aligned}$$

hence the measurement uncertainty is bounded from below by theoretical error measures,

$$\Delta_X(a_k) \geq \sqrt{\sigma_X^2 + |\bar{A} - \bar{X}|^2} = \sqrt{\langle |X - \bar{A}|^2 \rangle}.$$

---

<sup>18</sup>For unbounded operators  $A$  (for example the position operator vector  $q$  in a particular coordinate system), formula (10) and hence the bias is well-defined only for sufficiently regular density operators  $\rho$ . This reflects a slight deficiency of the standard textbook presentation of expectations. However, measurement equipment for unbounded operators such as position, say, only produces results in a bounded range. Hence it corresponds in fact to a measurement device for a clipped version  $X = F(q)$  of the position operator  $q$  with bounded  $F$ , resulting in a bounded  $X$ .

<sup>19</sup>Measuring time with a broken watch shows twice a day the exact time, whereas a watch that is slow 1 second per day shows the correct time at most once in a century.

In particular,  $\Delta_X(a_k)$  is always at least as large as the q-uncertainty of  $X$ , and larger if there is a nonzero bias. This gives an operational interpretation to these theoretical terms.

By changing the scale of a detector we may define measurements of many different quantities  $A$  based on the same POVM. By picking the scale carefully one can in many cases choose it such that  $A$  approximates a particular operator  $X$  of interest with small measurement uncertainty  $\Delta_X[a_k]$  for the collection of states of interest. There are several alternative ways to quantify what constitutes adequate approximations; see, e.g., APPLEBY [7, 6], BARCIELLI et al. [9], BUSCH et al. [17, 21], and references there.

Finding a good match of  $A$  and  $X$  by choosing a good scale  $a_k$  is the process called **tuning**. It corresponds to the classical situation of labeling the scale of a meter to optimally match a desired quantity. If the detector can also be tuned by adjusting parameters  $\theta$  affecting its responses, the operators  $P_k = P_k(\theta)$  depend on these parameters, giving

$$A = A(\theta) = \sum a_k P_k(\theta).$$

Now both the labels  $a_k$  and the parameters  $\theta$  can be tuned to improve the accuracy with which the desired  $X$  is approximated by  $A(\theta)$ , perfecting the tuning.

### 3.3 Reproducibility

*A student has read such and such a number on his thermometer. He has taken no precautions. It does not matter; he has read it, and if it is only the fact which counts, this is a reality [...] Experiment only gives us a certain number of isolated points. They must be connected by a continuous line, and this is a true generalisation. But more is done. The curve thus traced will pass between and near the points observed; it will not pass through the points themselves. Thus we are not restricted to generalising our experiment, we correct it; and the physicist who would abstain from these corrections, and really content himself with experiment pure and simple, would be compelled to enunciate very extraordinary laws indeed.*

Henri Poincaré, 1902 [77, p.142f]

*The purpose of measurements is the determination of properties of the physical system under investigation.*

Busch, Lahti and Mittelstaedt, 1996 ([20, p.25])

*Let us emphasize again that when a qubit is measured, it only ever gives 0 or 1 as the measurement result – probabilistically.*

Nielsen and Chuang, 2001 ([73, p.14])

*Measurements are regarded metrologically to be better the lower their uncertainty is. However, measurements must be reproducible, because otherwise they lose their objective character and therefore become meaningless.*

Semyon Rabinovich, 2005 ([78, p.22])



*In a fundamentally statistical theory like quantum mechanics the results of individual measurements tell us almost nothing: It is always the probability distribution of outcomes for a fixed experimental arrangement which can properly be called the result of an experiment.*

Busch, Lahti and Werner 2014 [22, p.5]

*The Born measure is a mathematical construction; what is its relationship to experiment? This relationship must be the source of the (alleged) randomness of quantum mechanics, for the Schrödinger equation is deterministic.*

Klaas Landsman, 2019 [57, p.20]

Let us consider the measurement of a quantity  $A \in \mathbb{C}^{2 \times 2}$  of an arbitrary 2-state system (a qubit). According to the experimental record, the response of a sufficiently good detector produces measurement results concentrated near two spots (or parallel lines) of the detector, just as what one gets when measuring a classical diffusion process in a double-well potential (see, e.g., HONGLER & ZHENG [46]). For example, this happens in the original Stern-Gerlach experiment; cf. the quote of BUSCH et al. [16] at the beginning of Section 1. This results in a bimodal distribution with two more or less sharp peaks, with details depending on the detection method used and its resolution.

In a frequently used idealization – e.g., in the typical textbook treatment of a spin measurement – one ignores the limited efficiency of a detector. Then the distribution may even be assumed to be 2-valued, with measurement results that take only one of two values  $\lambda'_1$  and  $\lambda'_2$ , corresponding to the two modes of the bimodal distribution.

In the standard formulation of **Born's statistical interpretation** of quantum mechanics, based on projective measurements, the measurement results are **quantized**: the measured result will be one of the eigenvalues  $\lambda_k$  of  $A$ . Multiple repetition of the measurement results in a random sequence of values  $\lambda_k$ , with probabilities computed from (12) if the system is in a pure state. In the limit of arbitrarily many repetitions, the mean value of this sequence approaches  $\bar{A}$  and the standard deviation approaches  $\sigma_A$ .

Returning to the qubit case, we assume that  $A$  has unknown but distinct eigenvalues  $\lambda_1, \lambda_2$ . The q-expectation and the q-uncertainty of  $A$  can be exactly calculated in terms of the probability  $p = p_1$ . We assume for simplicity that the system is in a pure state  $\alpha_1|\lambda_1\rangle + \alpha_2|\lambda_2\rangle$ , where the kets denote the eigenstates of  $A$  and  $|\alpha_1|^2 = p$ ,  $|\alpha_2|^2 = 1 - p$ . Then q-expectation and q-uncertainty are found to be

$$\bar{A} = p\lambda_1 + (1 - p)\lambda_2, \quad \sigma_A = |\lambda_1 - \lambda_2|\sqrt{p(1 - p)}.$$

The prediction made by Born's rule is that the observed bimodal distribution has point support at the nodes  $\lambda'_1 = \lambda_1$  and  $\lambda'_2 = \lambda_2$ . Clearly, Born's rule only describes idealized<sup>20</sup>

<sup>20</sup> Originally, Born's statistical interpretation was stated only for energy measurements for systems with discrete energy levels. Thus the measured quantity is the Hamiltonian  $H$ , and its eigenvalues are in general irrational. A measurement according to Born's rule in its standard form would produce these irrational numbers exactly. This is clearly not the case. Thus one is forced to use a more liberal reading of Born's rule, where some additional measurement error is acceptable. This means that Born's rule is no longer about real measurements but about idealized measurements whose observations are theoretical numbers, not actual results. Therefore, in this liberal reading, Born's rule is a purely theoretical construct, silent about actual measurement results.

measurement values, the eigenvalues  $\lambda_k$  of  $A$ . The deviations from the eigenvalues, e.g. in realistic Stern–Gerlach experiments, must be interpreted as measurement errors.

As an illustration we consider some piece of digital equipment with 3 digit display measuring some physical quantity  $A$  using  $N$  independent measurements. Suppose the measurement results were 6.57 in 20% of the cases and 6.58 in 80% of the cases. Every engineer or physicist would compute the mean  $\bar{A} = 6.578$ , the variance

$$\sigma_X^2 = \langle X^2 \rangle - \langle X \rangle^2 = 0.2 \cdot 0.008^2 + 0.8 \cdot 0.002^2 = 0.004^2,$$

and the standard deviation  $\sigma_A = 0.004$ , concluding that the true value of the quantity  $A$  deviates from 6.578 by an error of the order of  $0.004N^{-1/2}$ . Note that, as always when one measures something with a digital device, the error distribution is discrete, not Gaussian.

Now we consider the measurement of the Hermitian quantity  $A \in \mathbb{C}^{2 \times 2}$  of a 2-state quantum system in the pure up state, using  $N$  independent measurements, and suppose that we obtain exactly the same results. Now Born’s statistical interpretation proceeds differently and claims that there is no measurement error. Instead, each measurement result reveals one of the eigenvalues  $x_1 = 6.57$  or  $x_2 = 6.58$  in an unpredictable fashion with probabilities  $p = 0.2$  and  $1 - p = 0.8$ , up to statistical errors of order  $O(N^{-1/2})$ . In particular, the measurement results are not reproducible; only their statistic is.

For  $A = \begin{pmatrix} 6.578 & 0.004 \\ 0.004 & 6.572 \end{pmatrix}$ , both the engineering view and Born’s interpretation of the results for the 2-state quantum system are consistent with the data. However, Born’s statistical interpretation deviates radically from engineering practice, without any apparent necessity. It does not even conform to the notion of a measurement in the traditional sense since an essential element in the latter’s specification – the reproducibility of the result – is not guaranteed. Shouldn’t we rather proceed as before and draw the same conclusions as the engineer?

According to our discussion in Section 1 in a realistic measurement, the possible values obtained when measuring a particular quantity  $A$  depend on the decomposition  $A = P[a_k]$  used to construct the scale. That this decomposition is ambiguous follows from Subsection 1.4, where we saw that the scale is not determined by the quantity  $A$  measured. Since the scale determines the measurement results, this means that one can with equal right ascribe different results to the measurement of the same quantity  $A$ . Thus, in general, different detectors measuring the same quantity  $A$  have different sets of possible measurement results. In particular, the approach introduced in this paper gives projective measurements (and hence eigenvalues) no longer a special status. The same quantity  $A$  can be measured by detectors with different mathematical characteristics and in particular different measurement results that generally have nothing to do with the eigenvalues of  $A$ .

We conclude that not the individual observations but only their statistical properties – POVM probabilities and expectation values – are reproducible, hence the latter (and only these) correspond to objective properties of the source measured. Thus realistic rules for

measurement make the eigenvalue link of the traditional interpretation of quantum mechanics look artificial.

### 3.4 Measurement errors

*Von dem neuen Standpunkt schwebt nun zunächst dieser Fehlerbegriff in der Luft, denn er setzt doch nicht nur den Begriff eines beobachteten, sondern auch den Begriff eines 'wahren' Wertes voraus, und letzteren gibt es im physikalischen Sinne nicht mehr.*

Earle Kennard 1927 [51, p.340]

*This mean value  $x_0$  locates the wave packet in the crude sense that an observing apparatus must be placed near  $x_0$  if it is to have a significant chance of interacting with the particle.*

Carl Helstrom, 1974 [40, p.454]

*Measurement is the process of determinating the value of a physical quantity experimentally with the help of special technical means called measuring instruments. The value of a physical quantity [...] is found as the result of a measurement. The true value of a measurand is the value of the measured physical quantity, which, being known, would ideally reflect, both qualitatively and quantitatively, the corresponding property of the object.*

Semyon Rabinovich, 2005 ([78, p.1f])

*Measurement errors are in principle unavoidable, because a measurement is an experimental procedure and the true value of the measurable quantity is an abstract concept.*

Semyon Rabinovich, 2005 ([78, p.11])

Measurement errors are ubiquitous in physical practice; their definition requires, however, some care. A single measurement produces a number, the **measurement result**. The splitting of the measurement result into the sum of an intended result – the **true value** – and a **measurement error** (the deviation from it) depends on what one declares to be the true value. Thus what can be said about measurement errors depends on what one regards as the true value of something measured.

In general, the true value is necessarily a theoretical construct, an idealization arrived at by convention. Since measured are only actual results, never the hypothesized true values, there is no way to determine experimentally which convention is the right one. Both the quantum formalism and the experimental record are independent of what one declares to be the true value of a measurement. Different conventions only define different ways of bookkeeping, i.e., different ways of splitting the same actual measurement results into a sum of true values and errors, in the communication about quantum predictions and experiments. Nothing in the bookkeeping changes the predictions and the level of their agreement with experiment.

Thus the convention specifying what to consider as true values is entirely a matter of choice, an **interpretation**. The convention one chooses determines what one ends up with, and

each interpretation must be judged in terms of its implications for convenience and accuracy. Like conventions about defining measurement units [14], interpretations can be adjusted to improvements in theoretical and experimental understanding, in order to better serve the scientific community.

According to Born's statistical interpretation in the standard formulation (i.e., for projective measurements), each actual measurement result  $\lambda$  is claimed to be one of the eigenvalues, which is exactly (according to the literal reading<sup>21</sup> of most formulations) or approximately (in a more liberal reading) measured, with probabilities computed from  $A$  and the density operator  $\rho$  by the probability form of Born's rule. Since we saw that deviations from the eigenvalues in realistic experiments must be interpreted as measurement errors, we conclude that the eigenvalues are the true values of Born's statistical interpretation.

In the preceding subsections 3.1 and 3.2, we did not assume a notion of true value in the quantum case. However, (17) implies that the least uncertain value  $\xi$  is the q-expectation  $\bar{A}$ . Thus in a statistical sense, the best possible value that can be assigned is  $\bar{A}$ . This suggests that in the unbiased case,  $\bar{A}$  should be (in analogy to classical statistics) the true value of  $A$ , measured approximately.

### 3.5 The thermal interpretation of quantum physics

*We assume that (i) the density matrix is observable and (ii) any observable is a function of the density components.*

Lajos Diosi, 1988 [30, p.2887]

*The idea of unsharp objectification arises if one intends to leave quantum mechanics intact and still tries to maintain a notion of real and objective properties.*

Busch, Lahti and Mittelstaedt, 1996 ([20, p.127])

*We shall study a simple measurement problem – the measurement of the diameter of a disk. [...] It may happen that the difference of the measurements in different directions exceeds the permissible error of a given measurement. In this situation, we must state that within the required measurement accuracy, our disk does not have a unique diameter, as does a circle. Therefore, no concrete number can be taken, with prescribed accuracy, as an estimate of the true value of the measurable quantity.*

Semyon Rabinovich, 2005 ([78, p.11])

The preceding analysis suggests that we should perhaps reject the convention that declares the eigenvalues of operators to be the true values in a measurement. This is done explicitly in the **thermal interpretation of quantum physics** introduced in NEUMAIER

<sup>21</sup> The formulation appearing in WIKIPEDIA [88] is "the measured result will be one of the eigenvalues". GRIFFITHS & SCHROETER [39, p.133] declare, "If you measure an observable [...] you are certain to get one of the eigenvalues". PERES [74, p.95] defines, "each one of these outcomes corresponds to one of the eigenvalues of  $A$ ; that eigenvalue is then said to be the result of a measurement of  $A$ ". The only exceptions seem to be textbooks such as NIELSEN & CHUANG [73, p.84f] that start the formal exposition with the POVM approach rather than Born's interpretation. But in their informal introduction of qubits, even they give priority to projective measurements!

[65, 66, 67]; a detailed, definitive account is in my recent book *Coherent quantum physics* (NEUMAIER [68]); see especially Section 9.2. The thermal interpretation gives a new foundational perspective on quantum mechanics and suggests different questions and approaches than the traditional interpretations.

The thermal interpretation generalizes the well-known fact that in equilibrium statistical thermodynamics, all extensive quantities are represented by q-expectations. It proclaims – in direct opposition to the tradition created in 1927 by Jordan, Dirac, and von Neumann – the alternative convention that the true values are the q-expectations rather than the eigenvalues. In the thermal interpretation of quantum physics, given a particular instance of a quantum system described by a model at a given time, the state defines all its properties, and hence what **exists**<sup>22</sup> in the system at that time. The objective **properties** of the system are given by q-expectations and what is computable from these. All properties of a quantum system at a fixed time depend on the state  $\langle \cdot \rangle$  of the system at this time and are expressed in terms of definite but uncertain values of the quantities. As discussed in detail in my book (NEUMAIER [68]), the identification of these formal properties with real life properties is done by means of

**(CC) Callen’s criterion** (cf. CALLEN [23, p.15]): *Operationally, a system is in a given state if its properties are consistently described by the theory for this state.*

This is a concise version of the principle of identification suggested by EDDINGTON [33, p.222]; cf. the initial quote of this paper. Callen’s criterion is enough to find out in each single case how to approximately determine the uncertain value of a quantity of interest.

As we have seen, the simplest quantum system, a qubit, was already described by STOKES [83] in 1852, in terms essentially equivalent to the thermal interpretation – except that the intrinsic uncertainty was not yet an issue, being at that time far below the experimentally realizable accuracy. This alternative convention also matches the actual practice in quantum information theory, where the states are manipulated, transmitted, measured, hence their properties (i.e., whatever is computable from the state) are treated as objective properties. The density operator may be viewed simply as a calculational tool for obtaining these objective properties, in particular q-expectations and their uncertainties.

In the thermal interpretation, every observable scalar or vector quantity  $X$  has an associated intrinsic state-dependent uncertainty  $\sigma_X$  within which it can be (in principle) determined. The idea is that the q-expectation  $\bar{X}$  itself has no direct operational meaning; only the fuzzy region of  $\xi \in \mathbb{C}^m$  with  $|\xi - \bar{X}|$  bounded by the uncertainty  $\sigma_X$  or another small multiple of  $\sigma_X$  is meaningful. Statistics enters whenever a single value has too much uncertainty, and only then. In this case, the uncertainty can be reduced – as within classical physics – by calculating statistical means.

This is standard engineering practice when considering the diameter of a disk that is not

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<sup>22</sup>This gives a clear formal meaning to the notion of existence. Whether something that exists in this model sense also exists in Nature depends on how faithful the model is to the corresponding aspect of Nature.

perfectly circular. The uncertainty is in the imprecise definition, just as that in the position of an extended object such as a doughnut. In particular, the description of a quantum particle as having momentum  $p$  and being at position  $q$  is as unsharp as the description of a classical signal as having frequency  $\nu$  at time  $t$ . Even formally, the concepts are analogous and share the uncertainty relation, known in signal analysis as the Nyquist theorem, and discovered in the quantum context by Heisenberg. The analogy is especially clear in quantum field theory, where on the one hand position and time and on the other hand momentum and energy, related to frequency by the Rydberg–Ritz formula (16), are described on an equal footing. When measuring  $X$  it is meaningless to ask for more accuracy than the uncertainty  $\sigma_X$ , just as meaningless as to ask for the position of a doughnut to mm accuracy.

Thus the foundations invoked by the thermal interpretation are essentially the foundations used everywhere for uncertainty quantification, just slightly extended to accommodate quantum effects by not requiring that quantities commute.

### 3.6 Measurement in the thermal interpretation

*A satisfactory theory of the measuring process must start from a characterization of the macroscopic properties of a large body. Such properties must have an objective character.*

Daneri, Loinger and Prosperi, 1962 [25, p.305]

*Quantum mechanics has often been classified as a merely statistical ensemble theory, with not much bearings on the individual members of the ensembles. Yet there is an increasing variety of experiments exhibiting individual quantum processes which were conceived, devised and explained on the basis of this very theory.*

Busch and Lahti, 1996 [18, p.5899]

*Decoherence actually aggravates the measurement problem: where previously this problem was believed to be man-made and relevant only to rather unusual laboratory situations, it has now become clear that "measurement" of a quantum system by the environment (instead of by an experimental physicist) happens everywhere and all the time: hence it remains even more miraculous than before that there is a single outcome after each such measurement.*

Klaas Landsman, 2017 [56, p.443]

*We need to take the existence of measurement outcomes as a priori given, or otherwise give an account outside of decoherence of how measurement outcomes are produced, because the property of classicality is ultimately a statement about measurement statistics.*

Maximilian Schlosshauer, 2019 [80, p.72]

As argued in NEUMAIER [68, Section 10,5], decoherence tells roughly the same story as the thermal interpretation, but only in statistical terms, whereas the thermal interpretation refines this to a different, more detailed story for each single case. This is possible since in the thermal interpretation, measurement outcomes are defined as q-expectations of macroscopic detector quantities  $X'$  strongly correlated – in the way phenomenologically discussed in

Section 1 – with the microscopic quantities  $X$  to be measured. According to the thermal interpretation, a measurement  $a_k$  of  $X$  is treated as an approximation of the q-expectation  $\overline{X}$  of  $X$ .  $\overline{X}$  is (in principle, in general only inaccurately) **observable** if it varies sufficiently slowly with  $t$  and has a sufficiently small uncertainty  $\sigma_X$ . But it may require considerable experimental ingenuity to do so with an uncertainty close to  $\sigma_X$ . The uncertain value  $\overline{X}$  is considered informative only when its q-uncertainty  $\sigma_X$  is much less than  $|\overline{X}|$ .

Since q-expectations are always single-valued, this immediately resolves the unique outcome problem of quantum measurement theory. The thermal interpretation makes direct sense of individual events even at the theoretical level. It naturally gives an ontology for individual quantum systems – not only for thermal systems but also for microscopic systems and even the whole universe. In particular, because of the single-valuedness of the true values in the thermal interpretation, probabilities are not intrinsic to quantum physics but are emergent imperfections. In contrast, Born’s statistical interpretation needs probabilities in the very foundations, due to the multi-valuedness of the true values.

Both interpretations are in agreement with the experimental record. The same number  $a_k$  obtained by a measurement may be interpreted in two ways, depending on the convention used: (i) It measures the q-expectation to some accuracy. (ii) It measures some random eigenvalue to a possibly higher (in the idealization even infinite) accuracy. In both cases, the measurement involves an additional uncertainty related to the degree of reproducibility of the measurement, given by the standard deviation of the results of repeated measurements. Tradition and the thermal interpretation agree in that this uncertainty is – by (17) – at least  $\sigma_X$ . If the eigenvalues  $X_k$  of  $X$  (assumed to have discrete spectrum) are exactly known beforehand, one can calibrate the pointer scale to make  $a_k = X_k$  for all detector elements  $k$ . As long as one ignores the idealization error, the thermal interpretation and Born’s interpretation become experimentally indistinguishable. However, this is no longer so in the more realistic case where eigenvalues are only approximately known – the common situation in spectroscopy – and must therefore be inferred experimentally. In this case (cf. Footnote <sup>20</sup>), Born’s statistical interpretation paints an inadequate, idealized picture only. The thermal interpretation, however, still gives a correct account of the actual experimental situation. Measurements are regarded as fluctuating discrete readings of detector properties, defined as q-expectations of macroscopic pointer variables whose statistical mean agrees with the q-expectation of the system quantity measured, as discussed in Section 1.

Thus the thermal interpretation is in full agreement with the standard recipes for drawing inferences from inaccurate measurement results. The situation is precisely the same as in classical metrology, where observable quantities always have a true value determined by the theoretical description, and all randomness in measurements is assumed to be due to measurement noise.

The thermal interpretation regards each measurement result  $a_k$  as an approximation of the true value  $\overline{X}$ , with typical error  $|a_k - \overline{X}|$  of at least  $\sigma_X$ , by (17). In the limit of arbitrarily many repetitions, the statistical mean value of the approximations approaches the q-expectation  $\overline{X}$ , and their standard deviation approaches the q-uncertainty  $\sigma_X$ . The

observed discreteness is explained as an effect due to the recording device. The latter introduces a systematic discretization error, of the same nature as the rounding errors in the illustrative example given in Subsection 3.3.

For example, binary responses of the macroscopic detector elements may be explained as in NEUMAIER [68, Chapter 11] by a bistability (cf. BONIFACIO & LUGIATO [11], GEVORGYAN et al. [36]) of their coarse-grained microscopic dynamics analogous to the bistability that give rise to binary responses in classical coin tossing. The bimodal distribution of the measurement results may be due to environment-induced randomness and environment-induced dissipation, as for a classical, environment-induced diffusion process in a double-well potential. For a discussion of the dynamical aspects of the quantum chaos responsible for this see, e.g., INGRAHAM & ACOSTA [47], ZHANG & FENG [90], BELOT & EARMAN [10], GOMEZ et al. [37], and Chapter 11 of my book (NEUMAIER [68]). The bimodal distribution may also be due more immediately to the experimental setup. For example, the arrangement in a Stern–Gerlach experiment together with simple theory leads to two more or less focussed beams, which accounts for the approximate 2-valuedness of the response at the screen. The thermal interpretation attributes this discreteness to the detection setup, not to a true, discrete value of the spin.

For precision measurements, e.g., those related to the quantum Hall effect, relevant for preparing calibration states for metrology standards (LINDELEY ([59], KANEKO et al. [50]), one needs to be able to prepare states with tiny  $q$ -uncertainties, so that their measurement can be performed with very high accuracy. The possibility of achieving tiny uncertainties is linked to the spectrum. Let  $X$  be a scalar or vector quantity whose  $m$  components are defined on a common dense domain. We call the set of  $\xi \in \mathbb{C}^m$  for which no linear operator  $R(\xi)$  exists such that  $R(\xi)|X - \xi|^2$  is the identity the **spectrum**  $\text{Spec } X$  of  $X$ . The spectrum is always a closed set, but it may be empty, as, e.g., for the vector formed by position and momentum of a particle. The following result implies that the preparation of states such that  $X$  has arbitrarily small uncertainty is possible precisely when  $\bar{X}$  belongs to the spectrum of  $X$ .

**3.2 Theorem.**  $\xi \in \mathbb{C}^m$  belongs to the spectrum of a scalar or vector quantity  $X$  with  $m$  components iff states exist that have arbitrarily small positive  $\langle |X - \xi|^2 \rangle$ .

*Proof.* By definition, the operator  $B := |X - \xi|^2$  is Hermitian and positive semidefinite, hence essentially self-adjoint with nonnegative spectrum  $\Sigma$ . Note that  $\xi$  belongs to the spectrum of  $X$  iff  $0 \in \Sigma$ . We consider the spectral projector  $P(s)$  to the invariant subspace corresponding to the spectrum in  $[0, s]$ . If  $0 \notin \Sigma$  then  $P(s) = 0$  for some  $s > 0$ , and  $\langle |X - \xi|^2 \rangle = \langle B \rangle \geq s$  cannot be arbitrarily small. On the other hand, if  $0 \in \Sigma$  then, for all  $s > 0$ , the projector  $P(s)$  is nonzero. Thus we can find vectors  $\phi(s)$  such that  $\psi(s) := P(s)\phi(s)$  is nonzero. By scaling  $\phi(s)$  appropriately we can ensure that  $\psi(s)$  has norm one. Then in the corresponding pure state,  $\langle |X - \xi|^2 \rangle = \langle B \rangle \leq s$ . Since this works



for any  $s > 0$ , the claim follows. □

Thus though in the thermal interpretation eigenvalues no longer play the role they traditionally have in measurement, they continue to be essential for high precision measurements. Elsewhere in quantum physics, they play a role primarily in the case of the Hamiltonian, where the eigenvalues specify the possible energy levels of a quantum system. The latter is relevant not only on the theoretical level, where a spectral representation allows the explicit solution of the quantum Liouville equation and the time-dependent Schrödinger equation. It is also important in many contexts with experimentally relevant consequences: We have seen the Rydberg–Ritz formula (16) from spectroscopy, which can access a large number of energy levels. In many other cases, only few energy levels are experimentally accessible, in which case the quantum system can be modeled as a few level system, drastically simplifying the task of state tomography. In much of quantum chemistry, only the electronic ground state is considered relevant. Its energy, parameterized by the nuclear coordinates, determines the potential energy surface whose properties are sufficient to describe the shape of all molecules and their chemical reactions. The energy spectrum is also prominent in thermal equilibrium physics since it determines the partition function from which all other thermodynamic properties can be derived. For a treatment of thermal equilibrium quantum physics in the spirit of the thermal interpretation – i.e., based on  $q$ -expectations without making reference to statistical assumptions – see the online textbook by NEUMAIER & WESTRA [71].

## 4 Conclusion

In Chapter 1, assuming an intuitive informal notion of response, a suggestive definition was given of what constitutes a detector. From this, the standard POVM description of general quantum measurements was derived, including Born’s rule in its generalized POVM form. The traditional form of Born’s rule for projective measurement followed as a special case. Rather than postulating Born’s rule for projective measurements, as done in standard textbooks, it was derived here together with all its ramifications and its domain of validity, from simple, easily motivated definitions. In particular, the spectral notions appear not as postulated input as in traditional expositions, but as consequences of the derivation. The derivation shows that measurements satisfying Born’s rule (i.e., projective measurements) are always idealizations. Born’s rule in textbook form does not apply when one approximately measures quantities with a continuous spectrum, or simultaneously measures quantities corresponding to noncommuting operators – one needs a POVM that is not projective. Other limitations of Born’s rule were pointed out in my recent book (NEUMAIER [68, Section 14.3]).

Though very elementary, our notion of a detector and the associated formal notion of a discrete POVM is very flexible and accounts for all basic aspects of practical measurement processes. Apart from the simplicity and the straightforward motivation and derivation, another advantage of the POVM setting above is that it is absolutely clear what measure-

ment amounts to and how accurately it may represent given  $q$ -expectations in operational terms.

For qubits, the density operator and the dynamical equations (quantum Liouville, Lindblad, and Schrödinger equation) were derived from classical optics. Together with the present elementary approach to measurement, this provides a fully intelligible introduction to all basic features of quantum mechanics. In contrast to the usual treatments, where these basic features are addressed by just postulating the required items, this gives actual understanding, not only the appearance of it. To motivate and understand Born's rule for POVMs is much easier (one just needs simple linear algebra) than to motivate and understand Born's rule in its original form, where unfamiliar stuff about wave functions, probability amplitudes and spectral representations must be swallowed by the beginner – not to speak of the difficult notion of self-adjointness (which is usually simply suppressed in introductory treatments). Thus introductory courses on quantum mechanics would benefit from presenting the real thing in place of only a time-honored but too special, highly idealized case. An outline for a possible design of such a course was given in Section 2.6.

In Section 3, a thorough discussion was given of various aspects of uncertainty in quantum measurements. It culminated in a justification of the thermal interpretation of quantum physics, treated in detail in my recent book *Coherent quantum physics* (NEUMAIER [68]). This interpretation replaces the traditional fundamentally stochastic eigenvalue link to measurement by the assumption that the true properties of a quantum system are its  $q$ -expectations and what can be computed from these. This gives the stochastic aspects of measurements the same status as in classical mechanics, thus making the foundations of quantum physics much more intuitive.

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